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Thermodynamic Stabilization via Energy Dissipating Hybrid Controllers

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Abstract— **A novel class of fixed-order, energy-based hybrid controllers is proposed as a means for achieving enhanced energy dissipation in Euler-Lagrange, port-controlled Hamiltonian, and lossless dynamical systems. These dynamic controllers combine a logical switching architecture with continuous dynamics to guarantee that the system plant energy is strictly decreasing across switchings. The general framework leads to closed-loop systems described by impulsive differential equations. In addition, we construct hybrid dynamic controllers that guarantee that the closed-loop system is consistent with basic thermodynamic principles. In particular, the existence of an entropy function for the closed-loop system is established that satisfies a hybrid Clausius-type inequality. Special cases of energy-based hybrid controllers involving state-dependent switching are described.**

I. Introduction

Energy is a concept that underlies our understanding of all physical phenomena and is a measure of the ability of a dynamical system to produce changes (motion) in its own system state as well as changes in the system states of its surroundings. In control engineering, dissipativity theory [1], which encompasses passivity theory, provides a fundamental framework for the analysis and control design of dynamical systems using an input-output system description based on system energy related considerations [2]. The notion of energy here refers to abstract energy notions for which a physical system energy interpretation is not necessary. The dissipation hypothesis on dynamical systems results in a fundamental constraint on their dynamic behavior, wherein a dissipative dynamical system can only deliver a fraction of its energy to its surroundings and can only store a fraction of the work done to it. Thus, dissipativity theory provides a powerful framework for the analysis and control design of dynamical systems based on generalized energy considerations by exploiting the notion that numerous physical systems have certain input-output properties related to conservation, dissipation, and transport of energy. Such conservation laws are prevalent in dynamical systems such as mechanical systems, fluid systems, electromechanical systems, electrical systems, combustion systems, structural vibration systems, biological systems, physiological systems, power systems, telecommunications systems, and economic systems, to cite but a few examples.

Energy-based control for Euler-Lagrange dynamical systems and Hamiltonian dynamical systems based on passivity notions has received considerable attention in the literature [2]. This controller design technique achieves system stabilization by shaping the energy of the closed-loop system which involves the physical system energy and the controller emulated energy. Specifically, *energy shaping* is achieved by modifying the system potential energy in such a way so that the shaped potential energy function for the closed-loop system possesses a unique global minimum at a desired equilibrium point. Next, damping is *injected* via feedback control modifying the system dissipation to guarantee asymptotic stability of the closed-loop system. A central feature of this energy-based stabilization approach is that the Lagrangian system form is preserved at the closed-loop system level. Furthermore, the control action has a clear physical energy interpretation, wherein the total energy of the closed-loop Euler-Lagrange system corresponds to the difference between the physical system energy and the emulated energy supplied by the controller.

In this paper, we develop a novel energy dissipating hybrid control framework for Lagrangian, port-controlled Hamiltonian, and lossless dynamical systems. These dynamical systems cover a very broad spectrum of applications including mechanical, electrical, electromechanical, structural, biological, and power systems. The fixed-order, energy-based hybrid controller is a hybrid controller that emulates a hybrid Hamiltonian dynamical system and exploits the feature that the states of the dynamic controller may be reset to enhance the overall energy dissipation in the closed-loop system. An important feature of the hybrid controller is that its Hamiltonian structure can be associated with a kinetic and potential energy function. In a mechanical Euler-Lagrange system, positions typically correspond to elastic deformations, which contribute to the potential energy of the system, whereas velocities typically correspond to momenta, which contribute to the kinetic energy of the system. On the other hand, while our energy-based hybrid controller has dynamical states that emulate the motion of a physical Hamiltonian system, these states only "exist" as numerical representations inside the processor. Consequently, while one can associate an *emulated energy* with these states, this energy is merely a mathematical construct and does not correspond to any physical form of energy.

The concept of an energy-based hybrid controller can be viewed as a feedback control technique that exploits the coupling between a physical dynamical system and an energy-based controller to efficiently remove energy from the physical system. Specifically, if a dissipative or lossless plant is at high energy level, and a lossless feedback controller at a low energy level is attached to it, then energy will generally tend to flow from the plant into the controller, decreasing the plant energy and increasing the controller energy. Of course, emulated energy, and not physical energy, is accumulated by the controller. Conversely, if the attached controller is at a high energy level and a plant is at a low energy level, then energy can flow from the controller to the plant, since a controller can generate real, physical energy to effect the required energy flow. Hence, if and when the controller states coincide with a high emulated energy level, then we can *reset* these states to remove the emulated energy so that the emulated energy is not returned to the plant. In this case, the overall closed-loop system consisting of the plant and the

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controller possesses discontinuous flows since it combines logical switchings with continuous dynamics, leading to impulsive differential equations [3], [4], [5]. Within the context of vibration control using resetting virtual absorbers, these ideas were first explored in [6].

II. Hybrid Control and Impulsive Systems

In this section, we establish definitions, notation, and review some basic results on impulsive dynamical systems [4]. Let $\mathbb{R}$ denote the set of real numbers, let $\overline{\mathbb{R}}_+$ denote the set of nonnegative real numbers, let $\mathbb{R}^n$ denote the set of $n\times 1$ real column vectors, let $\overline{\mathbb{Z}}_+$ denote the set of nonnegative integers, let $(\cdot)^{\mathrm{T}}$ denote transpose, and let I_n denote the $n\times n$ identity matrix. Furthermore, let $\partial\mathcal{S}$, $\overset{\circ}{\mathcal{S}}$, and $\overline{\mathcal{S}}$ denote the boundary, the interior, and the closure of the subset $\mathcal{S}\subset\mathbb{R}^n$, respectively. We write $\|\cdot\|$ for the Euclidean vector norm, $\mathcal{B}_\varepsilon(\alpha)$, $\alpha\in\mathbb{R}^n$, $\varepsilon>0$, for the open ball centered at α with radius ε, and $V'(x)$ for the Fréchet derivative of V at x.

In this paper, we consider continuous-time nonlinear dynamical systems of the form

$$\dot{x}_{\mathrm{p}}(t) = f_{\mathrm{p}}(x_{\mathrm{p}}(t),u(t)), \quad x_{\mathrm{p}}(0)=x_{\mathrm{p}0}, \quad t\geq 0, \tag{1}$$

$$y(t) = h_{\mathrm{p}}(x_{\mathrm{p}}(t)), \tag{2}$$

where $t\geq 0$, $x_{\mathrm{p}}(t)\in\mathcal{D}_{\mathrm{p}}\subseteq\mathbb{R}^{n_{\mathrm{p}}}$, $\mathcal{D}_{\mathrm{p}}$ is an open set with $0\in\mathcal{D}_{\mathrm{p}}$, $u(t)\in\mathbb{R}^m$, $f_{\mathrm{p}}:\mathcal{D}_{\mathrm{p}}\times\mathbb{R}^m\to\mathbb{R}^{n_{\mathrm{p}}}$ is locally Lipschitz continuous on $\mathcal{D}_{\mathrm{p}}$ and satisfies $f_{\mathrm{p}}(0,0)=0$, and $h_{\mathrm{p}}:\mathcal{D}_{\mathrm{p}}\to\mathbb{R}^l$ is continuous and satisfies $h_{\mathrm{p}}(0)=0$. Furthermore, we consider hybrid (resetting) dynamic controllers of the form

$$\dot{x}_{\mathrm{c}}(t) = f_{\mathrm{cc}}(x_{\mathrm{c}}(t),y(t)), \quad x_{\mathrm{c}}(0)=x_{\mathrm{c}0}, \quad (x_{\mathrm{c}}(t),y(t))\notin\mathcal{Z}_{\mathrm{c}}, \tag{3}$$

