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Formal Distributed Port-Hamiltonian Representation of Field Equations

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Abstract— **The purpose of this study is to establish a unified modeling procedure of distributed port-Hamiltonian formulations for field equations. First, higher order Stokes-Dirac structures on variational complexes of jet bundles are introduced. Next, a one-to-one correspondence between Euler-Lagrange equations and distributed port-Hamiltonian systems is presented. Finally, in the case that the Lagrangian is given, the concrete transformation procedure for distributed port-Hamiltonian systems is explained by using two examples.**

I. INTRODUCTION

Port-Hamiltonian systems have been developed as a general control model for passivity [10]. A power-conserving property of port-Hamiltonian systems is written by a Dirac structure [1], [2]. The framework has been extended to a distributed parameter system with a Stokes-Dirac structure [1]. The Stokes-Dirac structure is defined on a spatial domain and its boundary of the system by differential forms. The structure makes clear the property that the change of the interior energy is equal to the power supplied to the system through its boundary. The internal energy variables can be stabilized by a damping injection on the energy balance of the boundary. Some physical models have been represented by the port-Hamiltonian formulation [4], [5]. On the other hand, the mathematical extension for the Stokes-Dirac structure itself have been considered, especially there are some suggestive variations in the original paper [1]. In the Timoshenko beam models, Hodge-star operators have been introduced into Stokes-Dirac structure [3]. And this concept have been generalized as the higher order structure [7]. As more general representation, the constant Stokes-Dirac structure for multi variable systems was presented with a constant differential matrix and its adjoint [6]. In the latest study [8], a new axis is introduced by contact forms on a 1-jet space.

In this paper a unified modeling procedures of distributed port-Hamiltonian formulations for field equations is presented. First, a higher order Stokes-Dirac structure on variational complexes of jet bundles is introduced. This structure rewritten version of the standard higher order Stokes-Dirac structure [7] with total differential operators is introduced to represent higher order energy variables. The mathematical background is the same as the concept in [6] basically. But a more concrete calculation has been presented to show a relationship between variations on extremal points and boundary ports. Next, a one-to-one correspondence between field Euler-Lagrange equations and distributed port-Hamiltonian systems is presented. One of distributed port-Hamiltonian formulations, called field port-Lagrangian systems, is defined by changing of variables in the Lagrange density functional. A fundamental volume-form, which gives the field port-Lagrangian system, is introduced. And we show that the form is an adjoint form of Cartan fundamental 1-form. As a result, if there exists a Lagrangian of systems, the Euler-Lagrange equations which given by variational problems can be written as distributed port-Hamiltonian systems, that is, field port-Lagrangian systems. Finally, in the case that the Lagrangian is given, the concrete transformation procedure for distributed port-Hamiltonian systems is explained by using two examples. One of them is the thin film equation on two-dimensional domain. The model is one example of systems that has complex energy variables. However the field port-Lagrangian system can be given by the Lagrangian systematically. The other is the potential Boussinesq equation, which expresses a wave in shallow waters. In this example a connection between variations on extremal points and higher order Stokes-Dirac structures is presented.

The following advantages can be considered as reasons why we introduce this relationship. Firstly, this approach clarifies the relation between the Euler-Lagrange equation, which is calculated by a variational principle, and the distributed port-Hamiltonian systems related to passivity. This relation provides a unified modeling method for the systems that have an infinite dimensional freedom such as a field equation. That is, if the systems have the Lagrangian and it is known, such power ports are given systematically. Generally, it is difficult to find such passive pairs with a manual observation except simple systems. Secondly, the variations on extremal points are fixed by boundary conditions in many cases of variational problems and then the total divergence terms can be eliminated. On the other hand, in the view of the distributed port-Hamiltonian systems the conservation laws that result from the total divergence terms form the new Stokes-Dirac structure. This matter can be unified by the higher order Stokes-Dirac structure. Thirdly, from a different angle of the above, we can see that the integration by parts formula on the variational complex yields the freedom of energy variable definitions. Then it is possible to formulate more extensive system.

