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Structural properties of $\mathcal{H}_\infty$ discrete-time controllers based on J-lossless factorisations

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***Abstract*— The paper deals with a problem of numerically reliable synthesis of $\mathcal{H}_\infty$-optimal discrete-time controllers based on J-lossless factorisations of the delta-domain chain-scattering models of continuous-time plants. Necessary and sufficient conditions for the existence of strictly proper $\mathcal{H}_\infty$ discrete-time controllers are given. Both the regular problem concerning models with no zeros on the stability boundary and the extended problem of models with such zeros are discussed. It is shown that in the latter case some structural constraints always appear. Controllers are obtained via solving two coupled delta-domain algebraic Riccati equations.**

I. INTRODUCTION

It is well known that the so-called delta (δ) operator approach to many discrete-time design problems in the area of control, estimation, modelling, and signal processing has a number of advantages as opposed to using the conventional forward shift operator (q) [1].

The δ-operator formulation has better numerical conditioning at higher sampling rate and is less sensitive to arithmetic round-off errors and allows for describing the asymptotic behaviour of discrete-time models of continuous-time systems as the sampling period converges to zero [1].

The δ-operator methodology has been widely accepted as an effective tool well matched to modern control system design procedures including those based on the $\mathcal{H}_\infty$ paradigm [2], [3]. In this paper, the structure of δ-operator $\mathcal{H}_\infty$ suboptimal controllers based on J-lossless approaches are studied. New conditions for the existence of strictly proper solutions are given (see also [4], [5]). Some structural constraints imposed by plant zeros located on the stability circle in the δ-domain are also investigated.

Both regular problems concerning models having no zeros on the stability circle and the extended problem of models with such zeros are examined. For the first case, an approach based on a dual J-lossless factorisation is proposed while in the second case an extended J-lossless factorisation of the process model is required. Suitable factorisations are obtained by solving two coupled algebraic Riccati equations given in the δ-domain formulation.

II. DELTA-DOMAIN MODELLING

The delta operator

$$\delta \;:\; l_2 \to l_2 \tag{1}$$

is defined as the first-order divided difference

$$\delta = \frac{q-1}{\Delta} \tag{2}$$

where q is the forward shift operator and $\Delta > 0$ is the sampling interval [1]. Let (q,z) and (δ,ζ) denote the pairs of discrete-time operators q and δ, and the corresponding complex variables z and ζ. Let

$$\mathcal{D}_\Delta = \{\zeta \;:\; |\zeta + 1/\Delta| < 1/\Delta\}$$

be the open shifted circle. The closed circle is denoted as $\bar{\mathcal{D}}_\Delta$ with the boundary $\partial\mathcal{D}_\Delta$. The set of all eigenvalues of a square matrix A is denoted by $\lambda(A)$. A matrix A is said to be stable if $\lambda(A) \subset \mathcal{D}_\Delta$ while a transfer matrix $G(\zeta)$ is stable if all its poles belong to $\mathcal{D}_\Delta$. The conjugate of $G(\zeta)$ is defined as

$$G^\sim(\zeta) = G^T(-\zeta/(1+\Delta\zeta))$$

The Hermitian conjugate is $G^*(\zeta) = G^T(\zeta^*)$.

Let a linear continuous-time ($\rho = d/dt$) system be described be the following state-space model

$$\begin{cases} \rho x(t) = A_\rho x(t) + B_\rho u(t) \\ y(t) = C_\rho x(t) + D_\rho u(t) \end{cases} \tag{3}$$

here $x(t)$ denotes the state vector, $u(t)$ is the input, and $y(t)$ is the output. If $u(t)$ is piece-wise constant and right-continuous the following state-space model can be derived

$$\begin{cases} \delta x_k = Ax_k + Bu_k \\ y_k = Cx_k + Du_k \end{cases} \tag{4}$$

where

$$\begin{array}{ll} A = \Delta^{-1}\Gamma_\Delta A_\rho & B = \Delta^{-1}\Gamma_\Delta B_\rho \\ C = C_\rho & D = D_\rho \\ \Gamma_\Delta = \int_0^\Delta e^{\tau A_\rho} d\tau & \end{array}$$

The q-domain model takes form of (A_q, B_q, C_q, D_q) with

$$\begin{array}{ll} A_q = I_n + \Delta A & B_q = \Delta B \\ C_q = C & D_q = D \end{array}$$

Consider the the discrete-time δ-domain Riccati equation (δARE) in $X \in \mathbb{R}^{n\times n}$

$$P^T X + XP + \Delta P^T XP - ((I_n + \Delta P^T)XQ + S)(T + \Delta Q^T XQ)^{-1} \times ((I_n + \Delta P^T)XQ + S)^T + R = 0_{n\times n} \quad (5)$$

where $P,\, R = R^T \in \mathbb{R}^{n\times n},\ Q,\, S \in \mathbb{R}^{n\times m}$ and $T = T^T \in \mathbb{R}^{m\times m}$. Let

$$(U, W) = \quad (6)$$

$$\left(\begin{bmatrix} P & 0_{n\times n} & Q \\ -R & -P^T & -S \\ S^T & Q^T & T \end{bmatrix}, \begin{bmatrix} I_n & 0_{n\times n} & 0_{n\times m} \\ 0_{n\times n} & I_n + \Delta P^T & 0_{n\times m} \\ 0_{m\times n} & -\Delta Q^T & 0_{m\times m} \end{bmatrix} \right)$$

