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Volterra model inversion using restored input feedback Application to an anaerobic digestor

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***Abstract*— This paper presents a way to obtain Volterra model inversion. A Volterra series expansion truncated to its first terms is used to represent a nonlinear system. An expansion of Volterra kernels on transfer functions is presented in order to obtain a more parsimonious model than the one expanded on generalized function bases. An inversion method is then proposed in order to restore Volterra model input from its output. At last, this method is experimentally validated in the case of a continuous anaerobic digestor.**

I. INTRODUCTION

Inverse methods were developed mainly because a lot of quantities are not directly measurable. Inverse problems consists in restoring such quantities from other directly measurable quantities and a particular model between them. In case of linear models, many inverse methods have been developed [22], [10], [17].

However, most of real systems currently considered are nonlinear. Consequently, nonlinear models must be used in order to correctly represent all kinds of nonlinearities. Some possibilities to model nonlinearities already exists such as NARMAX models [15], [9], multi-model approaches [8], neural networks [19] or Volterra series [23], [18], [11].

Volterra model is completely defined by its kernels but instead of obtaining their analytical expression, procedure which may prove difficult, indeed impossible, depending on nonlinear system complexity, their modelling is often preferred. In this paper, we propose to expand each Volterra kernel on transfer functions in order to obtain a parsimonious model, easier to invert [4]. Such a modelling is inspired by Fliess series expansion of Volterra kernels [14].

First section of this paper is about Volterra series and Volterra kernels expansion on transfer functions. Second section proposes an inverse method based on a restored input feedback scheme. At last, third section is an application of that inverse method to experimental data from an anaerobic digestor, a nonlinear biological system.

II. VOLTERRA MODEL

A. Volterra series

Let us consider an analytic single input/single output (SISO) nonlinear system of which state representation is given by

$$\begin{cases} \dot{x}(t) = f(x(t)) + u(t)g(x(t)) \\ y(t) = h(x(t)) \end{cases} \tag{1}$$

where $u(t)$ and $y(t)$ are respectively amplitude bounded input and output. Functions f, g and h are analytical ones and $x(t) \in \mathbb{R}^m$.

We assume that such a system can be represented by a Volterra series

$$y(t) = \sum_{i=1}^{\infty} \left(\int_0^t \dots \int_0^t h_i(\tau_1, ..., \tau_i) \prod_{j=1}^{i} u(t - \tau_j) d\tau_j \right) \tag{2}$$

where $h_i(\tau_1, ..., \tau_i)$ are called Volterra kernels under some conditions (causality, finite memory and stationarity) and are locally bounded and continuous functions. Moreover, it is assumed that initial condition and free response for Volterra model are zero.

Volterra series may be considered as a subset of functional series, which allow to represent a large number of nonlinear systems. Volterra series expansion (2) may be more easily written

$$y(t) = H[u(t)] = \sum_{i=1}^{\infty} H_i[u(t)] \tag{3}$$

where H_i is a regular and homogenous i^{th} order functional whose integral representation is given by

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$$H_i[u(t)] = \int_0^t \dots \int_0^t h_i(\tau_1, ..., \tau_i) \prod_{j=1}^{i} u(t-\tau_j) d\tau_j \quad (4)$$

For practical reasons, Volterra model (2) is often truncated to a finite number of terms in order to obtain an optimal number of kernels which correctly represent the nonlinear system behaviour. Therefore, Volterra series expansion may be a more efficient model for smooth nonlinearities. However, such a model allows to represent a large variety of polynomial nonlinearities. Let us note that the convergence of the Volterra series is only guaranteed on a bounded time interval and for an input signal which is sufficiently small.

Complete knowledge of functions f, g and h is necessary to obtain theoretical expressions for all Volterra kernels $h_i(\tau_1, ..., \tau_i)$. Because these functions may be complex, Volterra kernel modelling is often preferred to the calculation of a theoretical expression.

B. Volterra kernel modelling

Let us consider a Volterra series expansion truncated to its n^{th} term:

$$\widehat{y}(t) = \sum_{i=1}^{n} H_i[u(t)] \quad (5)$$

Generalized orthonormal basis functions [2], [1] may be used to represent Volterra kernels $h_i(\tau_1, ..., \tau_i)$ $i = 1, ..., n$. The resulting model [13] is particularly flexible in contrast to the well known Laguerre and Kautz bases but, in case of sufficiently strong nonlinearities, the global number of parameters increases and may makes difficult the identification procedure.