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II. Mathematical Preliminaries

In this section some required mathematical concepts are quoted from existing results [11], [12].Then our new results will be presented from the next section. Note that the basic concepts of distributed parameter port-Hamiltonian systems are not explained in this paper at all. The detailed definitions follow the original paper [1].

A. Jet bundle formalism

A *bundle* is a triple (M, ϖ, X) with the total manifold M, the base manifold X and the surjective submersion $\varpi : M \to X$. For each point $x \in X$, the subset $\varpi^{-1}(x) = M_x$ is called the fiber over x. A section σ of M is a map $\sigma : X \to M$ such that $\varpi \circ \sigma = \mathrm{id}_X$, where id_X denotes the identity map on X. Let σ_f be a smooth section of a bundle (M, ϖ, X) defined by $u^\alpha = f^\alpha(x)$ with coordinates (x^i, u^α) , where $x^i = (x^1, \cdots, x^m)$ are m independent variables and $u^\alpha = (u^1, \cdots, u^l)$ are l dependent variables.

The k-th order partial derivatives of f will be denoted by

$$\partial_J f(x) = \frac{\partial^k f(x)}{\partial x^{j_1} \partial x^{j_2} \cdots \partial x^{j_k}} \tag{1}$$

with $J = (j_1, \cdots, j_k)$ is a multi-index of order $k = \sharp J$. The *v-th prolongation* $u^{(v)} = f^{(v)}(x) : X \to U^{(v)}$ is defined by $u_J^\alpha = \partial_J f^\alpha(x) \subset U_k$ where $U^{(v)} := U \times U_1 \times \cdots \times U_v$. Now we introduce *$v$-th jet space* $M^{(v)} = X \times U^{(v)}$. Let $\mathscr{A}$ be a space of smooth functions $P(x, u^{(v)})$ called *differential functions*.

B. Variational complexes

A *total derivative* D_i (called formal differentials also) can be thought of as a kind of vector field on the infinite jet space.

$$D_i = \frac{\partial}{\partial x^i} + \sum_J u_{J,i} \frac{\partial}{\partial u_J} \tag{2}$$

where $u_{J,i} = \partial u_J / \partial x^i$. As such, we can allow it to act on the following vertical forms as a Lie derivative. In particular, D_i acts on the basic forms by $D_i\, du_J = d(D_i u_J) = du_{J,i}$.

The total r-forms concentrated on the horizontal variables x in $M \subset X \times U$ in that only the differentials dx^i appeared. Vertical forms is constructed by similarly concentrating on the vertical variables, which consist of the u's and all their derivatives. Specially, a *vertical k-form* is a finite sum

$$\hat{\omega} = \sum P_J^\alpha\, du_{J_1}^{\alpha_1} \wedge \cdots \wedge du_{J_k}^{\alpha_k} \tag{3}$$

in which the coefficients P_J^α are differential functions. Since only the differentials du_J^α appear in these forms, the analogue of the differential of the ordinary de Rham complex is the *vertical differential*:

$$\hat{d}\hat{\omega} = \sum \frac{\partial P_J^\alpha}{\partial u_K^\beta} du_K^\beta \wedge du_{J_1}^{\alpha_1} \wedge \cdots \wedge du_{J_k}^{\alpha_k} . \tag{4}$$

Since any given vertical form $\hat{\omega}$ can depend on only finitely many of the variables u_J^α, and hence exists on a finite jet space $M^{(v)}$, the vertical differential $\hat{d}\hat{\omega}$ is in reality the same as the de Rham differential in these variables, the remaining independent variables playing the role of parameters. Thus the vertical differential is readily seen to have the usual bilinearity, anti-derivation and closure properties of the ordinary differential.