Eigenvalues $\lambda(U, W)$ of the $(2n+m)\times(2n+m)$ extended pencil associated with (U, W) are defined by

$$\lambda(U, W) = \{z \in \mathbb{C} : \det(U - zW) = 0\} \quad (7)$$

Let $X_-(U, W)$ of dimension $n_- = \dim(X_-(U, W)) \leq n$ denote the invariant subspace corresponding to stable eigenvalues associated with (U, W). Let $[\ X_1^T \quad X_2^T \quad X_3^T\ ]^T \in \mathbb{R}^{(n+n+m)\times n_-}$ be a matrix of full column rank whose columns form a basis for $X_-(U, W)$. The domain of δRic, denoted by $\text{dom}(\delta\text{Ric})$, consists of all pairs (U, W) such that $n_- = n$ and $X_1 \in \mathbb{R}^{n\times n}$ is non-singular.

The numerical conditionig of the above discrete-time Riccati equations is discussed in [6], [7], [8] where the sensitivity properties are evaluated by using a suitable defined relative condition number.

III. $\mathcal{H}_\infty$ OPTIMISATION IN DELTA-DOMAIN

Let $\mathcal{RL}_\infty^{p\times r}$ denote the space of proper real-rational $p\times r$-matrix-valued functions of $\zeta \in \mathbb{C}$ that are analytical in $\partial\mathcal{D}_\Delta$. $\mathcal{RH}_\infty^{p\times r} \subset \mathcal{RL}_\infty^{p\times r}$ consists of all stable matrices. The set of all unitary bounded matrices in $\mathcal{RH}_\infty^{p\times r}$ is defined by

$$\mathcal{BH}_\infty^{p\times r} = \{\Phi \in \mathcal{RH}_\infty^{p\times r} \colon \|\Phi\|_\infty < 1\} \quad (8)$$

where $\|\cdot\|_\infty$ is the $\mathcal{RH}_\infty^{p\times r}$ infinity norm, while the units of $\mathcal{RH}_\infty^{p\times p}$ are denoted by $\mathcal{GH}_\infty^{p}$. Moreover, let

$$J_{mn} = I_m \oplus (-I_n) \in \mathbb{R}^{(m+n)\times(m+n)} \quad (9)$$

denote a signature matrix.

Consider a standard linear discrete-time generalised plant described by its *scattering matrix* [9], [10]

$$\begin{aligned} P \quad &: \quad \begin{bmatrix} w \\ u \end{bmatrix} \rightarrow \begin{bmatrix} z \\ y \end{bmatrix} \\ P(\zeta) \quad &= \quad \begin{bmatrix} P_{zw}(\zeta) & P_{zu}(\zeta) \\ P_{yw}(\zeta) & P_{yu}(\zeta) \end{bmatrix} \end{aligned} \quad (10)$$

with four input/output signals: w is the exogenous input of dimension r, u of dimension p is the controlling input, z of dimension m is the controlled output (objective) and y is the measured output of dimension q.

The closed-loop system equipped with a state feedback controller

$$K \ : \ y \rightarrow u$$

can be described by a *linear fractional transformation*

$$\begin{aligned} LF(P, K) \quad &: \quad w \rightarrow z \\ LF(P, K) \quad &= \quad P_{zw} + P_{zu}K(I_q - P_{yu}K)^{-1}P_{yw} \end{aligned}$$

of K with respect to P.

The standard problem of optimisation in $\mathcal{H}_\infty$ is to find a causal linear K, which internally stabilises the closed-loop system and enforces the norm bound

$$\|LF(P, K)\|_\infty < \gamma$$

for a prespecified parameter $\gamma > 0$ [9], [10].

The plant P of (10) with $q = r$ and an invertible $P_{yw}(\zeta)$ can be characterised via its *chain-scattering* representation while the plant with $m = p$ and an invertible $P_{zu}(\zeta)$ can be characterised via its *dual chain-scattering* representation, respectively [9]

$$\begin{aligned} G \quad &: \quad \begin{bmatrix} u \\ y \end{bmatrix} \rightarrow \begin{bmatrix} z \\ w \end{bmatrix} \\ G(\zeta) \quad &= \quad \begin{bmatrix} G_{zu}(\zeta) & G_{zy}(\zeta) \\ G_{wu}(\zeta) & G_{wy}(\zeta) \end{bmatrix} \end{aligned} \quad (11)$$

$$\begin{aligned} H \quad &: \quad \begin{bmatrix} z \\ w \end{bmatrix} \rightarrow \begin{bmatrix} u \\ y \end{bmatrix} \\ H(\zeta) \quad &= \quad \begin{bmatrix} H_{uz}(\zeta) & H_{uw}(\zeta) \\ H_{yz}(\zeta) & H_{yw}(\zeta) \end{bmatrix} \end{aligned} \quad (12)$$

The closed-loop system can thus be described by the *homographic transformation*

$$\begin{aligned} HM(G, K) \quad &: \quad w \rightarrow z \\ HM(G, K) \quad &= \quad (G_{zu}K + G_{zy})(G_{wu}K + G_{wy})^{-1} \end{aligned}$$

or the *dual homographic transformation*

$$\begin{aligned} HDM(G, K) \quad &: \quad w \rightarrow z \\ DHM(H, K) \quad &= \quad -(H_{uz} - KH_{yz})^{-1}(H_{uw} - KH_{yw}) \end{aligned}$$

of K with respect to G or H, respectively.

