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A Large Deviations Analysis of Scheduling in Wireless Networks

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Abstract— **We consider a cellular network consisting of a base station and *N* receivers. The channel to each receiver is assumed to be in one of two states (ON or OFF) and the channel states of the receivers are assumed to be independent of each other. The goal is to compare the throughput of two different scheduling policies given an upper bound on the queue overflow probability or the delay violation probability. The two scheduling policies that we consider are: (i) a greedy scheduling policy which chooses to serve any of the channels in the ON state, and (ii) a queue-length-based policy which serves the longest queue connected to an ON channel. We show that the total network throughput of the queue-length-based policy is no less than that of the greedy policy for all *N* and is strictly larger than the throughput of the greedy policy for large *N*. Further, given an upper bound on the delay violation probability, we show that the throughput of the queue-length-based policy is an increasing function of *N* while the throughput of the greedy policy eventually decreases with increasing *N* and goes to zero. Given an upper bound on the queue overflow probability, we show that the throughput of the queue-length-based policy is a strictly increasing function of *N* while the throughput of the greedy policy eventually goes to a constant.**

I. Introduction

Multiuser wireless scheduling has received much attention in recent years. Consider a cellular network consisting of a base station and N users (receivers), where the base station maintains N separate queues, one corresponding to each user. Assume time is slotted and the channel states of the receivers at each time slot are known at the base station. Then, the base station can determine which queues to serve according to their channels states. In this paper, we assume that the base station operates in a TDMA fashion, i.e., the base station can serve only one queue in each time slot. Two scheduling policies have been widely studied in the literature: (i) the base station serves the user with the best (weighted) channel state (opportunistic scheduling) [7], [4], or (ii) serve the one with the best queue-length-weighted channel state (queue-length based (QLB) scheduling) [6]. While QLB scheduling is throughput optimal (i.e., can stabilize any set of user throughputs that can be stabilized by any other algorithm), opportunistic scheduling maximizes the total network throughput if all queues are continuously backlogged. If the arrival rates to the users are identical and the channel state distribution to the receivers are identical, then these two scheduling policies have the same stability region.

Stability is the first concern of scheduling policies. However, quality-of-service (QoS) is important too. For example, we may require the queue overflow probability to be small or require small delays. The performance of different scheduling policies under QoS constraints has received much attention recently. For reasons of analytical tractability, much of the prior work assumes that the channels to all the receivers are independent and statistically identical. Under this assumption, and assuming identical user utilities, the opportunistic scheduling policies become greedy policies in which the base station transmits to the receiver with the best channel state. In [5], the author studies a simple network consisting of two users where the channels are assumed to be independent, identically distributed ON-OFF channels. Using large-deviations techniques, it is shown that the total network throughput of the QLB policy is larger than the greedy policy under the queue overflow constraint. In [3], a wireless network with N users and ON-OFF channels is considered. It is assumed that the arrivals are identical and Poisson, and the capacity when the channel is ON is one packet per time slot. It is then shown that, when the number of users increases from N to $2N$, the expected sum of queue lengths is non-increasing under the QLB policy, while it increases linearly under the greedy policy. Further, in [2], the behavior of the greedy policy for Rayleigh fading channels is studied and it shows that under a delay constraint, the total network throughput of the greedy policy increases initially with the number of users, but eventually decreases and goes to zero when the number of the users is sufficiently large.

Motivated by these prior results, in this paper, we study the performance of the two scheduling policies (greedy and QLB) for a wireless network with symmetric users and simple ON-OFF channels. The main contributions of this paper are as follows:

1) Assuming a constant arrival rate in each slot, we compute the large-deviations exponent of the probability that one queue in the network exceeds a large threshold. A key contribution here is the characterization of the queue-length trajectories that lead to queue overflow. It was conjectured that in [5] that the complexity of the calculation of the large-deviations exponent increases exponentially with increasing N, but we show here that a simple closed-form expression can be obtained.
2) In [3], under the assumption that the channel capacity

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is one packet per time slot, it is shown the expected sum of the queue lengths is nondecreasing when the number of users increases from N to $2N$. In this paper, we show that the large-deviations exponent is nondecreasing (strictly increasing) in N under the delay-violation constraint (overflow constraint). Our result does not only compare performance with N users and $2N$ users, but at all intermediate values as well. Further, our result holds even when the capacity of the network is greater than 1 packet-per-slot.