We consider an equivalence relation on the space of vertical forms, with $[\hat{\omega}] = \hat{\omega} + \operatorname{div} \hat{\eta},\ \hat{\omega}, \hat{\eta} \in \widehat{\bigwedge}^r$. The space of equivalence classes is the space of *functional r-forms* $\bigwedge_*^r = \widehat{\bigwedge}^r / \operatorname{div}(\widehat{\bigwedge}^r)$. The natural projection from $\widehat{\bigwedge}^r$ to $\bigwedge_*^r$ is denoted by an integral sign $\int \hat{\omega}\, dx$ stands for $[\hat{\omega}]$. This definition gives the integration by parts formula.

$$\int \hat{\psi} \wedge D_i \hat{\eta}\, dx = -\int (D_i \hat{\psi}) \wedge \hat{\eta}\, dx \tag{5}$$

where $\hat{\psi} \in \widehat{\bigwedge}^r$, $\hat{\eta} \in \widehat{\bigwedge}^s$ and D_i is the total derivative.

Let $\omega = \int \hat{\omega}\, dx$ be a functional r-form corresponding to the vertical r-form $\hat{\omega}$. The variational differential of ω is the functional $(r+1)$-form corresponding to the *vertical differential* of ω:

$$\delta\omega = \int \hat{d}\hat{\omega}\, dx . \tag{6}$$

The *variational complex* is defined as follows.

Theorem 2.1 ([11]): Let $M \subset X \times U$ be vertically star-shaped. Then the variational differential determines an exact complex

$$0 \longrightarrow \textstyle\bigwedge_*^0 \xrightarrow{\delta} \bigwedge_*^1 \xrightarrow{\delta} \bigwedge_*^2 \xrightarrow{\delta} \cdots \tag{7}$$

on the spaces of functional forms on M.

C. Euler-Lagrange equations

Let $Z \subset X$ denote a connected open set with smooth boundary ∂Z. A variational problem means the problem of finding the extremals of a functional, is referred to as the Lagrangian

$$\mathscr{L} = \int_Z L(x, u^{(v)})\, dx \tag{8}$$

over some space of functions $u = f(x),\ x \in Z$. Then its variational differential is the functional 1-form

$$\delta\mathscr{L} = \int \hat{d}L\, dx = \int \big\{E(L) \cdot du\big\} dx \tag{9}$$

where $E = (E_1, \cdots, E_l)$ is the *Euler operator* such that

$$E_\alpha = \sum_J (-D)_J \frac{\partial}{\partial u_J^\alpha} \tag{10}$$

and $E(L) \equiv 0$ yields *Euler-Lagrange equations*.

If we interpret the differentials du^α as infinitesimal variations of u^α with corresponding variations $du_J^\alpha = D_J du^\alpha$ in the derivatives (see *Remark 3.1*), the above computation (9) is the same as the traditional determination of the Euler-Lagrange equations from the definition of the variational derivatives. This interpretation leads to a natural correspondence between the standard Stokes-Dirac structure with differential forms and the definition on a variational complex (see *Remark 3.2*).

III. Stokes-Dirac Structures on Variational Complexes

At the start, a higher order Stokes-Dirac structure on variational complexes of jet bundles will be presented with the definitions mentioned above.

Let Z be an n-dimensional smooth manifold with a smooth $(n-1)$-dimensional boundary ∂Z. Here the space $\bigwedge_*^1$ is the center of topic because it is related to the calculus of variation yielding Euler-Lagrange equations.

From (9), the integrand of functionals $\hat{d}L$ in $\bigwedge_*^1$ can be defined by a product of differential functions $\partial L/\partial u_J^\alpha \in \mathscr{A}$ and differentials $du_J^\alpha \in \widehat{\bigwedge}^1$. Then let us consider the Stokes-Dirac structure that is written by elements of the two spaces.