In order to obtain a more parsimonious model, we choose to model each Volterra kernel as a transfer function product [4] as presented in figure 1.

Each expansion D_{ji} is given by a sum of transfer functions:

$$D_{ji}(a_{i,j}, p_{j,i}) = \sum_{k=0}^{p_{j,i}-1} \frac{\gamma_{k,j,i}}{(s-a_{i,j})^{k+1}} \quad (6)$$

for $j = 1, ..., n$ and $i = 1, ..., j$.

Each block output represents simple expansion D_{ji} response to the input $u(t)$. Each product of i responses models the global response $y_i(t)$ of the i^{th} order Volterra kernel. Unknown Volterra model parameters to be identified are gains $\gamma_{k,j,i} \in \mathbb{N}$ and poles $a_{i,j} \in \mathbb{C}$. We assume that each term $p_{j,i} \in \mathbb{N}$ value is fixed before identification in order to set up Volterra model structure.

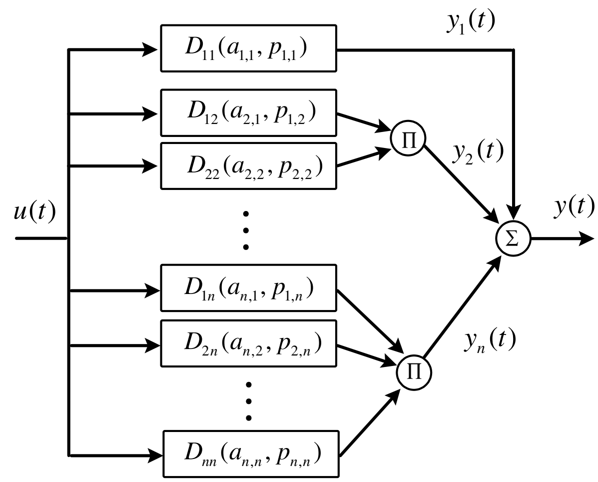

Fig. 1. Global model expanded on transfer functions

Volterra model with kernels expanded on transfer functions is nonlinear in parameters. Therefore, identification procedure used is a nonlinear identification algorithm, like Levenberg-Marquardt algorithm [16]. Depending on initial values given to parameters $\gamma_{k,j,i}$ and $a_{i,j}$, Levenberg-Marquardt algorithm may not guarantee a global convergence towards optimal values. Using first a genetic algorithm may be useful to obtain some satisfactory initial values for parameters in order to further guarantee their global convergence.

III. RESTORED INPUT FEEDBACK INVERSE METHOD

This section presents a way to obtain a restored input $\widehat{u}$ of a real input $u(t)$ from knowledge of the nonlinear system measured output $y(t)$ and of a Volterra model $H[u(t)]$. This approach is based on a closed loop inversion scheme, that is why it is considered as a restored input feedback inverse method [12].

We consider in this section a Volterra model expanded on transfer functions (6). All results presented in the section may be available for Volterra models expanded on orthonormal basis functions.

A. Method description

The method to obtain an acceptable reconstruction $\widehat{u}$ of real input $u(t)$ is given by scheme 2.

Volterra functional H is defined by (3) and (2). I may be called *identity* functional and is defined the following way:

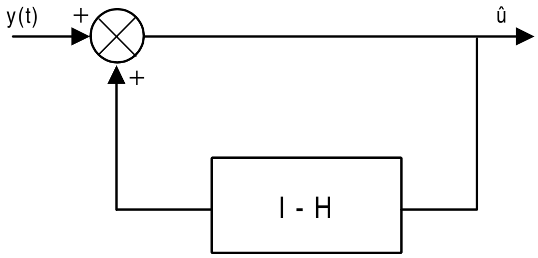

Fig. 2. Input restoration method

$$I[u(t)] = \sum_{i=1}^{\infty} I_i[u(t)] \tag{7}$$

Considering expression (4), all terms I_i of the identity functional may be defined the following way if they are applied to $u(t)$:

$$I_i[u(t)] = \int_0^t \dots \int_0^t \delta_i(\tau_1, ..., \tau_i) \prod_{j=1}^{i} u(t-\tau_j) d\tau_j \tag{8}$$

or

$$I_i[u(t)] = \int_0^t \dots \int_0^t \delta_i(t-\tau_1, ..., t-\tau_i) \prod_{j=1}^{i} u(t) d\tau_j \tag{9}$$

where each term δ_i may be considered as a multidimensional generalization of Dirac impulse δ. By definition, Dirac impulse is the neutral element for convolution product.