3) For the greedy policy, we analytically show that the throughput goes to a constant under the queue overflow constraint, and decreases to zero under the delay violation constraint. This result is consistent with the numerical results for Rayleigh fading channels in [2].
4) Under the QoS constraints, we show that the throughput of the QLB scheduling policy is strictly larger than the throughput of the greedy policy. This conclusion has been proved true in [5] for a two users system and under queue overflow constraint. Here, we prove that it is true for an N-user system.

While in this paper we only study ON-OFF channels, in a longer version of the paper, we show that the comparative performance of the greedy and QLB policies show a similar behavior even for more general channel models.

II. Basic Model

Consider a wireless network shared by N users in the downlink of a cellular network. We assume that the time is slotted and at each time slot, only one user can be chosen to transmit. Each user is associated with a channel and all channel-state processes $\eta_i[t]$ are statistically identical. In this paper, we will consider a simple channel model — ON-OFF channel, which has two states. We use "0" to indicate the channel is OFF and "1" to indicate the channel is ON. When the channel is OFF, no data can be transmitted. When the channel is ON, this channel can be selected to transmit. Furthermore, we let p be the probability that the channel is in the ON state. Also assume that the arrival rate is constant and is equal to λ/N bits/slot for each user. When a channel is ON, we can transmit at most F bits to the user of that channel.

For this simple model, we consider the throughput of two different scheduling policies under two different quantity of service (QoS) constraints. The two scheduling policies we will investigate are:

1) Queue-length based (QLB) policy: Choose user i to transmit if
$$i \in \arg\max_j \eta_j[t] Q_j[t],$$
where $Q_i[t]$ is the queue length of user i at time t. In our simple model, this policy chooses the user with the largest queue length from ON channels.
2) Greedy policy: Choose user i if
$$i \in \arg\max_j \eta_j[t].$$
In our model, we assume that the base station is equally likely to choose one of the ON channels.

The two QoS constraints we will consider are:

1) Queue overflow constraint:
$$\Pr(\max_i Q_i(0) > B) \leq \varepsilon,$$
where $Q_i(0)$ is the stationary queue length. So this QoS constraint requires the steady-state probability that the queue length is larger than B to be small. Instead of studying this constraint as above, we study the following approximation to the constraint:
$$\theta_B(N) := \lim_{B\to\infty} -\frac{1}{B}\log P(\max_i Q_i(0) > B) \geq \delta. \qquad (1)$$
The exponent δ can be related to ε for large B using the following approximation: $e^{-\delta B} = \varepsilon$.
2) Delay violation constraint: Define $D(t)$ to be the maximum delay experienced so far by any bit in any of the queues in slot t. Assuming that the system started at time $-\infty$, the steady-state delay violation constraint that we consider can be expressed as follows: $\Pr(D(0) > D) \leq \varepsilon$. Since the arrival rate is constant, it is easily seen that $\Pr(D(0) > D) = P\left(\max_i Q_i(0) > \frac{\lambda}{N} D\right)$. Thus, the delay violation constraint can be expressed as:
$$\Pr\left(\max_i Q_i(0) > \frac{\lambda}{N} D\right) \leq \varepsilon.$$
As before, we study the following approximation to the constraint:
$$\theta_D(N) := \lim_{D\to\infty} -\frac{1}{D}\log P(\max_i Q_i(0) > \frac{\lambda}{N} D) \geq \delta. \qquad (2)$$

In the rest of the paper, we will study the behavior of $\theta_B(N)$ and $\theta_D(N)$. From the above description of the two quantities, it is clear that if we obtain an expression for $\theta_B(N)$, then
$$\theta_D(N) = \frac{\lambda\,\theta_B(N)}{N}.$$
Thus, we will primarily consider the queue overflow problem when we analyze the wireless system using large deviations.

III. Queue-Length Based Policy

In this section, we use large deviations techniques to study the QLB policy. In Subsection III-A, we present a large deviations argument given in [5] to show that the probabilities of QoS violation can be related to an optimal control problem. In Subsection III-B, we prove two important properties of the optimal trajectory, which are then used in Subsection III-C to prove that the optimal trajectory is linear. Also, in Subsection III-C, we provide a formula to calculate the large-deviations exponent under the QLB policy.