$$\Delta x_{\mathrm{c}}(t) = f_{\mathrm{dc}}(x_{\mathrm{c}}(t),y(t)), \quad (x_{\mathrm{c}}(t),y(t))\in\mathcal{Z}_{\mathrm{c}}, \tag{4}$$

$$u(t) = h_{\mathrm{cc}}(x_{\mathrm{c}}(t),y(t)), \tag{5}$$

where $t\geq 0$, $x_{\mathrm{c}}(t)\in\mathcal{D}_{\mathrm{c}}\subseteq\mathbb{R}^{n_{\mathrm{c}}}$, $\mathcal{D}_{\mathrm{c}}$ is an open set with $0\in\mathcal{D}_{\mathrm{c}}$, $\Delta x_{\mathrm{c}}(t)\triangleq x_{\mathrm{c}}(t^+)-x_{\mathrm{c}}(t)$, $f_{\mathrm{cc}}:\mathcal{D}_{\mathrm{c}}\times\mathbb{R}^l\to\mathbb{R}^{n_{\mathrm{c}}}$ is locally Lipschitz continuous on $\mathcal{D}_{\mathrm{c}}$ and satisfies $f_{\mathrm{cc}}(0,0)=0$, $h_{\mathrm{cc}}:\mathcal{D}_{\mathrm{c}}\times\mathbb{R}^l\to\mathbb{R}^m$ is continuous and satisfies $h_{\mathrm{cc}}(0,0)=0$, $f_{\mathrm{dc}}:\mathcal{D}_{\mathrm{c}}\times\mathbb{R}^l\to\mathbb{R}^{n_{\mathrm{c}}}$ is continuous, and $\mathcal{Z}_{\mathrm{c}}\subset\mathcal{D}_{\mathrm{c}}\times\mathbb{R}^l$ is the resetting set. Note that, for generality, we allow the hybrid dynamic controller to be of fixed dimension n_{c} which may be less than the plant order n_{p}.

The equations of motion for the closed-loop dynamical system (1)–(5) have the form

$$\dot{x}(t) = f_{\mathrm{c}}(x(t)), \quad x(0)=x_0, \quad x(t)\notin\mathcal{Z}, \tag{6}$$

$$\Delta x(t) = f_{\mathrm{d}}(x(t)), \quad x(t)\in\mathcal{Z}, \tag{7}$$

where

$$x\triangleq\begin{bmatrix} x_{\mathrm{p}} \\ x_{\mathrm{c}}\end{bmatrix}\in\mathbb{R}^n, \quad f_{\mathrm{d}}(x)\triangleq\begin{bmatrix} 0 \\ f_{\mathrm{dc}}(x_{\mathrm{c}},h_{\mathrm{p}}(x_{\mathrm{p}}))\end{bmatrix}$$

$$f_{\mathrm{c}}(x)\triangleq\begin{bmatrix} f_{\mathrm{p}}(x_{\mathrm{p}},h_{\mathrm{cc}}(x_{\mathrm{c}},h_{\mathrm{p}}(x_{\mathrm{p}}))) \\ f_{\mathrm{cc}}(x_{\mathrm{c}},h_{\mathrm{p}}(x_{\mathrm{p}}))\end{bmatrix}, \tag{8}$$

and $\mathcal{Z}\triangleq\{x\in\mathcal{D}:(x_{\mathrm{c}},h_{\mathrm{p}}(x_{\mathrm{p}}))\in\mathcal{Z}_{\mathrm{c}}\}$, with $n\triangleq n_{\mathrm{p}}+n_{\mathrm{c}}$ and $\mathcal{D}\triangleq\mathcal{D}_{\mathrm{p}}\times\mathcal{D}_{\mathrm{c}}$. We refer to the differential equation (6) as the *continuous-time dynamics*, and we refer to the difference equation (7) as the *resetting law*. Note that although the closed-loop state vector consists of plant states and controller states, it is clear from (8) that only those states associated with the controller are reset. For convenience, we use the notation $s(t,x_0)$ to denote the solution $x(t)$ of (6) and (7) at time $t\geq 0$ with initial condition $x(0)=x_0$.

For a particular closed-loop trajectory $x(t)$, we let $t_k\triangleq\tau_k(x_0)$ denote the kth instant of time at which $x(t)$ intersects $\mathcal{Z}$, and we call the times t_k the *resetting times*. Thus, the trajectory of the closed-loop system (6) and (7) from the initial condition $x(0)=x_0$ is given by $\psi(t,x_0)$ for $0<t\leq t_1$, where $\psi(t,x_0)$ denotes the solution to the continuous-time dynamics (6). If and when the trajectory reaches a state $x_1\triangleq x(t_1)$ satisfying $x_1\in\mathcal{Z}$, then the state is instantaneously transferred to $x_1^+\triangleq x_1+f_{\mathrm{d}}(x_1)$ according to the resetting law (7). The trajectory $x(t)$, $t_1<t\leq t_2$, is then given by $\psi(t-t_1,x_1^+)$, and so on. Note that the solution $x(t)$ of (6) and (7) is left continuous, that is, it is continuous everywhere except at the resetting times t_k, $k=1,2,\ldots$.

To ensure well posedness of the resetting times, we make the following additional assumptions:

Assumption 1. If $x(t)\in\overline{\mathcal{Z}}\setminus\mathcal{Z}$, then there exists $\varepsilon>0$ such that, for all $0<\delta<\varepsilon$, $s(\delta,x(t))\notin\mathcal{Z}$.

Assumption 2. If $x\in\mathcal{Z}$, then $x+f_{\mathrm{d}}(x)\notin\mathcal{Z}$.

Assumption 1 ensures that if a trajectory reaches the closure of $\mathcal{Z}$ at a point that does not belong to $\mathcal{Z}$, then the trajectory must be directed away from $\mathcal{Z}$, that is, a trajectory cannot enter $\mathcal{Z}$ through a point that belongs to the closure of $\mathcal{Z}$ but not to $\mathcal{Z}$. Furthermore, Assumption 2 ensures that when a trajectory intersects the resetting set $\mathcal{Z}$, it instantaneously exits $\mathcal{Z}$. Finally, we note that if $x_0\in\mathcal{Z}$, then the system initially resets to $x_0^+=x_0+f_{\mathrm{d}}(x_0)\notin\mathcal{Z}$, which serves as the initial condition for the continuous-time dynamics (6).

It follows from Assumptions 1 and 2 that for a particular initial condition, the resetting times $t_k=\tau_k(x_0)$ are distinct and well defined [4]. Since the resetting times are well defined and distinct, and since the solution to (6) exists and is unique, it follows that the solution of the impulsive dynamical system (6) and (7) also exists and is unique over a forward time interval. However, it is important to note that the analysis of impulsive dynamical systems can be quite involved. In particular, such systems can exhibit *Zenoness*, *beating*, as well as *confluence*, wherein solutions exhibit infinitely many resettings in a finite-time, encounter the same resetting surface a finite or infinite number of times in zero time, and coincide after a certain point in time [4], [5]. In this paper we allow for the possibility of confluence and Zeno solutions, however, Assumption 2 precludes the possibility of beating. Furthermore, since *not* every bounded solution of an impulsive dynamical system over a forward time interval can be extended to infinity due to Zeno solutions, we assume that existence and uniqueness of solutions are satisfied in forward time. For details see [3].

For the statement of the next result the following key assumption is needed.