Definition 3.1: Let $\mathcal{F}$ and $\mathcal{E}$ be linear spaces as follows:

$$\begin{aligned} \mathcal{F} &:= \widehat{\bigwedge}^1(Z) \times \widehat{\bigwedge}^1(Z) \times \widehat{\bigwedge}^0(\partial Z)\,, \\ \mathcal{E} &:= \widehat{\bigwedge}^0(Z) \times \widehat{\bigwedge}^0(Z) \times \widehat{\bigwedge}^0(\partial Z)\,. \end{aligned} \tag{11}$$

A pairing between flows f and efforts e is defined by

$$\begin{aligned} &\langle\!\langle\, (f^1, e^1), (f^2, e^2)\, \rangle\!\rangle \\ &:= \int_Z \left(e_p^1 \wedge f_p^2 + e_q^1 \wedge f_q^2 + e_p^2 \wedge f_p^1 + e_q^2 \wedge f_q^1\right) dx \\ &\quad + \int_{\partial Z} \left(e_b^1 \wedge f_b^2 + e_b^2 \wedge f_b^1\right) dx \end{aligned} \tag{12}$$

where $f = (f_p, f_q, f_b) \in \mathcal{F}$ and $e = (e_p, e_q, e_b) \in \mathcal{E}$.

Now let us consider the higher order Stokes-Dirac structure on variational complexes from the above definitions.

Theorem 3.1: The linear subspace

$$\begin{aligned} \mathbb{D} = \Big\{ &(f_p, f_q, e_p, e_q, f_b, e_b) \in \mathcal{F} \times \mathcal{E} \,| \\ &\begin{bmatrix} f_p \\ f_q \end{bmatrix} = \begin{bmatrix} 0 & -(-D_x)^n \\ D_x^n & 0 \end{bmatrix} \begin{bmatrix} e_p \\ e_q \end{bmatrix}, \\ &f_b = \begin{bmatrix} D_x^{n-1} e_p|_{\partial Z} \\ \vdots \\ D_x^{n-i} e_p|_{\partial Z} \\ \vdots \\ e_p|_{\partial Z} \end{bmatrix}, \; e_b = \begin{bmatrix} -e_q|_{\partial Z} \\ \vdots \\ (-1)^i D_x^{i-1} e_q|_{\partial Z} \\ \vdots \\ (-1)^n D_x^{n-1} e_q|_{\partial Z} \end{bmatrix} \Big\} \end{aligned} \tag{13}$$

satisfies Dirac structure with the pairing (12), where D_x is a total differential operator concerning with spatial variables.

Proof: See appendix. ∎

Theorem 3.2: The linear subspace

$$\begin{aligned} \bar{\mathbb{D}} = \Big\{ &(f_p, f_q, e_p, e_q, f_b, e_b) \in \mathcal{F} \times \mathcal{E} \,| \\ &\begin{bmatrix} f_p \\ f_q \end{bmatrix} = \begin{bmatrix} 0 & \mp I \\ \pm I & 0 \end{bmatrix} \begin{bmatrix} e_p \\ e_q \end{bmatrix}, \; f_b = 0, \; e_b = 0 \Big\} \end{aligned} \tag{14}$$

satisfies Dirac structure with the pairing (12), where $I = \mathrm{id}_Z$ is an identity operator on the manifold.

Proof: The structure corresponds with the special Stokes-Dirac structure with Hodge-star operator '$*$' [7], [3]. Remainder omitted. ∎

Remark 3.1: If we consider the differentials du^α as infinitesimal variations in the u^α, the coefficient of du^α can be regard as a space of functions $\mathscr{A}$ to define flows. And efforts are equal to functions $\mathscr{A}$. Then the definitions of vertical forms and of differential functions are compatible. Now (11) are interpreted as two spaces of differential functions $\mathscr{A}$:

$$\hat{\mathcal{F}} = \mathscr{A} \times \mathscr{A} \times \mathscr{A}\,, \quad \hat{\mathcal{E}} = \mathscr{A} \times \mathscr{A} \times \mathscr{A}\,. \tag{15}$$

And this interpretation agrees with the original definition [1] of energy variables: $e \wedge \alpha := \partial \mathscr{H}/\partial \alpha \wedge \delta\alpha|_{\delta\alpha \to \alpha}$.