Let us now consider a multidimensional function $x(t_1, ..., t_i), i \in \mathbb{N}$. Each functional I_i applied to $x(t_1, ..., t_i)$ may be written the following way:

$$I_i[x(t_1, ..., t_i)] = \int_0^{t_1} \dots \int_0^{t_i} \delta_i(t_1-\tau_1, ..., t_i-\tau_i) x(t_1, ..., t_i) \prod_{j=1}^{i} d\tau_j \tag{10}$$

Dirac impulse generalization δ_i being the neutral element for i^{th} order terms of (7), expression (10) may be rewritten:

$$I_i[x(t_1, ..., t_i)] = x(t_1, ..., t_i) \tag{11}$$

Such a result allows us to obtain the following simplification for (8) and (9):

$$I_i[u(t)] = \prod_{j=1}^{i} u(t) = u^i(t) \tag{12}$$

provided we consider the particular case $t_1 = ... = t_i = t$.

By definition, a sum of functionals is a functional. Therefore, we may expand $(I-H)[u(t)]$ as a classical functional:

$$(I-H)[u(t)] = \sum_{i=1}^{\infty} (I_i - H_i)[u(t)] \tag{13}$$

with

$$(I_i - H_i)[u(t)] = u^i(t) - H_i[u(t)] \tag{14}$$

Let us now introduce scheme of figure 3, equivalent to figure 2, where L is a functional whose input and output are respectively $y(t)$ and $\widehat{u}$.

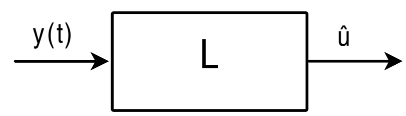

Fig. 3. Equivalent scheme to restoration method

Functional L may be classically expanded into a sum of functionals:

$$L[y(t)] = \sum_{i=1}^{\infty} L_i[y(t)] \tag{15}$$

We propose to show that functional L is in fact the exact inverse of Volterra functional H.

Let us consider a simple case where output $y(t)$ is measured without any disturbing noise. Scheme of figure 2 leads us to:

$$\widehat{u} = y(t) + (I-H)[\widehat{u}(t)] \tag{16}$$

Equivalent scheme of figure 3 gives relation $\widehat{u} = L[y(t)]$ which can be introduced into expression (16):

$$L[y(t)] = y(t) + (I-H)[L[y(t)]] \tag{17}$$

Functional $I-H$ may be expanded into a sum of two functionals:

$$\begin{aligned} L[y(t)] &= y(t) + I[L[y(t)]] - H[L[y(t)]] \\ &= y(t) + L[y(t)] - H[L[y(t)]] \end{aligned} \tag{18}$$

I being the identity functional. This leads us to the following result:

$$y(t) = H[L[y(t)]] \tag{19}$$

From this expression, we may deduce that functional L is functional H exact inverse. Therefore, such a scheme as

2 may be theoretically used to calculate the exact inverse $\widehat{u}$ of input $u(t)$ from knowledge of Volterra model H and measured output $y(t)$, without any disturbing noise.

B. Case of disturbing noise

In order to study the case of noise corruption influence on the inversion method, let us add a gaussian noise $b_2(t)$ to the nonlinear system output $y(t)$. Output signal $y^\star(t)$ used to restore the input $u(t)$ is different from $y(t)$ used without any disturbing noise.

Restored input signal $\widehat{u}^\star$ is also different from real input signal $u(t)$ and restored input signal $\widehat{u}$ obtain without any added noise. Such a restored input signal may be written

$$\widehat{u}^\star = u(t) + b_1(t) \tag{20}$$

where $b_1(t)$ may be defined as restoration error.

In case of disturbing noise, the restored input $\widehat{u}^\star$ may be written

$$\widehat{u}^* = L[y^*(t)] = L[y(t) + b_2(t)] \tag{21}$$

Let us expand this expression

$$\widehat{u}^* = L[H[u(t)] + b_2(t)] = u(t) + L[b_2(t)] \tag{22}$$

An analogy between expression 20 and expression 22 leads to the following result:

$$b_1(t) = L[b_2(t)] \tag{23}$$

Such a result confirms that restoration error $b_1(t)$ depends only on disturbing noise $b_2(t)$.

Next section will show that in nonlinear case, the inversion method may give satisfying results, even in case of disturbing noise.