Assumption 3. Consider the impulsive dynamical system (6) and (7), and let $s(t,x_0)$, $t\geq 0$, denote the solution to (6) and (7) with initial condition x_0. Then for every $x_0\notin\mathcal{Z}$ and every $\varepsilon>0$ and $t\neq t_k$, there exists $\delta(\varepsilon,x_0,t)>0$ such that if $\|x_0-y\|<\delta(\varepsilon,x_0,t)$, $y\in\mathcal{D}$, then $\|s(t,x_0)-s(t,y)\|<\varepsilon$.

Assumption 3 is a generalization of the standard continuous dependence property for dynamical systems with continuous flows to dynamical systems with left-continuous flows. Since solutions of impulsive dynamical systems are not continuous in time and solutions are not continuous functions of the system initial conditions, Assumption 3 is needed to apply the hybrid invariance principle developed in [4], [5] to hybrid closed-loop systems. The following result provides sufficient conditions for guaranteeing that the impulsive dynamical system (6) and (7) satisfies Assumption 3.

Proposition 2.1: Consider the impulsive dynamical system $\mathcal{G}$ given by (6) and (7). Assume that Assumptions 1 and 2 hold, assume that for all $x_0 \notin \overline{\mathcal{Z}}$, $0 < \tau_1(x_0) < \infty$, and $\tau_1(\cdot)$ is continuous, and assume that if $x_0 \in \mathcal{Z}$, then $x_0 + f_{\rm d}(x_0) \in \overline{\mathcal{Z}} \backslash \mathcal{Z}$. Furthermore, for every $x_0 \in \overline{\mathcal{Z}} \backslash \mathcal{Z}$ such that $0 < \tau_1(x_0) < \infty$, let $\{x_i\}_{i=1}^{\infty} \in \mathcal{D}$ be such that $\lim_{i\to\infty} x_i = x_0$ and $\lim_{i\to\infty} \tau_1(x_i)$ exists, and assume that if $\{x_i\}_{i=1}^{\infty} \in \overline{\mathcal{Z}} \backslash \mathcal{Z}$, then $\lim_{i\to\infty} \tau_1(x_i) = \tau_1(x_0)$; or otherwise, assume that *i)* $\lim_{i\to\infty} \tau_1(x_i) = \tau_1(x_0)$ or *ii)* $f_{\rm d}(x_0) = 0$ and $\lim_{i\to\infty} \tau_1(x_i) = 0$. Then $\mathcal{G}$ satisfies Assumption 3.

The following result provides sufficient conditions for establishing continuity of $\tau_1(\cdot)$ at $x_0 \notin \overline{\mathcal{Z}}$ and *sequential continuity* of $\tau_1(\cdot)$ at $x_0 \notin \overline{\mathcal{Z}} \backslash \mathcal{Z}$. For this result, the following definition is needed.

Definition 2.1: Let $\mathcal{M} \triangleq \{x \in \mathcal{D} : \ \mathcal{X}(x) = 0\}$, where $\mathcal{X} : \mathcal{D} \to \mathbb{R}$ is an infinitely-differentiable function. A point $x_0 \in \mathcal{M}$ is *transversal* to (6) if there exists $k \in \{1, 2, \ldots\}$ such that, for $r = 0, \ldots, 2k-2$,

$$\frac{\mathrm{d}^r}{\mathrm{d}t^r} \mathcal{X}(x(t))\Big|_{t=0} = 0, \quad \frac{\mathrm{d}^{2k-1}}{\mathrm{d}t^{2k-1}} \mathcal{X}(x(t))\Big|_{t=0} \neq 0, \qquad (9)$$

where $x(t)$ denotes the solution to (6) with $x(0) = x_0$.

Proposition 2.2: Consider the impulsive dynamical system (6) and (7). Let $x_0 \notin \overline{\mathcal{Z}}$, assume there exists an infinitely-differentiable function $\mathcal{X} : \mathcal{D} \to \mathbb{R}$ such that $\overline{\mathcal{Z}} = \{x \in \mathcal{D} : \mathcal{X}(x) = 0\}$, and assume every $x_0 \in \overline{\mathcal{Z}}$ is transversal to (6). Then $\tau_1(\cdot)$ is continuous at $x_0 \notin \overline{\mathcal{Z}}$, where $0 < \tau_1(x_0) < \infty$. Alternatively, let $x_0 \in \overline{\mathcal{Z}} \backslash \mathcal{Z}$, let $\{x_i\}_{i=1}^{\infty} \notin \mathcal{Z}$ be such that $\lim_{i\to\infty} x_i = x_0$ and $\lim_{i\to\infty} \tau_1(x_i)$ exists, and assume that either of the following statements hold: *i)* $\{x_i\}_{i=1}^{\infty} \in \overline{\mathcal{Z}} \backslash \mathcal{Z}$ or *ii)* $\lim_{i\to\infty} \tau_1(x_i) > 0$. Then $\lim_{i\to\infty} \tau_1(x_i) = \tau_1(x_0)$, where $0 < \tau_1(x_0) < \infty$.

III. Hybrid Control Design for Lossless Dynamical Systems

In this section, we present a hybrid controller design framework for lossless dynamical systems [1]. Specifically, we consider nonlinear dynamical systems $\mathcal{G}_{\rm p}$ of the form given by (1) and (2) where $u(\cdot)$ satisfies sufficient regularity conditions such that (1) has a unique solution forward in time. Furthermore, we consider hybrid resetting dynamic controllers $\mathcal{G}_{\rm c}$ of the form

$$\begin{aligned} \dot{x}_{\rm c}(t) &= f_{\rm cc}(x_{\rm c}(t), u_{\rm c}(t)), \quad x_{\rm c}(0) = x_{\rm c0}, \\ & \qquad (x_{\rm c}(t), y(t)) \notin \mathcal{Z}_{\rm c}, \quad (10) \\ \Delta x_{\rm c}(t) &= \eta(y(t)) - x_{\rm c}(t), \ (x_{\rm c}(t), y(t)) \in \mathcal{Z}_{\rm c}, \quad (11) \\ y_{\rm c}(t) &= h_{\rm cc}(x_{\rm c}(t), u_{\rm c}(t)), \quad (12) \end{aligned}$$

where $x_{\rm c}(t) \in \mathcal{D}_{\rm c} \subseteq \mathbb{R}^{n_{\rm c}}$, $\mathcal{D}_{\rm c}$ is an open set with $0 \in \mathcal{D}_{\rm c}$, $u_{\rm c}(t) \in \mathbb{R}^l$, $y_{\rm c}(t) \in \mathbb{R}^m$, $f_{\rm cc} : \mathcal{D}_{\rm c} \times \mathbb{R}^l \to \mathbb{R}^{n_{\rm c}}$ is locally Lipschitz continuous on $\mathcal{D}_{\rm c}$ and satisfies $f_{\rm cc}(0,0) = 0$, $\eta : \mathbb{R}^l \to \mathcal{D}_{\rm c}$ is continuous and satisfies $\eta(0) = 0$, and $h_{\rm cc} : \mathcal{D}_{\rm c} \times \mathbb{R}^l \to \mathbb{R}^m$ is continuous and satisfies $h_{\rm cc}(0,0) = 0$.