A. Large Deviations and Optimal Control

Consider a N-user system, and let $\gamma_i(t)$ be the state of channel i at time t, so that $\gamma_i(t) = 1$ or 0. The state of the system depends on the state of each channel, so there are 2^N

system states. Each state can be represented as a N-tuple in $\{0,1\}^N$. For example, consider a 2-user system, the channel states are: $(0,0)$, $(0,1)$, $(1,0)$ and $(1,1)$, where 0 means the corresponding channel is OFF and 1 means the channel is ON. To simplify our notation, we will use the decimal value of the binary representation to represent the system state. So for a 2-user system, the system states are: 0, 1, 2 and 3. Thus, define the system state variable $S(t)$ as follows:

$$S(t) := \sum_{i=0}^{N-1} \gamma_i(t) 2^i, \tag{3}$$

and further define the probability vector $\mathbf{p}$, where p_j is the probability the system is at state j.

For sufficiently large T, we define $\mathbf{s}^{(B)}(t)$ on $[-T,0]$ using $S(t)$ on $[0,BT]$ as follows:

$$s_j^{(B)}(t) := \frac{1}{B} \sum_{l=0}^{B(T+t)} 1_{S(l)=j}, \text{ for } t = \frac{k}{B} - T \text{ and } k = \{0,\dots,BT\}$$

where for values of t which are not of the form k/n, define $s_j^{(B)}(t)$ by linear interpolation. Notice that we have scaled and shifted time so that the discrete time units $0,1,\dots,BT$ have now become the continuous time interval $[-T,0]$. For each t, the variable $s_j^{(B)}(t)$ is the amount of (scaled) time in the interval $[-T,t]$ that the system is in state j. Next, define the system channel rate processes using a 2^N-tuple — $\mathbf{u}(t)$, where $\mathbf{u}(t)$ is nonnegative, integrable, $\sum_{j=0}^{2^N-1} u_j(t) = 1$, and given $\varepsilon > 0$, for all sufficiently large B, we have for any $t_1 < t_2$

$$\left| s_j^{(B)}(t_2) - s_j^{(B)}(t_1) - \int_{t_1}^{t_2} u_j(s)\, ds \right| \le \varepsilon.$$

Now we will consider the normalized queue length

$$q_i(t) = \frac{1}{B} Q_i(t).$$

Then we have

$$\Pr\left(\max_i Q_i(0) > B\right) = \Pr\left(\max_i q_i(0) > 1\right),$$

and the dynamics of the normalized queue length can be described using following differential equation:

$$\dot{q}_i(t) = \frac{\lambda}{N} - F \sum_{j \in A_i} u_j(t),$$

where A_i is the set such that if $j \in A_i$, then user i will be chosen to transmit when the system is at state j. The optimization problem that is used to find the large deviations exponents is defined in terms of the Kullback-Liebler distance from $\mathbf{p}$ to $\mathbf{u}(t)$:

$$D(\mathbf{u}(t)\|\mathbf{p}) = \sum_{j=0}^{2^N-1} u_j(t) \log \frac{u_j(t)}{p_j}.$$

Recall $\mathbf{u}(t)$ is nonnegative, integrable and $\sum_j u_j(t) = 1$, we define following optimization problem:

$$\theta_B^{\text{QLB}}(N) = \inf_{T,\mathbf{u}} \int_{-T}^{0} D(\mathbf{u}(s)||\mathbf{p})\, ds, \tag{4}$$

where $T \ge 0$, $q_i(-T) = 0$ for all i, $\max_i q_i(0) = 1$, and the QLB policy is used.

Theorem 1:

$$\theta_B^{\text{QLB}}(N) = \lim_{B\to\infty} \frac{-1}{B} \log \Pr(\max_i q_i(0) \ge 1),$$

where $\theta_B^{\text{QLB}}(N)$ is defined as (4), and queues are scheduled according to the QLB policy.

Proof: The proof is a straightforward extension of Theorem 6.1 in [5] for the case $N = 2$. ■

Note that the optimization problem is intuitively obvious: among all possible channel state trajectories that could lead of overflow, we pick the one that is "closest" to the mean value $\mathbf{p}$. Given $\mathbf{u}(t)$, we call $\int_{-T}^{0} D(\mathbf{u}(s)||\mathbf{p})\, ds$ the cost of the trajectory generated by $\mathbf{u}(t)$. Thus, Theorem 1 tells us that the probability of the QoS violation is related to the minimum cost of optimal control problem (4). Due to the obvious similarity with optimal control problems, we will often refer to $\mathbf{u}(t)$ as the control law and the objective in (4) as the cost function.