IV. Application to an anaerobic digestor

Wastewater treatment is currently an interesting and widely open research area. Different kinds of processes may be applied to industrial effluents in order to clean them from some useless (or dangerous) organic matter. There exists some biological methods such as anaerobic digestion, a process into which organic matter is degraded into a mixture of methane (CH_4) and carbon dioxide (CO_2). The proposed model is going to be applied to such a biological process, which may be considered as a nonlinear SISO phenomenon.

A. Process description

The pilot plant considered is a continuous anaerobic digestor in which raw industrial wine distillery vinasses are mixed with water into a $1\ m^3$ upflow fixed bed reactor. For more details about the plant, its online instrumentation and its modelling, see a complete description in [20].

In order to correctly model the process, it is assumed that the anaerobic digestion can be described by a two-stage process including two groups of bacterial populations, each group having homogeneous characteristics. During the first step, the acidogenic bacteria (X_1) consume the organic substrate (S_1) and produce CO_2 and volatile fatty acids (S_2) following the acidogenesis reaction (with reaction rate $r_1 = \mu_1 X_1$) :

$$k_1 S_1 \rightarrow X_1 + k_2 S_2 + k_4 CO_2 \tag{24}$$

Through the second step, the methanogenic bacteria (X_2) degrades the volatile fatty acids, thus producing CO_2 and CH_4. The methanization reaction (rate $r_2 = \mu_2 X_2$) is:

$$k_3 S_2 \rightarrow X_2 + k_5 CO_2 + k_6 CH_4 \tag{25}$$

μ_1 and μ_2 represent the specific growth rate of acidogenesis and methanization respectively. All terms y_i $(i = 1, ..., 6)$ are yield coefficients.

A model representing this two-stage process was developped, identified and experimentally validated [3]. It consists of a set of first order differential equations. Strong nonlinearities in the system behavior are due to the growth rates expressions:

$$\begin{cases} \mu_1 = \mu_{\max 1} \frac{S_1}{K_{S_1} + S_1} \\ \mu_2 = \mu_0 \frac{S_2}{K_{S_2} + S_2 + \left(\frac{S_2}{K_{I_2}}\right)^2} \end{cases} \tag{26}$$

where $\mu_{\max 1}$ and μ_0 are parameters associated with biomass growth rates, K_{S_1} and K_{S_2} are saturation parameters and K_{I_2} is an inhibition constant. Precise definitions and values of these parameters are given in [3].

The inversion of such a model may be very difficult. On the other hand, applying a behavior model like Volterra model may be of use because such a model may be inverted using the method previously proposed.

B. Volterra model application

Experimental data used to validate the Volterra model come from online measurements obtained in September 2001 and kindly provided by the Laboratoire de Biotechnologie de l'Environnement (Narbonne, France). For more details, see [21].

The inlet dilution rate $D(t)$ (in l/h) is represented by figure 4. The sample period is 30 minutes. The process total time is 30 days. The considered output, measured inside the reactor, is the Chemical Oxygen Demand (COD, in g/l), which represent the organic substrate concentration. The transfert between $D(t)$ and $COD(t)$ is strongly nonlinear and may be represented using a Volterra model H truncated to 2 terms. The first Volterra kernel is expanded on 1 transfer function, the second on 2 transfer functions. The global number of parameters is 8. Figure 5 shows a good adequation between experimental COD and Volterra model output.