Recall that for the dynamical system $\mathcal{G}_{\rm p}$ given by (1) and (2), a function $s(u, y)$, where $s : \mathbb{R}^m \times \mathbb{R}^l \to \mathbb{R}$ is such that $s(0,0) = 0$, is called a *supply rate* [1] if it is locally integrable for all input-output pairs satisfying (1) and (2), that is, for all input-output pairs $u \in \mathcal{U}$ and $y \in \mathcal{Y}$ satisfying (1) and (2), $s(\cdot,\cdot)$ satisfies $\int_t^{\hat{t}} |s(u(\sigma), y(\sigma))| \mathrm{d}\sigma < \infty$, $t, \hat{t} \geq 0$. Here, $\mathcal{U}$ and $\mathcal{Y}$ are input and output spaces, respectively, that are assumed to be closed under the shift operator. Furthermore, we assume that $\mathcal{G}_{\rm p}$ is *lossless with respect to the supply rate* $s(u, y)$, and hence, there exists a continuous, nonnegative-definite *storage function* $V_{\rm s} : \mathcal{D}_{\rm p} \to \overline{\mathbb{R}}_+$ such that $V_{\rm s}(0) = 0$ and

$$V_{\rm s}(x_{\rm p}(t)) = V_{\rm s}(x_{\rm p}(t_0)) + \int_{t_0}^{t} s(u(\sigma), y(\sigma)) \mathrm{d}\sigma, \qquad (13)$$

for all $t_0, t \geq 0$, where $x_{\rm p}(t)$, $t \geq t_0$, is the solution to (1) with $u \in \mathcal{U}$. In addition, we assume that the nonlinear dynamical system $\mathcal{G}_{\rm p}$ is *completely reachable* [1] and *zero-state observable* [1], and there exists a function $\kappa : \mathbb{R}^l \to \mathbb{R}^m$ such that $\kappa(0) = 0$ and $s(\kappa(y), y) < 0$, $y \neq 0$, so that all storage functions $V_{\rm s}(x_{\rm p})$, $x_{\rm p} \in \mathcal{D}_{\rm p}$, of $\mathcal{G}_{\rm p}$ are positive definite. Finally, we assume that $V_{\rm s}(\cdot)$ is continuously differentiable.

Consider the negative feedback interconnection of $\mathcal{G}_{\rm p}$ and $\mathcal{G}_{\rm c}$ given by $y = u_{\rm c}$ and $u = -y_{\rm c}$. In this case, the closed-loop system $\mathcal{G}$ is given by

$$\begin{aligned} \dot{x}(t) &= f_{\rm c}(x(t)), \ x(0) = x_0, \ x(t) \notin \mathcal{Z}, \ t \geq 0, \quad (14) \\ \Delta x(t) &= f_{\rm d}(x(t)), \ x(t) \in \mathcal{Z}, \quad (15) \end{aligned}$$

where $t \geq 0$, $x(t) \triangleq [x_{\rm p}^{\rm T}(t), x_{\rm c}^{\rm T}(t)]^{\rm T}$, $\mathcal{Z} \triangleq \{x \in \mathcal{D} : (x_{\rm c}, h_{\rm p}(x_{\rm p})) \in \mathcal{Z}_{\rm c}\}$,

$$f_{\rm c}(x) = \begin{bmatrix} f_{\rm p}(x_{\rm p}, -h_{\rm cc}(x_{\rm c}, h_{\rm p}(x_{\rm p}))) \\ f_{\rm cc}(x_{\rm c}, h_{\rm p}(x_{\rm p})) \end{bmatrix},$$

$$f_{\rm d}(x) = \begin{bmatrix} 0 \\ \eta(h_{\rm p}(x_{\rm p})) - x_{\rm c} \end{bmatrix}. \qquad (16)$$

Assume that there exists an infinitely-differentiable function $V_{\rm c} : \mathcal{D}_{\rm c} \times \mathbb{R}^l \to \overline{\mathbb{R}}_+$ such that $V_{\rm c}(x_{\rm c}, y) \geq 0$, $x_{\rm c} \in \mathcal{D}_{\rm c}$, $y \in \mathbb{R}^l$, and $V_{\rm c}(x_{\rm c}, y) = 0$ if and only if $x_{\rm c} = \eta(y)$ and

$$\dot{V}_{\rm c}(x_{\rm c}(t), y(t)) = s_{\rm c}(u_{\rm c}(t), y_{\rm c}(t)), \quad (x_{\rm c}(t), y(t)) \notin \mathcal{Z}, \ (17)$$

where $s_{\rm c} : \mathbb{R}^l \times \mathbb{R}^m \to \mathbb{R}$ is such that $s_{\rm c}(0,0) = 0$ and is locally integrable for all input-output pairs satisfying (10)–(12).

We associate with the plant a positive-definite, continuously differentiable function $V_{\rm p}(x_{\rm p}) \triangleq V_{\rm s}(x_{\rm p})$, which we will refer to as the *plant energy*. Furthermore, we associate with the controller a nonnegative-definite, infinitely-differentiable function $V_{\rm c}(x_{\rm c}, y)$ called the controller *emulated energy*. Finally, we associate with the closed-loop system the function

$$V(x) \triangleq V_{\rm p}(x_{\rm p}) + V_{\rm c}(x_{\rm c}, h_{\rm p}(x_{\rm p})), \qquad (18)$$

called the *total energy*.

Next, we construct the resetting set for the closed-loop system $\mathcal{G}$ in the following form

$$\mathcal{Z} = \Big\{(x_{\rm p}, x_{\rm c}) \in \mathcal{D}_{\rm p} \times \mathcal{D}_{\rm c} : \frac{\mathrm{d}}{\mathrm{d}t} V_{\rm c}(x_{\rm c}, h_{\rm p}(x_{\rm p})) = 0 \text{ and } V_{\rm c}(x_{\rm c}, h_{\rm p}(x_{\rm p})) > 0\Big\}. \qquad (19)$$

The resetting set $\mathcal{Z}$ is thus defined to be the set of all points in the closed-loop state space that correspond to decreasing controller emulated energy. By resetting the controller states, the plant energy can never increase after the first resetting event. Furthermore, if the closed-loop system total energy is conserved between resetting events, then a decrease in plant energy is accompanied by a corresponding increase in emulated energy. Hence, this approach allows the plant energy to flow to the controller, where it increases the emulated energy but does not allow the emulated energy to flow back to the plant after the first resetting event. This energy dissipating hybrid controller effectively enforces a one-way energy transfer between the plant and the controller after the first resetting event. For practical implementation, knowledge of $x_{\rm c}$ and y is sufficient to determine whether or not the closed-loop state vector is in the set $\mathcal{Z}$.

The next theorem gives sufficient conditions for asymptotic stability of the closed-loop system $\mathcal{G}$ using state-dependent hybrid controllers.

Theorem 3.1: Consider the closed-loop impulsive dynamical system $\mathcal{G}$ given by (14) and (15) with the resetting set $\mathcal{Z}$ given by (19). Assume that $\mathcal{D}_{\rm ci} \subset \mathcal{D}$ is a compact positively invariant set with respect to $\mathcal{G}$ such that $0 \in \overset{\circ}{\mathcal{D}}_{\rm ci}$, assume that $\mathcal{G}_{\rm p}$ is lossless with respect to the supply rate $s(u, y)$ and with a positive definite, continuously differentiable storage function $V_{\rm p}(x_{\rm p})$, $x_{\rm p} \in \mathcal{D}_{\rm p}$, and assume there exists a smooth (i.e., infinitely-differentiable) function $V_{\rm c} : \mathcal{D}_{\rm c} \times \mathbb{R}^l \to \overline{\mathbb{R}}_+$ such that $V_{\rm c}(x_{\rm c}, y) \geq 0$, $x_{\rm c} \in \mathcal{D}_{\rm c}$, $y \in \mathbb{R}^l$, and $V_{\rm c}(x_{\rm c}, y) = 0$ if and only if $x_{\rm c} = \eta(y)$ and (17) holds. Furthermore, assume that every $x_0 \in \overline{\mathcal{Z}}$ is transversal to (14) and $s(u, y) + s_{\rm c}(u_{\rm c}, y_{\rm c}) = 0$, $x \notin \mathcal{Z}$. Then the zero solution $x(t) \equiv 0$ to the closed-loop system $\mathcal{G}$ is asymptotically stable. In addition, the total energy function $V(x)$ of $\mathcal{G}$ given by (18) is strictly decreasing across resetting events. Finally, if $\mathcal{D}_{\rm p} = \mathbb{R}^{n_{\rm p}}$, $\mathcal{D}_{\rm c} = \mathbb{R}^{n_{\rm c}}$, and $V(\cdot)$ is radially unbounded, then the zero solution $x(t) \equiv 0$ to $\mathcal{G}$ is globally asymptotically stable.