Now, the standard $\mathcal{H}_\infty$ optimisation problem is to find a causal K, which internally stabilises the closed-loop system and enforces the norm bounds

$$\|HM(G, K)\|_\infty < \gamma$$

or

$$\|DHM(G, K)\|_\infty < \gamma$$

for a prespecified $\gamma > 0$ [9].

IV. J-LOSSLESS FACTORISATION SOLUTIONS

The key role in the theory of $\mathcal{H}_\infty$ control is played by the so-called J-lossless factorisations of plant models [3], [9], [11], [12].

Definition 1:

(*i*) A $G(\zeta) \in \mathcal{RL}_\infty^{(m+r)\times(p+r)}$ is said to be (J_{mr}, J_{pr})-*unitary*, if

$$G^\sim(\zeta)J_{mr}G(\zeta) = J_{pr}, \quad \forall\zeta.$$

(*ii*) A (J_{mr}, J_{pr})-unitary $G(\zeta)$ is said to be (J_{mr}, J_{pr})-*lossless*, if

$$G^*(\zeta)J_{mr}G(\zeta) \geq J_{pr}, \quad \forall\zeta \notin \mathcal{D}_\Delta.$$

(*iii*) A $H(\zeta) \in \mathcal{RL}_\infty^{(m+q)\times(m+r)}$ is said to be *dual* (J_{mq}, J_{mr})-*unitary*, if

$$H(\zeta)J_{mr}H^\sim(\zeta) = J_{mq}, \quad \forall\zeta.$$

(*iv*) A dual (J_{mq}, J_{mr})-unitary $H(\zeta)$ is said to be *dual* (J_{mq}, J_{mr})-*lossless*, if

$$H(\zeta)J_{mr}H^*(\zeta) \geq J_{mq}, \quad \forall\zeta \notin \mathcal{D}_\Delta. \ \square$$

A. Basic J-lossless solutions

Definition 2:

(*i*) If $G(\zeta) \in \mathcal{RL}_\infty^{(m+r)\times(p+r)}$ can be represented as

$$G(\zeta) = \Theta(\zeta)\Pi(\zeta)$$

where $\Theta(\zeta) \in \mathcal{RL}_\infty^{(m+r)\times(p+r)}$ is (J_{mr}, J_{pr})-lossless and $\Pi(\zeta) \in \mathcal{GH}_\infty^{p+r}$, then $G(\zeta)$ is said to have a (J_{mr}, J_{pr})-*lossless factorisation*.

(*ii*) If $H(\zeta) \in \mathcal{RL}_\infty^{(m+q)\times(m+r)}$ can be represented as

$$H(\zeta) = \Omega(\zeta)\Psi(\zeta)$$

where $\Psi(\zeta) \in \mathcal{RL}_\infty^{(m+q)\times(m+r)}$ is dual (J_{mq}, J_{mr})-lossless and $\Omega(\zeta) \in \mathcal{GH}_\infty^{m+q}$, then $G(\zeta)$ is said to have a *dual* (J_{mq}, J_{mr})-*lossless factorisation*. $\square$

The following theorems [3], [8] can be proved in a similar way to the q-domain cases [11].

Theorem 1: *Let (A, B, C, D) be a minimal realisation of $G(\zeta) \in \mathcal{RL}_\infty^{(m+r)\times(p+r)}$ with no zeros on $\partial\mathcal{D}_\Delta$. It has a (J_{mr}, J_{pr})-lossless factorisation iff the conditions hold*:

(*i*) $(U_x, W_x) \in \mathrm{dom}(\delta\mathrm{Ric})$ *and* $X = \delta\mathrm{Ric}(U_x, W_x) \geq 0$*, where*

$$\begin{array}{c} P_x = A,\ Q_x = B,\ R_x = C^TJ_{mr}C, \\ S_x = C^TJ_{mr}D,\ T_x = D^TJ_{mr}D; \end{array} \tag{13}$$

(*ii*) $(U_{\bar{x}}, W_{\bar{x}}) \in \mathrm{dom}(\delta\mathrm{Ric})$ *and* $\bar{X} = \delta\mathrm{Ric}(U_{\bar{x}}, W_{\bar{x}}) \geq 0$*, where*

$$\begin{array}{c} P_{\bar{x}} = A^T, \quad Q_{\bar{x}} = C^T, \quad R_{\bar{x}} = 0_{n\times n}, \\ S_{\bar{x}} = 0_{n\times(m+r)},\ \ T_{\bar{x}} = -J_{mr}; \end{array} \tag{14}$$

(*iii*) $\|X\bar{X}\|_s < 1$*;*

(*iv*) *there exists a non-singular M_x such that*

$$M_x^T(T_x + \Delta Q_x^TXQ_x)M_x = J_{pr}. \ \square \tag{15}$$

Theorem 2: *Let (A, B, C, D) be a minimal realisation of $H(\zeta) \in \mathcal{RL}_\infty^{(m+q)\times(m+r)}$ with no zeros on $\partial\mathcal{D}_\Delta$. It has a dual (J_{mq}, J_{mr})-lossless factorisation iff the conditions hold*:

(*i*) $(U_y, W_y) \in \mathrm{dom}(\delta\mathrm{Ric})$ *and* $Y = \delta\mathrm{Ric}(U_y, W_y) \geq 0$*, where*

$$\begin{array}{c} P_y = A^T,\ Q_y = C^T,\ R_y = -BJ_{mr}B^T, \\ S_y = -BJ_{mr}D^T,\ T_y = -DJ_{mr}D^T; \end{array} \tag{16}$$