B. Properties of the Optimal Trajectory

In general, the optimal control problem (4) can be hard to solve. In this paper, since we only consider the simple ON-OFF channels and assume that all channels are symmetric, the problem turns out to be tractable. In this subsection, we will show two important properties of the optimal trajectory: piecewise linearity and another property which we call the *order property*. Then, in the next subsection, we will solve the optimal control problem. To show piecewise linearity, we segment the state space into regions such that the differential equations describing the system dynamics are unchanged in each region. For example, consider a 2-user system, there are three regions [5]:

1) If $q_1(t) > q_2(t)$, then

$$\dot{q}_1(t) = \frac{\lambda}{2} - F(u_1(t) + u_3(t))$$
$$\dot{q}_2(t) = \left(\frac{\lambda}{2} - F u_2(t)\right)_0^+.$$

2) If $q_2(t) > q_1(t)$, then

$$\dot{q}_2(t) = \frac{\lambda}{2} - F(u_2(t) + u_3(t))$$
$$\dot{q}_1(t) = \left(\frac{\lambda}{2} - F u_1(t)\right)_0^+.$$

3) If $q_1(t) = q_2(t)$, then

$$\dot{q}_1(t) + \dot{q}_2(t) = (\lambda - F(u_1(t) + u_2(t) + u_3(t)))_0^+.$$

Here $\dot{q}_i(t) = (a)_0^+$ is defined as follows:

$$\dot{q}_i = \begin{cases} a, & \text{if } q_i(t) > 0; \\ \max\{0,a\}, & \text{if } q_i(t) = 0. \end{cases}$$

This approach of diving the state space into regions where dynamics are invariant was first considered in [1]. We now prove following two lemmas. The first lemma is a straightforward extension of the corresponding result in [5] for the two-user case, but the second lemma is crucial to solving the problem for $N > 2$.

Lemma 2 (Piecewise Linearity): In a region of fixed system dynamics, the optimal control of (4) is constant. Thus, the optimal trajectory is piecewise linear.

Proof: Given arbitrarily control $\mathbf{u}(t)$, and consider a fixed system dynamics region in time interval $[t_1, t_2]$. Define

$$K_j = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} u_j(s)\, ds.$$

Since $D(\mathbf{u}||\mathbf{p})$ is convex in $\mathbf{u}$, from Jensen's inequality, it follows that

$$\begin{aligned}\int_{t_1}^{t_2} D(\mathbf{u}(s)||\mathbf{p})\, ds &\geq (t_2 - t_1) D\left(\frac{1}{t_2 - t_1}\int_{t_1}^{t_2} \mathbf{u}(s)\, ds||\mathbf{p}\right) \\ &= (t_2 - t_1) D(\mathbf{K}||\mathbf{p}).\end{aligned}$$

Furthermore, the queue lengths at $t = t_2$ are the same under both controls. Thus, the optimal control law in this region is a constant control law, and the optimal (scaled) queue length trajectory is piecewise linear. ■

Lemma 3 (Order Property): Given any trajectory, we can find another trajectory which has the same cost and the property such that if $i \geq j$, then

$$q_i(t) \geq q_j(t).$$

Proof: This lemma exploits the fact that all channels are symmetric. First we prove following statement: given control $\bar{\mathbf{u}}(t)$ and suppose $\bar{q}_i(\bar{t}) = \bar{q}_k(\bar{t})$ at time $\bar{t}$. Then, there exists a new trajectory $\hat{\mathbf{q}}(t)$ such that $\hat{q}_i(t) = \bar{q}_k(t)$ and $\hat{q}_k(t) = \bar{q}_i(t)$ for $t \geq \bar{t}$. Furthermore, this new trajectory is identical to the original one except the indexes of the queues, and two trajectories have the same cost.

Now suppose that $\bar{q}_i(\bar{t}) = \bar{q}_k(\bar{t})$ and define a new control $\hat{\mathbf{u}}(t)$ such that

$$\hat{u}_j(t) = \begin{cases} \bar{u}_j(t), & \text{if } t < \bar{t}; \\ \bar{u}_{l_j}(t), & \text{if } t \geq \bar{t}, \end{cases}$$

where l_j is obtained from j by exchanging the i^{th} and k^{th} digits of the binary expression of j. For example, for the two-user system, we will have $\bar{u}_{(0,1)}(t) = \hat{u}_{(1,0)}(t)$, $\bar{u}_{(1,0)}(t) = \hat{u}_{(0,1)}(t)$, $\bar{u}_{(0,0)}(t) = \hat{u}_{(0,0)}(t)$, and $\bar{u}_{(1,1)}(t) = \hat{u}_{(1,1)}(t)$. Then, since $\bar{q}_i(\bar{t}) = \bar{q}_k(\bar{t})$, the dynamics of queue i and queue j are exchanged after $\bar{t}$ under the new control, and we will have $\hat{q}_i(t) = \bar{q}_k(t)$ and $\hat{q}_k(t) = \bar{q}_i(t)$ for $t \geq \bar{t}$ in the new trajectory.