Remark 3.2: The energy variables of the standard Stokes-Dirac structure are defined by differential forms: efforts $e \in \Omega^{n-p}(Z)$ and flows $f \in \Omega^p(Z)$ shapes the volume form $d\sigma \in \Omega^n(Z)$. If we adopt the interpretation (15), then there differential functions $\mathscr{A}$ are regarded as coefficients of differential forms $dx^1 \wedge \cdots \wedge dx^p$ and $dx^{n-p} \wedge \cdots \wedge dx^n$ respectively. This is the compatible definition. Note that total differential operators D_x projected to base manifolds X is equivalent to exterior differential operators d.

IV. Relationship of Distributed Port-Hamiltonian Systems to Field Equations

In this section, the main result, that is, the relationship of distributed port-Hamiltonian systems to field equations will be presented by means of the previous preparations. Please note that the subjects of the classical field theory are quoted from [14].

Euler-Lagrange equations used as practical physical models are defined by variational problems of action integrals from the Hamilton's principle. If a Lagrange function $\mathcal{L}$ is regular, that is, a Hessian is not zero: $\det A_{ij} \neq 0, A_{ij} = \partial^2 \mathcal{L}/(\partial \dot{q}^i \partial \dot{q}^j)$, then the relation of Hamiltonian systems to Lagrange equations is one-to-one. On the other hand, many physical objects distributed on a space continuously (e.g. a gravity field, an electromagnetic field or a Yang-Mills field, etc.) have an infinite degree of freedom. Then these systems are described by the field theory.

Now we introduce the following concrete definition. Field quantities $\phi^a(x)$ are used instead of dynamic variables of particles $q(t)$, where 'a' is an index of independent components of the field, 'x' is a set of special coordinates $x^i, i = 1, \cdots, n$ and a time coordinate $x^0 = t$. We simplify the notation and write as $\phi^a_{,\mu} = \partial\phi^a/\partial x^\mu = \partial_\mu \phi^a$, $\mu = 0, \cdots, n$. Variables in a velocity state space are ϕ^a, $\phi^a_{,0}$ and variables in a phase space are ϕ^a, $\pi_a = \partial\mathcal{L}/\partial\phi^a_{,0}$.

A. Field equations via variational calculus

First of all, we refer to a calculation method of Euler-Lagrange equations extended to the field theory [14].

An action integral of Lagrange density functions $\mathcal{L}$ is given as a functional

$$\mathscr{L} = \int \mathcal{L}(\phi^a, \phi^a_{,\mu})\, dx \tag{16}$$

where $d\sigma = dx^1 \wedge \cdots \wedge dx^m, dx = dx^0 \wedge d\sigma$. The variational derivation of (16) yields

$$\delta\mathscr{L} = \int \left[\left(\frac{\partial \mathcal{L}}{\partial \phi^a} - \frac{\partial}{\partial x^\mu} \frac{\partial \mathcal{L}}{\partial \phi^a_{,\mu}} \right) d\phi^a + \frac{\partial}{\partial x^\mu} \left(\frac{\partial \mathcal{L}}{\partial \phi^a_{,\mu}} d\phi^a \right) \right] dx \tag{17}$$

where $d\phi^a_{,\mu} = (\partial/\partial x^\mu) d\phi^a$. The stationary condition of the first term of (17) results Euler-Lagrange equations. The second term of (17) relates to Noether's theorem of field situation. If a Lagrangian is invariant, such a quantity, that is, Noether current is conserved.

B. *Field port-Lagrangian systems*

From the above, we present the relationship of distributed port-Hamiltonian systems to field equations.