(*ii*) $(U_{\bar{y}}, W_{\bar{y}}) \in \mathrm{dom}(\delta\mathrm{Ric})$ *and* $\bar{Y} = \delta\mathrm{Ric}(U_{\bar{y}}, W_{\bar{y}}) \geq 0$*, where*

$$\begin{array}{c} P_{\bar{y}} = A, \quad Q_{\bar{y}} = B, \quad R_{\bar{y}} = 0_{n\times n}, \\ S_{\bar{y}} = 0_{n\times(m+q)},\ \ T_{\bar{y}} = J_{mr}; \end{array} \tag{17}$$

(*iii*) $\|Y\bar{Y}\|_s < 1$*;*

(*iv*) *there exists a non-singular M_y such that*

$$M_y(T_y + \Delta Q_y^TYQ_y)M_y^T = -J_{mq}. \ \square \tag{18}$$

Let $G_\gamma(\zeta)$ denote the plant model scaled with γ and assume that $G_\gamma(\zeta)$ has a (J_{mr}, J_{pr})-lossless factorisation $G_\gamma(\zeta) = \Theta(\zeta)\Pi(\zeta)$. The set of controllers $K(\zeta)$, for which $\|HM(G_\gamma, K)\|_\infty < 1$ holds, is parameterised with an arbitrary $\Phi(\zeta) \in \mathcal{BH}_\infty^{p\times r}$

$$K = HM(\Pi^{-1}, \Phi) \tag{19}$$

On the other hand, let $H_\gamma(\zeta)$ denote the plant model scaled with γ and assume that $H_\gamma(\zeta)$ has a dual (J_{mq}, J_{mr})-lossless factorisation $H_\gamma(\zeta) = \Omega(\zeta)\Psi(\zeta)$. The set of controllers $K(\zeta)$, for which $\|DHM(H_\gamma, K)\|_\infty < 1$ holds, is parameterised with an arbitrary $\Phi(\zeta) \in \mathcal{BH}_\infty^{m\times q}$

$$K = DHM(\Omega^{-1}, \Phi) \tag{20}$$

B. Extended J-lossless approach

A necessary condition for the existence of the stabilising solution X and Y of Theorems 1 and 2 is that $G(\zeta)$ and $H(\zeta)$ have no zeros on $\partial\mathcal{D}_\Delta$, respectively. We will discuss the case in which this assumption does not hold [8].

Definition 3:

(*i*) If $G(\zeta) \in \mathcal{RL}_\infty^{(m+r)\times(p+r)}$ is represented as a product $G(\zeta) = \Theta(\zeta)\Pi(\zeta)$ where $\Theta(\zeta) \in \mathcal{RL}_\infty^{(m+r)\times(p+r)}$ is (J_{mr}, J_{pr})-lossless and $\Pi(\zeta) \in$

$\mathcal{RH}_\infty^{(p+r)\times(p+r)}$ does not have any zeros outside $\bar{\mathcal{D}}_\Delta$, then $G(\zeta)$ is said to have an *extended* (J_{mr}, J_{pr})*-lossless factorisation*.

(*ii*) If $H(\zeta) \in \mathcal{RL}_\infty^{(m+q)\times(m+r)}$ is represented as a product $H(\zeta) = \Omega(\zeta)\Psi(\zeta)$ where $\Psi(\zeta) \in \mathcal{RL}_\infty^{(m+q)\times(m+r)}$ is dual (J_{mq}, J_{mr})-lossless and $\Omega(\zeta) \in \mathcal{RH}_\infty^{(m+q)\times(m+q)}$ does not have any zeros outside $\bar{\mathcal{D}}_\Delta$, then $H(\zeta)$ is said to have an *extended dual* (J_{mq}, J_{mr})*-lossless factorisation*. □

The set of all controllers $K(\zeta)$ satisfying $\|HM(G_\gamma, K)\|_\infty < 1$ is parameterised by $K = HM(\Pi^{-1}, \Phi)$ where $\Phi(\zeta) \in \mathcal{BH}_\infty^{p\times r}$ is such that $K(\zeta) \in \mathcal{RH}_\infty^{p\times r}$. Similarly, the set of all controllers $K(\zeta)$ satisfying $\|DHM(H_\gamma, K)\|_\infty < 1$ is given by $K = DHM(\Omega^{-1}, \Phi)$ with $\Phi(\zeta) \in \mathcal{BH}_\infty^{m\times q}$ chosen is such a way that $K(\zeta) \in \mathcal{RH}_\infty^{m\times q}$.

Extended J-lossless factorisations can be obtained by using a modified 'zero compensation' technique [8] based on [13]. Assume that $G(\zeta) \in \mathcal{RL}_\infty^{(m+r)\times(p+r)}$ has n_z invariant zeros on $\partial\mathcal{D}_\Delta$. Let $S_G(\zeta)$ denote the system matrix of $G(\zeta)$. Employing a *QZ* transformation with unitary matrices Q_z and Z_z gives

$$Q_z^T S_G(\zeta) Z_z = \begin{bmatrix} S_z - \zeta T_z & * \\ 0_{(n+m+r-n_z)\times n_z} & \star \end{bmatrix} \qquad (21)$$

where $S_z - \zeta T_z$ with $S_z, T_z \in \mathbb{R}^{n_z\times n_z}$ is a regular pencil containing all the elementary divisors associated with the $\partial\mathcal{D}_\Delta$ zeros of $G(\zeta)$:

$$\lambda(S_z, T_z) = \lambda(T_z^{-1}S_z) \subset \partial\mathcal{D}_\Delta$$

Let Z_z be partitioned in conformity with $S_G(\zeta)$

$$Z_z = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \qquad (22)$$

Analogous assumptions concerning $S_{H^T}(\zeta)$ can be made for an $H(\zeta) \in \mathcal{RL}_\infty^{(m+q)\times(m+r)}$ with n_z invariant zeros on $\partial\mathcal{D}_\Delta$.