Furthermore, the channels are symmetric, it is easy to see $p_j = p_{l_j}$ because the binary expressions of j and l_j have the same number of "0"s and "1"s. So we can conclude that the new trajectory have the same cost as the original one because

$$\begin{aligned}&\int_{\bar{t}}^{0} D(\hat{\mathbf{u}}(s)\|\mathbf{p})\, ds = \int_{\bar{t}}^{0} \sum_j \hat{u}_j(s) \log \frac{\hat{u}_j(s)}{p_j}\, ds \\ &= \int_{\bar{t}}^{0} \sum_j \bar{u}_{l_j}(s) \log \frac{\bar{u}_{l_j}(s)}{p_j}\, ds = \int_{\bar{t}}^{0} D(\bar{\mathbf{u}}(s)\|\mathbf{p})\, ds.\end{aligned}$$

Now, we have proved that if two queues have the same queue length at time $\bar{t}$, there exists a new trajectory with the same cost such that the lengths of the two queues are swapped after $\bar{t}$. It is also easy to see that if we have more than two queues with the same length at time $\bar{t}$, then we can swap any two of them after $\bar{t}$ to get a new trajectory with the same cost. Thus, give any trajectory, we can get a new trajectory with the same cost such that $q_i(t) \geq q_j(t)$ if $i \geq j$. ■

We have proved two important properties of the optimal trajectory: piecewise linearity and order property. In the next subsection, we use these two properties to prove that the linearity of the optimal trajectory. Also, we provide the closed-form expressions of $\theta_B^{\text{QLB}}(N)$ and $\theta_D^{\text{QLB}}(N)$.

C. The Optimal Solution

Before we solve the optimal control problem (4), we first consider a simpler optimization problem $OP(M,N)$ as follows:

$$\begin{aligned} OP(M,N): \quad & C_M^N(h) = \inf_{\mathbf{u},T} T D(\mathbf{u}||\mathbf{p}) && (5) \\ \text{Subject to}: \quad & T\left(\tfrac{M}{N}\lambda - F \textstyle\sum_{j=2^{N-M}}^{2^N-1} u_j\right) = Mh && (6) \\ & \textstyle\sum_{j=0}^{2^N-1} u_j = 1 && (7) \\ & u_j \geq 0 \quad \forall j. && (8) \end{aligned}$$

This problem is related to the (4) as follows: (5) uses the same cost function as (4), but it is assumed that a constant control is used, the lengths of the queues $N-M$ through $N-1$ are all equal and larger than the remaining queue lengths and that

$$q_{N-M}(0) = \ldots = q_{N-1}(0) = h.$$

We do not impose any restrictions on q_1 through q_{N-M-1}, other than the fact that these are smaller than $q_{N-M}(t)$ for all $t \in [-T, 0]$. However, in problem (5), we do not verify that the resulting policy is a QLB policy, i.e., we give priority to the users indexed $N-M$ through $N-1$ without verifying that their queue lengths are larger than the queue lengths of the remaining users (we simply assume this in stating the optimization problem, but do not verify it).

Notice that

$$T = \frac{Mh}{\frac{M}{N}\lambda - F\sum_{j=2^{N-M}}^{2^N-1} u_j},$$

so

$$C_M^N(h) = \inf_{\mathbf{u}} \frac{Mh}{\frac{M}{N}\lambda - F\sum_{j=2^{N-M}}^{2^N-1} u_j} D(\mathbf{u}||\mathbf{p}) = hC_M^N(1).$$

Now we focus on $C_M^N(1)$ and obtain a closed-form expression for it in Lemma 4. Then in Theorem 5, we prove that the linearity of the optimal trajectory, and that $\theta_B^{\text{QLB}}(N) = \min_M C_M^N(1)$.