Lemma 4.1: The linear subspace $\mathbb{D}$ such that

$$\mathbb{D} = \big\{ (f_p, f_q, f_r, e_p, e_q, e_r, f_b, e_b) \in \mathcal{F} \times \mathcal{E} | \begin{bmatrix} f_p \\ f_r \\ f_q \end{bmatrix} = \begin{bmatrix} 0 & -I & D_i \\ I & 0 & 0 \\ D_i & 0 & 0 \end{bmatrix} \begin{bmatrix} e_p \\ e_r \\ e_q \end{bmatrix}, \begin{bmatrix} f_b \\ e_b \end{bmatrix} = \begin{bmatrix} e_p|_{\partial Z} \\ -e_q|_{\partial Z} \end{bmatrix} \big\} \tag{18}$$

satisfies the Dirac structure where D_i is a total differential operator as regards spatial variables, $I = \mathrm{id}_Z$ is an identity operator.

Proof: First, it is easy to see the structure on 'I' in the small matrix does not affect the boundary energy structure by *Theorem 3.2*. Then we see that (18) satisfies a Stokes-Dirac structure through *Theorem 3.1*. ■

Theorem 4.2: In (18) let us consider the definitions:

$$f = (f_p, f_r, f_q) = \left(-\partial_0 \frac{\partial \mathcal{L}}{\partial \phi^a_{,0}},\ -\partial_0 \phi^a,\ -\partial_0 \phi^a_{,i} \right), \tag{19}$$

$$e = (e_p, e_r, e_q) = \left(-\phi^a_{,0},\ \frac{\partial \mathcal{L}}{\partial \phi^a},\ \frac{\partial \mathcal{L}}{\partial \phi^a_{,i}} \right). \tag{20}$$

Then the first row of (18) corresponds to Euler-Lagrange equations. The second and the third row of (18) define an identity relation of higher order variables.

Proof: Let us consider the energy density function:

$$\mathcal{E}(\phi^a, \phi^a_{,\mu}) = \phi^a_{,0} \frac{\partial \mathcal{L}}{\partial \phi^a_{,0}} - \mathcal{L}. \tag{21}$$

The energy functional $\mathscr{E}$ is defined by the integral of (21). The variation of $\mathscr{E}$ is obtained as follows.

$$\begin{aligned} \delta\mathscr{E} &= \int \hat{d} \left(\phi^a_{,0} \frac{\partial \mathcal{L}}{\partial \phi^a_{,0}} - \mathcal{L} \right) dx \\ &= \int \left(\phi^a_{,0}\, d \frac{\partial \mathcal{L}}{\partial \phi^a_{,0}} - \frac{\partial \mathcal{L}}{\partial \phi^a} d\phi^a - \frac{\partial \mathcal{L}}{\partial \phi^a_{,i}} d\phi^a_{,i} \right) dx \\ &= \int \left(e_p \cdot \alpha_p + e_r \cdot \alpha_r + e_q \cdot \alpha_q \right) dx \\ &\equiv \int e_p\, dx^{\bar{p}} \wedge \alpha_p\, dx^p + e_r\, dx^{\bar{r}} \wedge \alpha_r\, dx^r \\ &\qquad + e_q\, dx^{\bar{q}} \wedge \alpha_q\, dx^q \\ &= \int \sum_{i=p,r,q} e_i\, dx^{\bar{i}} \wedge *(e_i\, dx^{\bar{i}}), \end{aligned} \tag{22}$$

where $f_\bullet\, dx^\bullet = -\partial\alpha_\bullet/\partial t\, dx^\bullet$ and $dx^{\bar{i}}$ is a $(m-i)$-form for any i-form dx^i. Actually, $r = \bar{p}$, $\bar{r} = p$. By *Remark 3.1*, vertical 1-forms $d\phi^a$ on a variational complex are identified as a differential function ϕ^a. Then the direct calculation with the energy variables (19)–(20) leads to a conclusion. ■

The system (18) is called *a field port-Lagrangian system*. And more the justification of *Theorem 4.2* is given as follows.