Theorem 3: *Let* (A, B, C, D) *be a minimal realisation of* $G(\zeta) \in \mathcal{RL}_\infty^{(m+r)\times(p+r)}$ $(H(\zeta) \in \mathcal{RL}_\infty^{(m+q)\times(m+r)})$ *having* n_z *zeros on* $\partial\mathcal{D}_\Delta$. *Let* $[\ S_1^T \ \ S_2^T \ \ S_3^T\ ]^T \in \mathbb{R}^{(n+n+(p+r))\times(n-n_z)}$ $(\in \mathbb{R}^{(n+n\ +(m+q))\times(n-n_z)})$ *denote a basis of a stable invariant* $(n-n_z)$*-dimensional subspace of the extended pencil* $U_x - \zeta W_x$ $(U_y - \zeta W_y)$. $G(\zeta)$ $(H(\zeta))$ *has an extended* (J_{mr}, J_{pr})*-lossless (dual* (J_{mr}, J_{pr})*-lossless) factorisation iff the conditions hold:*

(*i*) $[\ S_1 \ \ Z_{11}\ ] \in \mathbb{R}^{n\times n}$ *is non-singular and* X (Y) $= [\ S_2 \ \ 0_{n\times n_z}\ ][\ S_1 \ \ Z_{11}\ ]^{-1} \geq 0$, X (Y) $\in \mathbb{R}^{n\times n}$;

(*ii*) $(U_{\bar{x}}, W_{\bar{x}})$ $(U_{\bar{y}}, W_{\bar{y}}) \in \mathrm{dom}(\delta\mathrm{Ric})$ *and* $\bar{X} = \delta\mathrm{Ric}(U_{\bar{x}}, W_{\bar{x}}) \geq 0$ $(\bar{Y} = \delta\mathrm{Ric}(U_{\bar{y}}, W_{\bar{y}}) \geq 0)$;

(*iii*) $\|X\bar{X}\|_s$ $(\|Y\bar{Y}\|_s) < 1$;

(*iv*) *there exists a non-singular* M_x (M_y) *satisfying (15) ((18)).* □

V. STRUCTURE OF CONTROLLERS

Necessary and sufficient conditions for the existence of J-lossless based strictly proper $\mathcal{H}_\infty$ controllers are given.

A. Basic attributes of controllers

Consider $K = HM(\Pi^{-1}, \Phi)$. It is easily seen that

$$\Pi(\zeta)^{-1} = \Pi_x(\zeta) \cdot N_{x\bar{x}}$$

where [8]

$$\Pi_x(\zeta) = \left[\begin{array}{c|c} A + BF_x & (I_n - \bar{X}X)^{-1}B_{\bar{x}} \\ \hline F_x & I_{p+r} \end{array}\right] \qquad (23)$$

$$\begin{aligned} B_{\bar{x}} &= B + H_{\bar{x}}D \\ F_x &= -(T_x + \Delta Q_x^T X Q_x)^{-1}((I_n + \Delta P_x^T)XQ_x + S_x)^T \\ H_{\bar{x}} &= -(I_n + \Delta P_{\bar{x}}^T)\bar{X}Q_{\bar{x}}(T_{\bar{x}} + \Delta Q_{\bar{x}}^T \bar{X} Q_{\bar{x}})^{-1} \end{aligned}$$

while $N_{x\bar{x}} \in \mathbb{R}^{(p+r)\times(p+r)}$ denotes a non-singular matrix satisfying

$$N_{x\bar{x}}^{-T} J_{pr} N_{x\bar{x}}^{-1} = D^T(J_{mr} - \Delta C\bar{X}C^T)^{-1}D + \Delta B_{\bar{x}}^T X (I_n - \bar{X}X)^{-1}B_{\bar{x}}$$

Factor $\Pi_x(\zeta) \in \mathcal{RH}_\infty^{p+r}$ can be represented as

$$\begin{aligned} \Pi_x(\zeta) &= \left[\begin{array}{c|cc} A + BF_x & U_{\bar{x}x}B_{\bar{x}}^a & U_{\bar{x}x}B_{\bar{x}}^b \\ \hline F_x^u & I_p & 0_{p\times r} \\ F_x^y & 0_{r\times p} & I_r \end{array}\right] \qquad (24) \\ U_{\bar{x}x} &= (I_n - \bar{X}X)^{-1} \in \mathbb{R}^{n\times n} \\ B_{\bar{x}} &= \begin{bmatrix} B_{\bar{x}}^a & B_{\bar{x}}^b \end{bmatrix}, \quad B_{\bar{x}}^a \in \mathbb{R}^{n\times p}, \quad B_{\bar{x}}^b \in \mathbb{R}^{n\times r} \\ F_x &= \begin{bmatrix} F_x^u \\ F_x^y \end{bmatrix}, \quad F_x^u \in \mathbb{R}^{p\times n}, \quad F_x^y \in \mathbb{R}^{r\times n} \end{aligned}$$