Lemma 4: Consider the optimization problem (5), we have

$$C_M^N(1) = \inf_{0 \leq x < \frac{\lambda M}{FN}} \frac{N}{\lambda - xF\frac{N}{M}} D(\mathbf{u}^x(M)||\mathbf{p}), \qquad (9)$$

where

$$D(\mathbf{u}^x(M)||\mathbf{p}) = x \log \frac{x}{1-(1-p)^M} + (1-x) \log \frac{1-x}{(1-p)^M}.$$

Proof: Let $\frac{M}{F}\left(\frac{\lambda}{N}-\frac{1}{T}\right)=x$, we first solve (5) for fix T. Since $D(\mathbf{u}||\mathbf{p})$ is a strictly convex function, the optimal solution of (5) for fixed T is unique and equation (9) can be obtained by Lagrange multipliers. For more detail, please refer to Lemma 4 of [8]. ■

In the following theorem, we prove that $\theta_B^{\text{QLB}}(N) = \min_M C_M^N(1)$. Then, using the formula for $C_M^N(1)$, it is easy to obtain $\theta_B^{\text{QLB}}(N)$.

Theorem 5: For a N-user network, the optimal control that solves (4) is a constant, and hence the optimal queue length trajectories are linear, and further

$$\theta_B^{\text{QLB}}(N) = \min_M C_M^N(1).$$

Proof: From Lemma 2 and Lemma 3, we only need to consider the trajectories which are piece-wise linear and $q_i(t) \geq q_k(t)$ for $i \geq k$. Pick any piece of this trajectory which is in a fixed dynamic region. Suppose this piece of trajectory is in the time interval $[t_1,t_2]$, $q_{N-1}(t_2)-q_{N-1}(t_1)=h$, and $q_{N-1}(t) = \ldots = q_{N-M}(t) > q_{N-M-1}(t)$ for any $t \in (t_1,t_2)$. Thus, in this region, the dynamics of queue $N-1,\ldots,N-M$ are as follows:

$$\sum_{i=N-M}^{N-1} \dot{q}_i(t) = \frac{M}{N}\lambda - F\left(\sum_{j=2^{N-M}}^{2^N-1} u_j\right).$$

Then, it is easy to see that

$$(t_2-t_1)\left(\frac{M}{N}\lambda - F\sum_{j=2^{N-M}}^{2^N-1} u_j\right) = Mh. \tag{10}$$

Define $T=t_2-t_1$, then (10) is similar to (6). Let $D(\mathbf{u}||\mathbf{p})$ be the cost of this piece of the trajectory, we have

$$(t_2-t_1)D(\mathbf{u}||\mathbf{p}) \geq C_M^N(h).$$

Define M^* to be any one element of $\arg\min_M C_M^N(1)$, then we have

$$(t_2-t_1)D(\mathbf{u}||\mathbf{p}) \geq hC_M^N(1) \geq hC_{M^*}^N(1).$$

Now consider a piecewise linear and ordered trajectory, and divide $[-T,0]$ into subintervals $[t_i,t_{i+1}]$ such that the trajectory in each subinterval is in a fixed system dynamic region. So the control in $[t_i,t_{i+1}]$ is constant — we call it $\mathbf{u}_i$. Also define $h_i = q_{N-1}(t_{i+1}) - q_{N-1}(t_i)$. Then, we have

$$\begin{aligned}\int_{-T}^{0} D(\mathbf{u}(s)||\mathbf{p})\,ds &= \sum_i (t_{i+1}-t_i)D(\mathbf{u}_i||\mathbf{p}) \geq \sum_i h_i C_{M^*}^N(1)\\ &= C_{M^*}^N(1)\end{aligned}$$

This is true for each piece-wise linear and ordered trajectory, and thus,

$$\theta_B^{\text{QLB}}(N) \geq C_{M^*}^N(1).$$

We have shown that $\min_M C_M^N(1)$ is a lower bound on $\theta_B^{\text{QLB}}(N)$. Furthermore, it can be shown (Theorem 5 of [8]) that there exists a trajectory that has the cost $\min_M C_M^N(1)$, then we can conclude that $\theta_B^{\text{QLB}}(N) = \min_M C_M^N(1)$. ■

In the next theorem, we characterize the behavior of $\theta_B^{\text{QLB}}(N)$ and $\theta_D^{\text{QLB}}(N)$ as a function of N.