Proposition 4.3: An energy balance of (18) is equivalent to an energy balance of Euler-Lagrange equations which given by $\delta\mathscr{L} \equiv 0$ in (17).

Proof: The energy-momentum tensor [14]

$$T^{\mu\nu} := -\mathcal{L}\, \delta^\mu_\nu + \frac{\partial \mathcal{L}}{\partial \phi^a_{,\mu}} \phi^a_{,\nu} \tag{23}$$

is defined by the conservation law which given by taking an exterior differentiation of $\mathcal{L}$. This satisfies $\partial T^{\mu\nu}/\partial x^\mu = 0$. If $\mu = \nu = 0$, then (23) means Hamiltonian density. Then, an integration of $\nu = 0$ components yields a total divergence equation of energy.

$$\begin{aligned} \frac{\partial}{\partial t} \int_\sigma \mathcal{H}\, d\sigma &= -\int_\sigma \frac{\partial}{\partial x^i} \left(\frac{\partial \mathcal{L}}{\partial \phi^a_{,i}} \phi^a_{,0} \right) d\sigma \\ &= -\int_{\partial\sigma} \frac{\partial \mathcal{L}}{\partial \phi^a_{,i}} \phi^a_{,0}\, ds = \int_{\partial\sigma} e_q \wedge e_p\, ds \end{aligned} \tag{24}$$

where $ds = \sum_i dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n$ and the caret denotes omission. Indeed, this is equal to the energy balance of (18). ■

Even if field equations are known, Lagrangian density, which gives the equations, is not unique. If total divergence terms $\partial_\mu W(x) = \mathrm{div}\, W(x)$ are added to the Lagrangian density, same equations are obtained by the variational calculation. If we identify conservation laws, which yield total divergence terms as an equivalence class, then the quotient space, that is, the variational complex can be defined. Such a conservation law has been considered in relation to the distributed port-Hamiltonian system [1], [9].

Usually, when we calculate extremals, the boundary conditions of variations are fixed and then the total divergence terms are eliminated by the integration by parts formula. If let the variations be free, these terms remain in the variations and we should treat them explicitly. In the framework of field port-Lagrangian systems, the situation corresponds with an appearance of the higher order Stokes-Dirac structure. The concrete example will be presented in the next section.

C. *Field equations via fundamental forms*

Let us consider the alternate method that defines equations of motion in the theory of field. The field equations can be calculated with Cartan fundamental form [15]

$$\Omega = \int \left(\pi_a d\phi^a - \mathcal{E}\, dt \right) d\zeta, \tag{25}$$

$$\mathcal{E} = \pi_a \phi^a_{,0} - \mathcal{L} \tag{26}$$

where $\zeta \in \Omega^n(Z)$ and $\pi_a = \partial\mathcal{L}/\partial\phi^a_{,0}$. Indeed, we have

$$d\Omega = \int \theta_a(x) \wedge \rho^a(x)\, d\zeta \qquad (27)$$

$$\theta_a(x) = d\pi_a - \left[\frac{\partial\mathcal{L}}{\partial\phi^a} - \frac{\partial}{\partial x^i}\left(\frac{\partial\mathcal{L}}{\partial\phi^a_{,i}}\right)\right] dt \qquad (28)$$

$$\rho^a(x) = d\phi^a - \phi^a_{,0}\, dt \qquad (29)$$

where we assume that $(\partial\mathcal{L}/\partial\phi^a_{,i})\, d\phi^a$ is zero on the boundary. If we consider $X_\pi = \partial/\partial\pi_a$ and $X_\phi = \partial/\partial\phi^a$, then

$$i_{X_\pi}(d\Omega) = 0\,, \quad i_{X_\phi}(d\Omega) = 0 \qquad (30)$$

determine Euler-Lagrange equations.