Consequently, the controller can be implemented in the following manner

$$\begin{aligned} \delta\hat{x}(t) &= (A + BF_x)\hat{x}(t) + \qquad (25) \\ &\quad U_{\bar{x}x}(B_{\bar{x}}^a HM(N_{x\bar{x}}, \Phi) + B_{\bar{x}}^b)b_x(t) \\ b_x(t) &= -F_x^y\hat{x}(t) + y(t) \\ u(t) &= F_x^u\hat{x}(t) + HM(N_{x\bar{x}}, \Phi)b_x(t) \end{aligned}$$

Consider $K = DHM(\Omega^{-1}, \Phi)$. It is easy to check that

$$\Omega(\zeta)^{-1} = N_{y\bar{y}} \cdot \Omega_y(\zeta)$$

where [8]

$$\Omega_y(\zeta) = \left[\begin{array}{c|c} A + H_yC & H_y \\ \hline C_{\bar{y}}(I_n - Y\bar{Y})^{-1} & I_{m+q} \end{array}\right] \qquad (26)$$

$$\begin{aligned} C_{\bar{y}} &= C + DF_{\bar{y}} \\ F_{\bar{y}} &= -(T_{\bar{y}} + \Delta Q_{\bar{y}}^T \bar{Y} Q_{\bar{y}})^{-1}((I_n + \Delta P_{\bar{y}}^T)\bar{Y}Q_{\bar{y}})^T \\ H_y &= -((I_n + \Delta P_y^T)YQ_y + S_y)(T_y + \Delta Q_y^T Y Q_y)^{-1} \end{aligned}$$

while $N_{y\bar{y}} \in \mathbb{R}^{(m+q)\times(m+q)}$ is a non-singular solution to

$$N_{y\bar{y}}^{-1}J_{mq}N_{y\bar{y}}^{-T} = D(J_{mr}+\Delta B^T\bar{Y}B)^{-1}D^T - \Delta C_{\bar{y}}(I_n - Y\bar{Y})^{-1}YC_{\bar{y}}^T$$

Factor $\Omega_y(\zeta) \in \mathcal{RH}_\infty^{m+q}$ takes the form

$$\begin{aligned}
\Omega_y(\zeta) &= \left[\begin{array}{c|cc} A+H_yC & H_y^u & H_y^y \\ \hline C_{\bar{y}}^a U_{y\bar{y}} & I_m & 0_{m\times q} \\ C_{\bar{y}}^b U_{y\bar{y}} & 0_{q\times m} & I_q \end{array}\right] \quad (27)\\
U_{y\bar{y}} &= (I_n - Y\bar{Y})^{-1} \in \mathbb{R}^{n\times n}\\
H_y &= \left[\begin{array}{cc} H_y^u & H_y^y \end{array}\right], \quad H_y^u \in \mathbb{R}^{n\times m}, \quad H_y^y \in \mathbb{R}^{n\times q}\\
C_{\bar{y}} &= \left[\begin{array}{c} C_{\bar{y}}^a \\ C_{\bar{y}}^b \end{array}\right], \quad C_{\bar{y}}^a \in \mathbb{R}^{m\times n}, \quad C_{\bar{y}}^b \in \mathbb{R}^{q\times n}
\end{aligned}$$

and the controller is implemented as

$$\begin{aligned}
\delta\hat{x}(t) &= (A+H_yC)\hat{x}(t) + H_y^u u(t) + H_y^y y(t)\\
b_y(t) &= C_{\bar{y}}^b U_{y\bar{y}}\hat{x}(t) + y(t) \quad (28)\\
u(t) &= -C_{\bar{y}}^a U_{y\bar{y}}\hat{x}(t) + DHM(N_{y\bar{y}},\Phi)b_y(t)
\end{aligned}$$

B. Strictly proper controllers

Controller

$$K = HM(\Pi_x, HM(N_{x\bar{x}},\Phi))$$

based on a J-lossless solution is strictly proper iff $K(\infty) = 0_{p\times r}$. Since $\Pi_x(\infty) = I_{p+r}$, it follows that

$$K(\infty) = HM(N_{x\bar{x}},\Phi)(\infty)$$

Consequently, assuming a static termination parameter

$$\Phi = HM(N_{x\bar{x}}^{-1}, 0_{p\times r}) \in \mathbb{R}^{p\times r}$$

we obtain a strictly proper controller. The $(p+r)\times(p+r)$ partition

$$N_{x\bar{x}}^{-1} = \left[\begin{array}{cc} \bar{N}_{x11} & \bar{N}_{x12} \\ \bar{N}_{x21} & \bar{N}_{x22} \end{array}\right] \quad (29)$$

gives

$$\Phi = HM(N_{x\bar{x}}^{-1}, 0_{p\times r}) = \bar{N}_{x12}\bar{N}_{x22}^{-1}$$

Remembering that $\Phi \in \mathcal{BH}_\infty^{p\times r}$ is required we should also demand

$$\|\bar{N}_{x12}\bar{N}_{x22}^{-1}\|_s < 1$$

This condition is satisfied by a non-singular block lower triangular $N_{x\bar{x}}^{-1}$ with $\bar{N}_{x12} = 0_{p\times r}$, which assures zeroing of $\Phi = 0_{p\times r}$ and leads to a central controller (see [14]).