Theorem 6: Under the QLB policy, the large-deviations exponents behave as follows:

$$\theta_B^{\text{QLB}}(N) < \theta_B^{\text{QLB}}(N+1) \quad \text{and} \quad \theta_D^{\text{QLB}}(N) < \theta_D^{\text{QLB}}(N+1).$$

Proof: The proof is omitted due to lack of space, please refer to Theorem 7 of [8] for detail. ■

IV. Greedy Policy

In this section, we will consider the greedy policy. We first obtain expressions for $\theta_B^{\text{Greedy}}(N)$ and $\theta_D^{\text{Greedy}}(N)$. For the system under the greedy policy, we use the notation (j,i) to indicate the state of the system; here, $j=0,\ldots,2^N-1$ represents the composite state of all the channels as in the QLB policy and i indicates the channel which is chosen to transmit. If there are M channels in the ON state, we are equally likely pick any one of them to transmit. Let S^j be the binary expansion of j and S_k^j be the kth entry of S^j. Then, for $j>0$, the probability of being in system state $p_{j,i}$ is

$$p_{j,i} = p_j \frac{S_i^j}{\sum_{k=0}^N S_k^j}.$$

For example, if M channels are ON, then $\sum_{k=0}^{N-1} S_k^j = M$. If channel i is OFF in the composite channel state j, then $p_{j,i} = 0$ because $S_i^j = 0$. Otherwise, $p_{j,i} = \frac{1}{M}p_j$, which means that every ON channel is equally likely to be chosen. When $j = 0$, no channel can be scheduled. Also, define $\tilde{\mathbf{p}}$ to be the probability vector $(p_0, \{p_{j,i}\}_{j>0,i=0,1,\ldots,N-1})$.

From the symmetry of the system and the scheduling policy, the differential equation describing the dynamics of the system are as follows:

$$\dot{q}_i(t) = \frac{\lambda}{N} - \sum_{j=0}^{2^N-1} F u_{j,i}(t),$$

where $q_i(t) = Q_i(t)/B$. Similar to the QLB policy, we define

$$\theta_B^{\text{Greedy}}(N) = \inf_{T,\mathbf{u}} \int_{-T}^{0} D\left(\mathbf{u}(s)||\tilde{\mathbf{p}}\right) ds, \tag{11}$$

where $q_i(-T) = 0$ for all i, $\max_i q_i(0) = 1$, and the greedy policy is used. Then, we have following theorem.

Theorem 7:

$$\theta_B^{\text{Greedy}}(N) = \lim_{B\to\infty} \frac{-1}{B} \log \Pr(\max_i q_i(0) \geq 1),$$

where queues are scheduled according to the greedy policy.

Proof: Same as the proof of Theorem 1. ■

Next, we calculate the large-deviations exponents $\theta_B^{\text{Greedy}}(N)$ and $\theta_D^{\text{Greedy}}(N)$.

Theorem 8: For a N-user system,

$$\theta_B^{\text{Greedy}}(N) = \inf_{0\leq x<\frac{\lambda}{NF}} \frac{N}{\lambda - xNF}\left(x\log\frac{x}{\hat{p}} + (1-x)\log\frac{1-x}{1-\hat{p}}\right),$$

and

$$\theta_D^{\text{Greedy}}(N) = \inf_{0\leq x<\frac{\lambda}{NF}} \frac{\lambda}{\lambda - xNF}\left(x\log\frac{x}{\hat{p}} + (1-x)\log\frac{1-x}{1-\hat{p}}\right),$$

where

$$\hat{p} = \frac{1}{N}(1-(1-p)^N).$$

Furthermore, the following limits hold:

$$\lim_{N\to\infty} \theta_B^{\text{Greedy}}(N) = \inf_a \frac{1}{\lambda - aF}\left(a\log a + 1 - a\right)$$

and

$$\lim_{N\to\infty} \theta_D^{\text{Greedy}}(N) = 0.$$

Proof: The proof is omitted due to lack of space. ∎

V. QLB Policy *vs* Greedy Policy

Under the delay constraint, we have shown that the large-deviations exponent of QLB policy is non-decreasing in N; while the large-deviations exponent of the greedy policy decreases to 0. Also because $\theta_B(N) = N\theta_D(N)$, we can conclude that the total throughput of the QLB policy is larger than the greedy policy under the queue overflow constraint or delay constraint for large N. In the following theorem, we will show that this relationship between the throughput of the two policies holds for all N.