IV. Hybrid Control Design for Euler-Lagrange Systems

Consider the governing equations of motion of an $\hat{n}_{\rm p}$ degree-of-freedom dynamical system given by the *Euler-Lagrange* equation

$$\frac{\rm d}{{\rm d}t}\left[\frac{\partial \mathcal{L}}{\partial \dot{q}}(q(t), \dot{q}(t))\right]^{\rm T} - \left[\frac{\partial \mathcal{L}}{\partial q}(q(t), \dot{q}(t))\right]^{\rm T} = u(t),$$
$$q(0) = q_0, \quad \dot{q}(0) = \dot{q}_0, \quad (20)$$

where $t \geq 0$, $q \in \mathbb{R}^{\hat{n}_{\rm p}}$ represents the generalized system positions, $\dot{q} \in \mathbb{R}^{\hat{n}_{\rm p}}$ represents the generalized system velocities, $\mathcal{L} : \mathbb{R}^{\hat{n}_{\rm p}} \times \mathbb{R}^{\hat{n}_{\rm p}} \to \mathbb{R}$ denotes the system Lagrangian given by $\mathcal{L}(q, \dot{q}) = T(q, \dot{q}) - U(q)$, where $T : \mathbb{R}^{\hat{n}_{\rm p}} \times \mathbb{R}^{\hat{n}_{\rm p}} \to \mathbb{R}$ is the system kinetic energy and $U : \mathbb{R}^{\hat{n}_{\rm p}} \to \mathbb{R}$ is the system potential energy, and $u \in \mathbb{R}^{\hat{n}_{\rm p}}$ is the vector of generalized control forces acting on the system. Furthermore, let $\mathcal{H} : \mathbb{R}^{\hat{n}_{\rm p}} \times \mathbb{R}^{\hat{n}_{\rm p}} \to \mathbb{R}$ denote the *Legendre transformation* of the Lagrangian function $\mathcal{L}(q, \dot{q})$ with respect to the generalized velocity $\dot{q}$ defined by $\mathcal{H}(q, p) \triangleq \dot{q}^{\rm T} p - \mathcal{L}(q, \dot{q})$, where p denotes the vector of generalized momenta given by

$$p(q, \dot{q}) = \left[\frac{\partial \mathcal{L}}{\partial \dot{q}}(q, \dot{q})\right]^{\rm T}, \quad (21)$$

where the map from the generalized velocities $\dot{q}$ to the generalized momenta p is assumed to be *bijective* (i.e., one-to-one and onto).

Next, we present a hybrid feedback control framework for Euler-Lagrange dynamical systems. Consider the Lagrangian system (20) with outputs

$$y = \begin{bmatrix} h_1(q) \\ h_2(\dot{q}) \end{bmatrix} = \begin{bmatrix} h_1(q) \\ h_2\left(\frac{\partial \mathcal{H}}{\partial p}(q, p)\right) \end{bmatrix}, \quad (22)$$

where $h_1 : \mathbb{R}^{\hat{n}_{\rm p}} \to \mathbb{R}^{l_1}$ and $h_2 : \mathbb{R}^{\hat{n}_{\rm p}} \to \mathbb{R}^{l-l_1}$ are continuously differentiable, $h_1(0) = 0$, $h_2(0) = 0$, and $h_1(q) \not\equiv 0$. We assume that the system kinetic energy is such that $T(q, \dot{q}) = \frac{1}{2}\dot{q}^{\rm T}[\frac{\partial T}{\partial \dot{q}}(q, \dot{q})]^{\rm T}$, $T(q, 0) = 0$, and $T(q, \dot{q}) > 0$, $\dot{q} \neq 0$, $\dot{q} \in \mathbb{R}^{\hat{n}_{\rm p}}$. We also assume that the system potential energy $U(\cdot)$ is such that $U(0) = 0$ and $U(q) > 0$, $q \neq 0$, $q \in \mathcal{D}_{\rm q} \subseteq \mathbb{R}^{\hat{n}_{\rm p}}$, which implies that $\mathcal{H}(q, p) = T(q, \dot{q}) + U(q) > 0$, $(q, \dot{q}) \neq 0$, $(q, \dot{q}) \in \mathcal{D}_{\rm q} \times \mathbb{R}^{\hat{n}_{\rm p}}$.

Next, consider the energy-based hybrid controller

$$\frac{\rm d}{{\rm d}t}\left[\frac{\partial \mathcal{L}_{\rm c}}{\partial \dot{q}_{\rm c}}(q_{\rm c}(t), \dot{q}_{\rm c}(t), y_q(t))\right]^{\rm T}$$
$$- \left[\frac{\partial \mathcal{L}_{\rm c}}{\partial q_{\rm c}}(q_{\rm c}(t), \dot{q}_{\rm c}(t), y_q(t))\right]^{\rm T} = 0, \quad q_{\rm c}(0) = q_{\rm c0},$$
$$\dot{q}_{\rm c}(0) = \dot{q}_{\rm c0}, \quad (q_{\rm c}(t), \dot{q}_{\rm c}(t), y(t)) \notin \mathcal{Z}_{\rm c}, \quad (23)$$

$$\begin{bmatrix} \Delta q_{\rm c}(t) \\ \Delta \dot{q}_{\rm c}(t) \end{bmatrix} = \begin{bmatrix} \eta(y_q(t)) - q_{\rm c}(t) \\ -\dot{q}_{\rm c}(t) \end{bmatrix},$$
$$(q_{\rm c}(t), \dot{q}_{\rm c}(t), y(t)) \in \mathcal{Z}_{\rm c}, \quad (24)$$

$$u(t) = \left[\frac{\partial \mathcal{L}_{\rm c}}{\partial q}(q_{\rm c}(t), \dot{q}_{\rm c}(t), y_q(t))\right]^{\rm T}, \quad (25)$$