On the other hand, considering $N_{x\bar{x}}^{-1}$ with $\bar{N}_{x12} = 0_{p\times r}$ in the equality

$$\Phi = HM(N_{x\bar{x}}^{-1}, HM(N_{x\bar{x}},\Phi))$$

gives

$$\Phi = \bar{N}_{x11}HM(N_{x\bar{x}},\Phi)(\bar{N}_{x21}HM(N_{x\bar{x}},\Phi) + \bar{N}_{x22})^{-1}$$

Hence, taking $\Phi = 0_{p\times r}$ with a non-singular $\bar{N}_{x11}$ we observe that $HM(N_{x\bar{x}},\Phi) = 0_{p\times r}$. In conlusion, the corresponding central strictly proper solution $K = HM(\Pi_x, 0_{p\times r})$ can be implemented by the following simple algorithm

$$\begin{aligned}
\delta\hat{x}(t) &= (A+BF_x - U_{\bar{x}x}B_{\bar{x}}^b F_x^y)\hat{x}(t) + U_{\bar{x}x}B_{\bar{x}}^b y(t)\\
u(t) &= F_x^u\hat{x}(t) \quad (30)
\end{aligned}$$

For a dual J-lossless solution we have $\Omega_y(\infty) = I_{m+q}$. Hence, letting

$$\Phi = DHM(N_{y\bar{y}}^{-1}, 0_{m\times q}) \in \mathbb{R}^{m\times q}$$

assures that

$$K = DHM(\Omega_y, DHM(N_{y\bar{y}},\Phi))$$

is strictly proper. Next, considering the $(m+q)\times(m+q)$ partition

$$N_{y\bar{y}}^{-1} = \left[\begin{array}{cc} \bar{N}_{y11} & \bar{N}_{y12} \\ \bar{N}_{y21} & \bar{N}_{y22} \end{array}\right] \quad (31)$$

leads to the following rule for tuning the termination

$$\Phi = -\bar{N}_{y11}^{-1}\bar{N}_{y12}$$

with $\|\bar{N}_{y11}^{-1}\bar{N}_{y12}\|_s < 1$. Clearly, the existence of a non-singular solution $N_{y\bar{y}}^{-1}$ with the zero submatrix $\bar{N}_{y12} = 0_{m\times q}$ allows for employing $\Phi = 0_{m\times q}$, which is equivalent to the use of a central controller.

Moreover, starting from the equality

$$\Phi = DHM(N_{y\bar{y}}^{-1}, DHM(N_{y\bar{y}},\Phi))$$

and taking $N_{y\bar{y}}^{-1}$ with $\bar{N}_{y12} = 0_{m\times q}$ gives

$$\Phi = (\bar{N}_{y11} - DHM(N_{y\bar{y}},\Phi)\bar{N}_{y21})^{-1}DHM(N_{y\bar{y}},\Phi)\bar{N}_{y22}$$

Hence, for $\Phi = 0_{m\times q}$ and non-singular $\bar{N}_{y22}$ we have $DHM(N_{y\bar{y}},\Phi) = 0_{m\times q}$. Consequently, the resulting central controller with the strictly proper transfer function $K = DHM(\Omega_y, 0_{m\times q})$ can be implemented by the following simple algorithm

$$\begin{aligned}
\delta\hat{x}(t) &= (A+H_yC - H_y^u C_{\bar{y}}^a U_{y\bar{y}})\hat{x}(t) + H_y^y y(t)\\
u(t) &= -C_{\bar{y}}^a U_{y\bar{y}}\hat{x}(t) \quad (32)
\end{aligned}$$

VI. STRUCTURAL CONSTRAINTS-$\mathcal{H}_\infty$ ESTIMATION IN DELTA-DOMAIN

Consider a linear discrete-time plant with three vector-valued input/output signals: w_1 and w_2 are the exogenous inputs (disturbances) of dimension r_1 and r_2, respectively, and y is the measured output of dimension q. Let x denote the observed state vector of dimension n_1 and $v = Lx$ be a weighted state vector of dimension m, where $L \in \mathbb{R}^{m\times n_1}$ stands for a weighting matrix. A measurement noise channel is represented by $C_2(\zeta I_{n_2} - A_2)^{-1}B_2 + D_2$. An approximate weighted state $\hat{v}$ is generated via employing the filter (estimator) $K(\zeta)$. Defining a residual $z = v - \hat{v}$ as the objective we obtain the generalised plant with $u = \hat{v}$, $w = [\ w_1^T \quad w_2^T\ ]^T \in R^r$, and $r = r_1 + r_2$. A disturbance d acts directly on the output. The corresponding dual chain-scattering model is

$$H_\gamma(\zeta) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \tag{33}$$

$$\left[\begin{array}{cc|c} A_1 & 0_{n_1\times n_2} & \multirow{2}{*}{$0_{n\times m}$} \\ 0_{n_2\times n_1} & A_2 & \\ \hline L & 0_{m\times n_2} & -\gamma I_m \\ C_1 & C_2 & 0_{q\times m} \end{array}\begin{array}{c} \left[\begin{array}{cc} B_1 & 0_{n_1\times r_2} \\ 0_{n_2\times r_1} & B_2 \end{array}\right] \\ \hline 0_{m\times r} \\ \left[\ 0_{q\times r_1} \quad D_2\ \right] \end{array}\right]$$