Theorem 9: For any N-user system the throughput under QLB policy is no less than the throughput under the greedy policy:

$$\theta_B^{\text{QLB}}(N) \geq \theta_B^{\text{Greedy}}(N) \quad \text{and} \quad \theta_D^{\text{QLB}}(N) \geq \theta_D^{\text{Greedy}}(N).$$

Proof: Since $\theta_B(N) = \frac{N}{\lambda}\theta_D(N)$, we only need to show one of the inequalities holds. We consider $\theta_B(N)$ and show $\theta_B^{\text{QLB}}(N) \geq \theta_B^{\text{Greedy}}(N)$.

Consider $M^* \in U = \{M : M = \arg\min_M C_M^N(1)\}$ and $\mathbf{u}(M^*)$ is the corresponding control, so

$$T(M^*)\left(\frac{M^*}{N}\lambda - F\sum_{j=2^{N-M^*}}^{2^N-1} u_j(M^*)\right) = M^*.$$

Now define a control $\tilde{\mathbf{u}}$ for the system under greedy policy such that, for $j \geq 1$,

$$\tilde{u}_{j,i} = \frac{S_i^j}{\sum_{k=0}^{N-1} S_k^j} u_j(M^*).$$

Then, under this control $\tilde{\mathbf{u}}$ and the greedy policy, we have

$$\begin{aligned}
\sum_{i=N-M^*}^{N-1} \dot{q}_i(t) &= \frac{M^*\lambda}{N} - F\sum_{i=N-M^*}^{N-1}\sum_{j=0}^{2^N-1} \tilde{u}_{j,i} \\
&= \frac{M^*\lambda}{N} - F\sum_{j=0}^{2^N-1}\left(\frac{\sum_{i=N-M^*}^{N-1} S_i^j}{\sum_{k=0}^{N-1} S_k^j} u_j(M^*)\right) \\
&= \frac{M^*\lambda}{N} - F\sum_{j=2^{N-M^*}}^{2^N-1}\left(\frac{\sum_{i=N-M^*}^{N-1} S_i^j}{\sum_{k=0}^{N-1} S_k^j} u_j(M^*)\right),
\end{aligned}$$

where the last equation holds because $\sum_{i=N-M^*}^{N-1} S_i^j = 0$ for $j = 0, \ldots, 2^{N-M^*} - 1$. Let $\tilde{\mathbf{p}}$ be the probability vector corresponding to $p_{j,i}$. Because $p_{j,i} = \frac{S_i^j}{\sum_{k=0}^{N-1} S_k^j} p_j$, it is easy to verify that

$$D(\tilde{\mathbf{u}}||\tilde{\mathbf{p}}) = D(\mathbf{u}(M^*)||\mathbf{p}).$$

Furthermore, $\frac{\sum_{i=N-M^*}^{N-1} S_i^j}{\sum_{k=0}^{N-1} S_k^j} \leq 1$, so we can conclude that

$$F\sum_{j=2^{N-M^*}}^{2^N-1} u_j(M^*) \geq F\sum_{j=2^{N-M^*}}^{2^N-1}\left(\frac{\sum_{i=N-M^*}^{N-1} S_i^j}{\sum_{k=0}^{N-1} S_k^j} u_j(M^*)\right). \quad (12)$$

Let $\tilde{T}$ be the time to overflow under $\tilde{\mathbf{u}}$, i.e., $\tilde{T}$ is the last time before $t = 0$ that $q_i(\tilde{T}) = 0\ \forall i$. Then,

$$\tilde{T} \leq T(M^*) \qquad \text{and} \qquad \tilde{T}D(\tilde{\mathbf{u}}\|\tilde{p}) \leq T(M^*)D(\mathbf{u}(M^*)\|\mathbf{p}).$$

From the definition (11), we can conclude

$$\theta_B^{\text{Greedy}}(N) \leq \tilde{T}D(\tilde{\mathbf{u}}||\tilde{\mathbf{p}}) \leq C_{M^*}^N(1) = \theta_B^{\text{QLB}}(N).$$

∎

From this theorem, we can conclude that the performance of QLB policy is better than the Greedy policy under the QoS constraints.

VI. Conclusions

In this paper, we use a large deviations analysis to investigate the performance of different scheduling policies for the downlink of a cellular network under QoS constraints. For a simple "ON-OFF" channel model, we prove that the throughput of queue-length based policy is larger than that of the greedy policy. Furthermore, when the number of users increases, the throughput of the greedy policy decreases while the QLB policy increases. In a longer version, we extend these results to more general channel models.

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