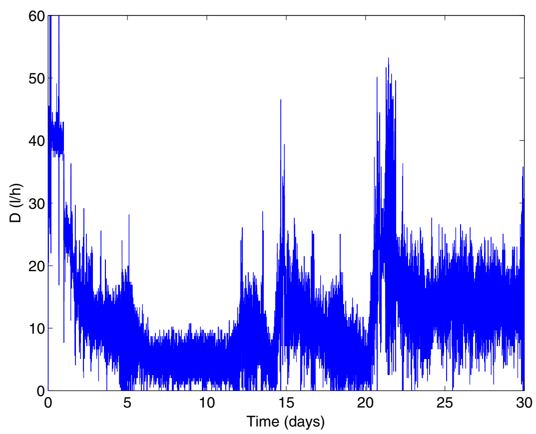

Fig. 4. Inlet dilution rate $D(t)$

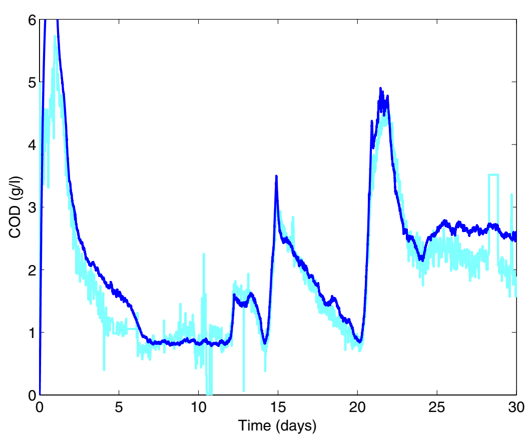

Fig. 5. Experimental COD (thin clear curve) and Volterra model output (thick dark curve)

Volterra model is therefore validated with these experimental measurements. Let us note that adding other terms into Volterra model H or into each kernel expansion does not significantly improve its precision. Moreover, in order to prove its good adequation, the Volterra model may be tested by means of cross-correlation functions [7], [6].

C. Inversion method application

Volterra model precision is considered sufficient by the user to apply the inversion method previously proposed in order to restore inlet dilution rate $D(t)$ from another COD experimental measurement (figure 6).

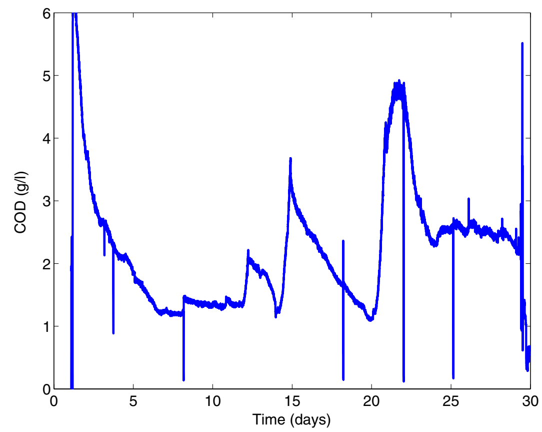

Fig. 6. Experimental COD (g/l)

The Volterra model H is again truncated to 2 terms and expanded on transfer functions. The global number of parameters remains 8. Restoration results are represented by figure 7.

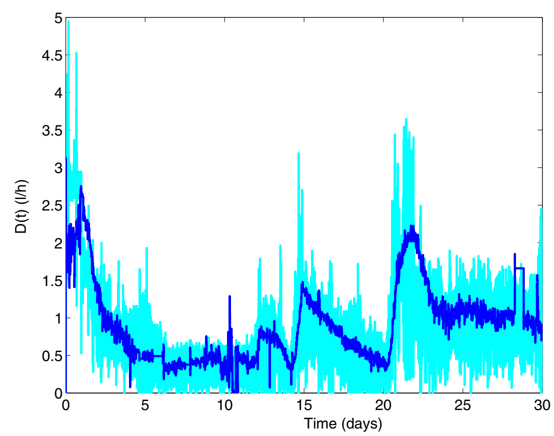

Fig. 7. Experimental inlet dilution rate $D(t)$ (thin clear curve) and its restoration (thick dark curve)

Despite a noisy experimental inlet dilution rate $D(t)$, the restoration quality remains satisfactory. Let us note that the magnitude of the restoration error on restored dilution rate is smaller than the amplitude of disturbing noise on experimental COD. The inversion method tends to attenuate noise magnitude, acting implicitly as a lowpass filter.

One of the main advantage of the method is its relatively low computational time (about 1 minute), compared to some others inversion methods such as Tikhonov regularization [5]. Such a difference may be explained because no iterative optimization algorithm (which needs a lot of computational time) is necessary during the whole inversion procedure.

V. Conclusion

In this paper, Volterra series kernels are expanded on transfer functions in order to obtain a quite simple and parsimonious model.

The main objective was to introduce a way to obtain a nonlinear system inversion in order to restore its input from knowledge of its output and of a sufficiently precise Volterra model. The inverse method is based on a closed loop scheme, thus justifying the name of restored input feedback inversion method. Compared to some other inverse methods, one of the main advantages is its low necessary calculation time, which allows to further consider real time applications.

Experimental results obtained with a continuous anaerobic digestor allow to further consider the application of the inversion method to restore some quantities whose measurement procedure is too complex, or even impossible at the present time. As an example, inlet COD restoration is difficult and costly to measure directly but its restoration could be very useful in order to obtain more informations about the inlet organic substrate to be degraded.

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