Assume that $H_\gamma(\zeta) \in \mathcal{RH}_\infty^{(m+q)\times(m+r)}$ has n_z invariant zeros on $\partial\mathcal{D}_\Delta$. Since $H_\gamma(\zeta)$ is stable, $\bar{Y} = 0_{n\times n}$, $F_{\bar{y}} = 0_{(m+r)\times n}$, $C_{\bar{y}} = C$ and $N_{y\bar{y}} = M_y$. Taking $\Phi = 0_{m\times q}$ gives

$$\Omega(\zeta)^{-1} = \tag{34}$$

$$M_y \left[\begin{array}{cc|c} \Sigma_{11} & 0_{(n-n_z)\times n_z} & S_3^T \\ 0_{n_z\times(n-n_z)} & T_z^{-1} S_z & \left[\ 0_{n_z\times m} \quad \underline{Z}_{21}^T\ \right] \\ \hline \multicolumn{2}{c|}{C\left[\ S_1 \quad Z_{11}\ \right]^{-T}} & I_{m+q} \end{array}\right]$$

where a stable $\Sigma_{11} \in \mathbb{R}^{(n-n_z)\times(n-n_z)}$ corresponds to the stable invariant subspace of the pencil $U_y - \zeta W_y$. It follows that unstable modes $\lambda(T_z^{-1} S_z) \subset \partial\mathcal{D}_\Delta$ are the uncontrollable eigenvalues of the pair $(A + H_y C, H_y^u)$. Note that zeroing of $\underline{Z}_{21} \in \mathbb{R}^{q\times n_z}$ is excluded. On account of the above remarks we have the following lemma.

Lemma 1: *In the case of $H_\gamma(\zeta) \in \mathcal{RH}_\infty^{(m+q)\times(m+r)}$ with invariant zeros on $\partial\mathcal{D}_\Delta$, using of central estimators is to be excluded since the resulting $K(\zeta) = -\bar{\Omega}_{11}(\zeta)^{-1}\bar{\Omega}_{12}(\zeta)$ has poles on $\partial\mathcal{D}_\Delta$.* $\square$

Considering $K(\zeta) = DHM(\Omega(\zeta)^{-1}, \Phi)$ with a non-zero static parameter $\Phi \in \mathbb{R}^{m\times q}$ we obtain

$$K(\zeta) = \tag{35}$$

$$\left[\begin{array}{c|c} A + H_y C - H_y^u D_1^{-1}\hat{C} & H_y^u D_1^{-1} D_2 - H_y^y \\ \hline D_1^{-1}\hat{C} & -D_1^{-1} D_2 \end{array}\right]$$

$$\begin{aligned} \hat{C} &= M_{11}\bar{L} + M_{12}\bar{C} - \Phi(M_{21}\bar{L} + M_{22}\bar{C}) \in \mathbb{R}^{m\times n} \\ D_1 &= M_{11} - \Phi M_{21} \in \mathbb{R}^{m\times m} \\ D_2 &= M_{12} - \Phi M_{22} \in \mathbb{R}^{m\times q} \end{aligned}$$

Since

$$\lambda(T_z^{-1} S_z) \subset \lambda(A + H_y C - H_y^u D_1^{-1}\hat{C})$$

it follows that making modes $\lambda(T_z^{-1} S_z)$ non-observable is a necessary condition for $K(\zeta)$ to be stable. Choosing Φ in such a way that

$$\hat{C}\left[\ S_1 \quad Z_{11}\ \right]^{-T} = \left[\ \star \quad 0_{m\times n_z}\ \right] \in \mathbb{R}^{m\times n} \tag{36}$$

ensures required both the block-diagonal structure of the corresponding state matrix of a realisation of $K(\zeta)$ and zeroing of the suitable part of the output matrix of this model. For this to happen, the following linear equation in Φ should be solved $\Phi\underline{V} = \overline{V}$ where $\overline{V} \in \mathbb{R}^{m\times n_z}$ and $\underline{V} \in \mathbb{R}^{q\times n_z}$ are defined by

$$M_y C\left[\ S_1 \quad Z_{11}\ \right]^{-T}\left[\begin{array}{c} 0_{(n-n_z)\times n_z} \\ I_{n_z} \end{array}\right] = \left[\begin{array}{c} \overline{V} \\ \underline{V} \end{array}\right] \tag{37}$$

Since $\|\Phi\| < 1$ is obligatory, it is a rational choice to examine the minimum-norm solution $\Phi = \overline{V}\underline{V}^+$ where $\underline{V}^+ \in \mathbb{R}^{n_z\times q}$.

The following lemma summarises the above development.

Lemma 2: *Any unitary bounded solution Φ leading to a stable $K(\zeta) = DHM(\Omega(\zeta)^{-1}, \Phi) \in \mathcal{RH}_\infty^{m\times q}$ of a minimal order $(n - n_z)$ is satisfying with respect to the corresponding $\mathcal{H}_\infty$ problem. The necessary condition takes the form of conjunction: $n_z \le q$ and $\mathrm{Im}\,(\overline{V}^T) \subset \mathrm{Im}\,(\underline{V}^T)$.* $\square$

VII. CONCLUSION

It has been shown, that utilising the chain-scattering modelling and J-lossless factorisation methodology allow for a relatively easy formulation of the set of conditions for the existence of the (strictly) proper $\mathcal{H}_\infty$ discrete-time δ-domain controllers.

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