where $t \geq 0$, $q_{\rm c} \in \mathbb{R}^{\hat{n}_{\rm c}}$ represents virtual controller positions, $\dot{q}_{\rm c} \in \mathbb{R}^{\hat{n}_{\rm c}}$ represents virtual controller velocities, $y_q \triangleq h_1(q)$, $\mathcal{L}_{\rm c} : \mathbb{R}^{\hat{n}_{\rm c}} \times \mathbb{R}^{\hat{n}_{\rm c}} \times \mathbb{R}^{l_1} \to \mathbb{R}$ denotes the controller Lagrangian given by $\mathcal{L}_{\rm c}(q_{\rm c}, \dot{q}_{\rm c}, y_q) \triangleq T_{\rm c}(q_{\rm c}, \dot{q}_{\rm c}) - U_{\rm c}(q_{\rm c}, y_q)$, where $T_{\rm c} : \mathbb{R}^{\hat{n}_{\rm c}} \times \mathbb{R}^{\hat{n}_{\rm c}} \to \mathbb{R}$ is the controller kinetic energy, $U_{\rm c} : \mathbb{R}^{\hat{n}_{\rm c}} \times \mathbb{R}^{l_1} \to \mathbb{R}$ is the controller potential energy, $\eta(\cdot)$ is a continuously differentiable function such that $\eta(0) = 0$, $\mathcal{Z}_{\rm c} \subset \mathbb{R}^{\hat{n}_{\rm c}} \times \mathbb{R}^{\hat{n}_{\rm c}} \times \mathbb{R}^l$ is the resetting set, $\Delta q_{\rm c}(t) \triangleq q_{\rm c}(t^+) - q_{\rm c}(t)$, and $\Delta \dot{q}_{\rm c}(t) \triangleq \dot{q}_{\rm c}(t^+) - \dot{q}_{\rm c}(t)$. We assume that the controller kinetic energy $T_{\rm c}(q_{\rm c}, \dot{q}_{\rm c})$ is such that $T_{\rm c}(q_{\rm c}, \dot{q}_{\rm c}) = \frac{1}{2}\dot{q}_{\rm c}^{\rm T}[\frac{\partial T_{\rm c}}{\partial \dot{q}_{\rm c}}(q_{\rm c}, \dot{q}_{\rm c})]^{\rm T}$, with $T_{\rm c}(q_{\rm c}, 0) = 0$ and $T_{\rm c}(q_{\rm c}, \dot{q}_{\rm c}) > 0$, $\dot{q}_{\rm c} \neq 0$, $\dot{q}_{\rm c} \in \mathbb{R}^{\hat{n}_{\rm c}}$. Furthermore, we assume that $U_{\rm c}(\eta(y_q), y_q) = 0$ and $U_{\rm c}(q_{\rm c}, y_q) > 0$ for $q_{\rm c} \neq \eta(y_q)$, $q_{\rm c} \in \mathcal{D}_{{\rm q}_{\rm c}} \subseteq \mathbb{R}^{\hat{n}_{\rm c}}$.

Similarly, note that $V_{\rm p}(q, \dot{q}) \triangleq T(q, \dot{q}) + U(q)$ is the plant energy and $V_{\rm c}(q_{\rm c}, \dot{q}_{\rm c}, y_q) \triangleq T_{\rm c}(q_{\rm c}, \dot{q}_{\rm c}) + U_{\rm c}(q_{\rm c}, y_q)$ is the controller emulated energy. Finally, $V(q, \dot{q}, q_{\rm c}, \dot{q}_{\rm c}) \triangleq V_{\rm p}(q, \dot{q}) + V_{\rm c}(q_{\rm c}, \dot{q}_{\rm c}, y_q)$, is the total energy of the closed-loop system. It is important to note that the Lagrangian dynamical system (20) is *not* lossless with outputs y_q or y. Next, we study the behavior of the total energy function $V(q, \dot{q}, q_{\rm c}, \dot{q}_{\rm c})$ along the trajectories of the closed-loop system dynamics. For the closed-loop system, we define our resetting set as

$$\mathcal{Z} \triangleq \{(q, \dot{q}, q_{\rm c}, \dot{q}_{\rm c}) : (q_{\rm c}, \dot{q}_{\rm c}, y) \in \mathcal{Z}_{\rm c}\}. \quad (26)$$

To obtain an expression for $\frac{\rm d}{{\rm d}t}V_{\rm c}(q_{\rm c}, \dot{q}_{\rm c}, y_q)$ when $(q, \dot{q}, q_{\rm c}, \dot{q}_{\rm c}) \notin \mathcal{Z}$, define the controller Hamiltonian by

$$\mathcal{H}_{\rm c}(q_{\rm c}, \dot{q}_{\rm c}, p_{\rm c}, y_q) \triangleq \dot{q}_{\rm c}^{\rm T} p_{\rm c} - \mathcal{L}_{\rm c}(q_{\rm c}, \dot{q}_{\rm c}, y_q), \quad (27)$$

where the virtual controller momentum $p_{\rm c}$ is given by $p_{\rm c}(q_{\rm c}, \dot{q}_{\rm c}, y_q) = \left[\frac{\partial \mathcal{L}_{\rm c}}{\partial \dot{q}_{\rm c}}(q_{\rm c}, \dot{q}_{\rm c}, y_q)\right]^{\rm T}$.

Next, note that the controller (23) and (25) can be written in Hamiltonian form. Specifically, it follows from (23) and (27) that

$$\dot{p}_{\rm c}(t) = -\left[\frac{\partial \mathcal{H}_{\rm c}}{\partial q_{\rm c}}(q_{\rm c}(t), \dot{q}_{\rm c}(t), p_{\rm c}(t), y_q(t))\right]^{\rm T},$$
$$(q(t), \dot{q}(t), q_{\rm c}(t), \dot{q}_{\rm c}(t)) \notin \mathcal{Z}, \quad (28)$$
$$\dot{q}_{\rm c}(t) = \left[\frac{\partial \mathcal{H}_{\rm c}}{\partial p_{\rm c}}(q_{\rm c}(t), \dot{q}_{\rm c}(t), p_{\rm c}(t), y_q(t))\right]^{\rm T},$$

$$(q(t),\dot{q}(t),q_{\rm c}(t),\dot{q}_{\rm c}(t))\notin\mathcal{Z}, \quad (29)$$

$$u(t) = -\left[\frac{\partial\mathcal{H}_{\rm c}}{\partial q}(q_{\rm c}(t),\dot{q}_{\rm c}(t),p_{\rm c}(t),y_q(t))\right]^{\rm T}, \quad (30)$$

where $\mathcal{H}_{\rm c}(q_{\rm c},\dot{q}_{\rm c},p_{\rm c},y_q) = T_{\rm c}(q_{\rm c},\dot{q}_{\rm c}) + U_{\rm c}(q_{\rm c},y_q)$. Now, it follows from (23) and the structure of $T_{\rm c}(q_{\rm c},\dot{q}_{\rm c})$ that, for $t\in(t_k,t_{k+1}]$,

$$\begin{aligned}&\frac{\rm d}{{\rm d}t}V(q(t),\dot{q}(t),q_{\rm c}(t),\dot{q}_{\rm c}(t))\\ &\quad= u(t)^{\rm T}\dot{q}(t) - \frac{\partial\mathcal{L}_{\rm c}}{\partial q}(q_{\rm c}(t),\dot{q}_{\rm c}(t),y_q(t))\dot{q}(t)\\ &\quad= 0,\quad (q(t),\dot{q}(t),q_{\rm c}(t),\dot{q}_{\rm c}(t))\notin\mathcal{Z},\quad t_k<t\le t_{k+1},\end{aligned} \quad (31)$$

which implies that the total energy of the closed-loop system between resetting events is conserved.

The total energy difference across resetting events is given by

$$\begin{aligned}&\Delta V(q(t_k),\dot{q}(t_k),q_{\rm c}(t_k),\dot{q}_{\rm c}(t_k))\\ &\quad= T_{\rm c}(q_{\rm c}(t_k^+),\dot{q}_{\rm c}(t_k^+)) + U_{\rm c}(q_{\rm c}(t_k^+),y_q(t_k))\\ &\qquad -V_{\rm c}(q_{\rm c}(t_k),\dot{q}_{\rm c}(t_k),y_q(t_k))\\ &\quad= -V_{\rm c}(q_{\rm c}(t_k),\dot{q}_{\rm c}(t_k),y_q(t_k)),\\ &(q(t_k),\dot{q}(t_k),q_{\rm c}(t_k),\dot{q}_{\rm c}(t_k))\in\mathcal{Z},\quad k\in\overline{\mathbb{Z}}_+,\end{aligned} \quad (32)$$

which implies that the resetting law (24) ensures the total energy decrease across resetting events by an amount equal to the accumulated emulated energy.

Here, we concentrate on an energy dissipating state-dependent resetting controller that affects a one-way energy transfer between the plant and the controller. Specifically, consider the closed-loop system (20), (22)–(25), where $\mathcal{Z}$ is defined by

$$\mathcal{Z} \triangleq \left\{(q,\dot{q},q_{\rm c},\dot{q}_{\rm c}) : \frac{\rm d}{{\rm d}t}V_{\rm c}(q_{\rm c},\dot{q}_{\rm c},y_q) = 0 \text{ and } V_{\rm c}(q_{\rm c},\dot{q}_{\rm c},y_q) > 0\right\}. \quad (33)$$

The next theorem gives sufficient conditions for stabilization of Euler-Lagrange dynamical systems using state-dependent hybrid controllers. For this result define the closed-loop system states $x \triangleq [q^{\rm T},\dot{q}^{\rm T},q_{\rm c}^{\rm T},\dot{q}_{\rm c}^{\rm T}]^{\rm T}$.

Theorem 4.1: Consider the closed-loop dynamical system $\mathcal{G}$ given by (20), (22)–(25), with the resetting set $\mathcal{Z}$ given by (33). Assume that $\mathcal{D}_{\rm ci}\subset\mathcal{D}_{\rm q}\times\mathbb{R}^{\hat{n}_{\rm p}}\times\mathcal{D}_{\rm q_c}\times\mathbb{R}^{\hat{n}_{\rm c}}$ is a compact positively invariant set with respect to $\mathcal{G}$ such that $0\in\overset{\circ}{\mathcal{D}}_{\rm ci}$. Furthermore, assume that the transversality condition (9) holds with $\mathcal{X}(x) = \frac{\rm d}{{\rm d}t}V_{\rm c}(q_{\rm c},\dot{q}_{\rm c},y_q)$. Then the zero solution $x(t)\equiv 0$ to $\mathcal{G}$ is asymptotically stable. In addition, the total energy function $V(x)$ of $\mathcal{G}$ is strictly decreasing across resetting events. Finally, if $\mathcal{D}_{\rm q} = \mathbb{R}^{\hat{n}_{\rm p}}$, $\mathcal{D}_{\rm q_c} = \mathbb{R}^{\hat{n}_{\rm c}}$, and the total energy function $V(x)$ is radially unbounded, then the zero solution $x(t)\equiv 0$ to $\mathcal{G}$ is globally asymptotically stable.

V. Thermodynamic Stabilization of Euler-Lagrange Systems

In this section, we use the recently developed notion of system thermodynamics [7] to develop thermodynamically consistent hybrid controllers for Euler-Lagrange systems. Specifically, since our energy-based hybrid controller architecture involves the exchange of energy with conservation laws describing transfer, accumulation, and dissipation of energy between the controller and the plant, we construct a modified hybrid controller that guarantees that the closed-loop system is consistent with basic thermodynamic principles after the first resetting event. To develop thermodynamically consistent hybrid controllers consider the closed-loop system $\mathcal{G}$ given by (20), (22)–(25), with $\mathcal{Z}$ given by

$$\mathcal{Z} \triangleq \{x\in\mathcal{D}_{\rm q}\times\mathbb{R}^{\hat{n}_{\rm p}}\times\mathcal{D}_{\rm q_c}\times\mathbb{R}^{\hat{n}_{\rm c}} : \phi(x)(V_{\rm p}(x)-V_{\rm c}(x)) = 0 \text{ and } V_{\rm c}(x)>0\}, \quad (34)$$

where $\phi(x) \triangleq \frac{\partial\mathcal{L}_{\rm c}}{\partial q}(q_{\rm c},\dot{q}_{\rm c},y_q)\dot{q}$, $V_{\rm p}(x)\triangleq V_{\rm p}(q,\dot{q})$, and $V_{\rm c}(x)\triangleq V_{\rm c}(q_{\rm c},\dot{q}_{\rm c},y_q)$. It follows from (31) that $\phi(\cdot)$ is the net energy flow from the plant to the controller, and hence, we refer to $\phi(\cdot)$ as the *net energy flow* function.

We assume that the energy flow function $\phi(x)$ is infinitely-differentiable and the transversality condition (9) holds with $\mathcal{X}(x) = \phi(x)(V_{\rm p}(x)-V_{\rm c}(x))$. To ensure a thermodynamically consistent energy flow between the plant and controller after the first resetting event, the controller resetting logic must be designed in such a way so as to satisfy three key thermodynamic axioms on the closed-loop system level. Namely, between resettings the energy flow function $\phi(\cdot)$ must satisfy the following two axioms [7]:

Axiom *i*) For the connectivity matrix $\mathcal{C}\in\mathbb{R}^{2\times 2}$ associated with the closed-loop system $\mathcal{G}$ defined by

$$\mathcal{C}_{(i,j)} \triangleq \begin{cases}0, & \text{if } \phi(x(t))\equiv 0\\ 1, & \text{otherwise}\end{cases},\quad i\ne j,\quad i,j=1,2,\quad t\ge t_1^+, \quad (35)$$

and

$$\mathcal{C}_{(i,i)} = -\mathcal{C}_{(k,i)},\quad i\ne k,\quad i,k=1,2, \quad (36)$$

$\operatorname{rank}\mathcal{C} = 1$, and for $\mathcal{C}_{(i,j)} = 1$, $i\ne j$, $\phi(x(t)) = 0$ if and only if $V_{\rm p}(x(t)) = V_{\rm c}(x(t))$, $x(t)\notin\mathcal{Z}$, $t\ge t_1^+$.

Axiom *ii*) $\phi(x(t))(V_{\rm p}(x(t))-V_{\rm c}(x(t)))\le 0$, $x(t)\notin\mathcal{Z}$, $t\ge t_1^+$.

Furthermore, across resettings the energy difference between the plant and the controller must satisfy the following axiom [8]:

Axiom *iii*) $[V_{\rm p}(x+f_{\rm d}(x)) - V_{\rm c}(x+f_{\rm d}(x))][V_{\rm p}(x) - V_{\rm c}(x)]\ge 0$, $x\in\mathcal{Z}$.

The fact that $\phi(x(t)) = 0$ if and only if $V_{\rm p}(x(t)) = V_{\rm c}(x(t))$, $x(t)\notin\mathcal{Z}$, $t\ge t_1^+$, implies that the plant and the controller are *connected*; alternatively, $\phi(x(t))\equiv 0$, $t\ge t_1^+$, implies that the plant and the controller are *disconnected*. Axiom i) implies that if the energies in the plant and the controller are equal, then energy exchange between the plant and controller is not possible unless a resetting event occurs. This statement is consistent with the *zeroth law of thermodynamics*, which postulates that temperature equality is a necessary and sufficient condition for thermal equilibrium of an isolated system. Axiom ii) implies that energy flows from a more energetic system to a less energetic system and is consistent with the *second law of thermodynamics*, which states that heat (energy) must flow in the direction of lower temperatures. Finally, Axiom iii) implies that the energy difference between the plant and the controller across resetting instants is monotonic, that is, $[V_{\rm p}(x(t_k^+)) - V_{\rm c}(x(t_k^+))][V_{\rm p}(x(t_k)) - V_{\rm c}(x(t_k))]\ge 0$ for all $V_{\rm p}(x)\ne V_{\rm c}(x)$, $x\in\mathcal{Z}$, $k\in\overline{\mathbb{Z}}_+$.

With the resetting law given by (34), it follows that the closed-loop dynamical system $\mathcal{G}$ satisfies Axioms i)–iii) for all $t \geq t_1$. To see this, note that since $\phi(x) \not\equiv 0$, the connectivity matrix $\mathcal{C}$ is given by

$$\mathcal{C} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}, \tag{37}$$

and hence, $\operatorname{rank} \mathcal{C} = 1$. The second condition in Axiom i) need not be satisfied since the case where $\phi(x) = 0$ or $V_{\rm p}(x) = V_{\rm c}(x)$ corresponds to a resetting instant. Furthermore, it follows from the definition of the resetting set (34) that Axiom ii) is satisfied for the closed-loop system for all $t \geq t_1^+$. Finally, since $V_{\rm c}(x + f_{\rm d}(x)) = 0$ and $V_{\rm p}(x+f_{\rm d}(x)) = V_{\rm p}(x)$, $x \in \mathcal{Z}$, it follows from the definition of the resetting set that

$$\begin{aligned}[V_{\rm p}(x + f_{\rm d}(x)) - V_{\rm c}(x + f_{\rm d}(x))][V_{\rm p}(x) - V_{\rm c}(x)] \\ = V_{\rm p}(x)[V_{\rm p}(x) - V_{\rm c}(x)] \geq 0, \quad x \in \mathcal{Z},\end{aligned} \tag{38}$$

and hence, Axiom *iii*) is satisfied across resettings. Hence, the closed-loop system $\mathcal{G}$ is thermodynamically consistent after the first resetting event in the sense of [7], [8].

Next, we give a hybrid definition of entropy for the closed-loop system $\mathcal{G}$ that generalizes the continuous-time and discrete-time entropy definitions established in [7], [8].

Definition 5.1: For the impulsive closed-loop system $\mathcal{G}$ given by (20), (22)–(25), a function $S : \overline{\mathbb{R}}_+^2 \to \mathbb{R}$ satisfying

$$S(E(x(T))) \geq S(E(x(t_1))) - \frac{1}{c} \sum_{k \in \mathbb{Z}_{[t_1,T)}} V_{\rm c}(x(t_k)), \quad T \geq t_1, \tag{39}$$

where $k \in \mathbb{Z}_{[t_1,T)} \triangleq \{k : t_1 \leq t_k < T\}$, $E \triangleq [V_{\rm p}, V_{\rm c}]^{\rm T}$, $c > 0$, is called the *entropy* function of $\mathcal{G}$.

The next result gives necessary and sufficient conditions for establishing the existence of an entropy function of $\mathcal{G}$ over an interval $t \in (t_k, t_{k+1}]$ involving the consecutive resetting times t_k and t_{k+1}, $k \in \overline{\mathbb{Z}}_+$.

Theorem 5.1: For the impulsive closed-loop system $\mathcal{G}$ given by (20), (22)–(25), a function $S : \overline{\mathbb{R}}_+^2 \to \mathbb{R}$ is an entropy function of $\mathcal{G}$ if and only if

$$S(E(x(\hat{t}))) \geq S(E(x(t))), \quad t_k < t \leq \hat{t} \leq t_{k+1}, \tag{40}$$

$$S(E(x(t_k) + f_{\rm d}(x(t_k)))) \geq S(E(x(t_k))) - \frac{V_{\rm c}(x(t_k))}{c}, \quad k \in \overline{\mathbb{Z}}_+. \tag{41}$$

The next theorem establishes the existence of an entropy function for the closed-loop system $\mathcal{G}$.

Theorem 5.2: Consider the impulsive closed-loop system $\mathcal{G}$ given by (20), (22)–(25), with $\mathcal{Z}$ given by (34). Then the function $S : \overline{\mathbb{R}}_+^2 \to \mathbb{R}$ given by

$$S(E) = \log_e(c + V_{\rm p}) + \log_e(c + V_{\rm c}) - 2\log_e c, \quad E \in \overline{\mathbb{R}}_+^2, \tag{42}$$

where $c > 0$, is a continuously differentiable entropy function of $\mathcal{G}$. In addition,

$$\dot{S}(E(x(t))) > 0, \quad x(t) \notin \mathcal{Z}, \quad t_k < t \leq t_{k+1}, \tag{43}$$

$$-\frac{V_{\rm c}(x(t_k))}{c} < \Delta S(E(x(t_k))) < -\frac{V_{\rm c}(x(t_k))}{c + V_{\rm c}(x(t_k))}, \quad x(t_k) \in \mathcal{Z}, \quad k \in \overline{\mathbb{Z}}_+. \tag{44}$$

Note that it follows from (43) that the entropy of the closed-loop system strictly increases between resetting events after the first resetting event, which is consistent with thermodynamic principles. This is not surprising since in this case the closed-loop system is *adiabatically isolated* (i.e., the system does not exchange energy (heat) with the environment) and the total energy of the closed-loop system is conserved between resetting events. Alternatively, it follows from (44) that the entropy of the closed-loop system strictly decreases across resetting events since the total energy strictly decreases at each resetting instant, and hence, energy is not conserved across resetting events.

Using Theorem 5.2, the resetting set $\mathcal{Z}$ given by (34) can be rewritten as

$$\mathcal{Z} \triangleq \Big\{x \in \mathcal{D}_{\rm q} \times \mathbb{R}^{\hat{n}_{\rm p}} \times \mathcal{D}_{\rm q_c} \times \mathbb{R}^{\hat{n}_{\rm c}} : \frac{\rm d}{{\rm d}t} S(E(x)) = 0 \text{ and } V_{\rm c}(x) > 0\Big\}, \tag{45}$$

where $\mathcal{X}(x) \triangleq \frac{\rm d}{{\rm d}t} S(E(x))$ is a continuously differentiable function that defines the resetting set as its zero level set. The resetting set (34) or, equivalently, (45) is motivated by thermodynamic principles and guarantees that the energy of the closed-loop system is always flowing from regions of higher to lower energies after the first resetting event, which is in accordance with the second law of thermodynamics. As shown in Theorem 5.2, this guarantees the existence of entropy function $S(E)$ for the closed-loop system that satisfies the Clausius-type inequality (43) between resetting events. If $\phi(x) = 0$ or $V_{\rm p}(x) = V_{\rm c}(x)$, then inequality (43) would be subverted, and hence, we reset the compensator states in order to ensure that the second law of thermodynamics is not violated.

Finally, if $\mathcal{D}_{\rm ci} \subset \mathcal{D}_{\rm q} \times \mathbb{R}^{\hat{n}_{\rm p}} \times \mathcal{D}_{\rm q_c} \times \mathbb{R}^{\hat{n}_{\rm c}}$ is a compact positively invariant set with respect to the closed-loop dynamical system $\mathcal{G}$ given by (20), (22)–(25) such that $0 \in \overset{\circ}{\mathcal{D}}_{\rm ci}$, and the transversality condition (9) holds with $\mathcal{X}(x) = \frac{\rm d}{{\rm d}t} S(E(x))$, then it follows from Theorem 4.1 that the zero solution $x(t) \equiv 0$ of the closed-loop system $\mathcal{G}$, with resetting set $\mathcal{Z}$ given by (34), is asymptotically stable. Furthermore, in this case, the hybrid controller (23) and (24), with resetting set (34), is a *thermodynamically stabilizing* compensator. Analogous thermodynamically stabilizing compensators can be constructed for lossless and port-controlled Hamiltonian dynamical systems.

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