D. Fundamental form for distributed port-Hamiltonian systems

We introduce a fundamental form for distributed port-Hamiltonian systems. The following correspondence is obtained.

Theorem 4.4: Let Ω_V be a functional n-form on an $(n+1)$-dimensional manifold with a coordinate $\{x^\mu;\ \mu = 0, \cdots, n\}$ such that

$$\Omega_V = \int (-1)^{n-q} dt \wedge e_q\, dx^{\bar{q}} \wedge e_p\, dx^{\bar{p}} - \mathcal{H}\, d\sigma \qquad (31)$$

where $\mathcal{H} \in \mathscr{A}$ is a differential function, $dx^{\bar{q}} \in \Omega^{n-q}(Z)$ and $dx^{\bar{p}} \in \Omega^{n-p}(Z)$. In this case, $d\Omega_V \equiv 0$ yields a distributed port-Hamiltonian system.

Proof: Indeed, we have

$$\begin{aligned}
&(-1)^n d\Omega_V \\
&= (-1)^n \int d\left[(-1)^{(n-q)+(n-1)} e_q\, dx^{\bar{q}} \wedge e_p\, dx^{\bar{p}} \wedge dt - \mathcal{H}\, d\sigma\right] \\
&= \int \Big[-(-1)^r e_p\, dx^{\bar{p}} \wedge d(e_q\, dx^{\bar{q}}) - e_q\, dx^{\bar{q}} \wedge d(e_p\, dx^{\bar{p}}) \\
&\quad - \frac{\partial\mathscr{H}}{\partial\alpha_q} dx^{\bar{q}} \wedge \frac{\partial\alpha_q}{\partial t} dx^q - \frac{\partial\mathscr{H}}{\partial\alpha_p} dx^{\bar{p}} \wedge \frac{\partial\alpha_p}{\partial t} dx^p\Big] dt
\end{aligned} \qquad (32)$$

where $r = pq + 1$, $dx^q \in \Omega^q(Z)$, $dx^p \in \Omega^p(Z)$ and the following relation is used.

$$\begin{aligned}
&\int d\mathcal{H}\, d\sigma \\
&= \int \Big[\left(\frac{\partial\mathcal{H}}{\partial\phi^a} - \frac{\partial}{\partial x^i}\frac{\partial\mathcal{H}}{\partial\phi^a_{,i}}\right)\frac{\partial\phi^a}{\partial t} + \left(\frac{\partial\mathcal{H}}{\partial\pi_a} - \frac{\partial}{\partial x^i}\frac{\partial\mathcal{H}}{\partial\pi_{a,i}}\right)\frac{\partial\pi_a}{\partial t} \\
&\quad + \frac{\partial}{\partial x^i}\left(\frac{\partial\mathcal{H}}{\partial\phi^a_{,i}}\frac{\partial\phi^a}{\partial t} + \frac{\partial\mathcal{H}}{\partial\pi_{a,i}}\frac{\partial\pi_a}{\partial t}\right)\Big] dt \wedge d\sigma \\
&\equiv \int \left[\frac{\partial\mathscr{H}}{\partial\phi^a}\frac{\partial\phi^a}{\partial t} + \frac{\partial\mathscr{H}}{\partial\pi_a}\frac{\partial\pi_a}{\partial t}\right] dt \wedge d\sigma \\
&= \int (-1)^n \left[\frac{\partial\mathscr{H}}{\partial\alpha_q} dx^{\bar{q}} \wedge \frac{\partial\alpha_q}{\partial t} dx^q + \frac{\partial\mathscr{H}}{\partial\alpha_p} dx^{\bar{p}} \wedge \frac{\partial\alpha_p}{\partial t} dx^p\right] dt\,.
\end{aligned} \qquad (33)$$

Here we used the relation that the third term in the first line of (33) vanish at infinity. The second line of (33) corresponds with the original definition of [1] as a variational derivative [11, p.245]. The relation of *Remark 3.1* is used in the last line of (33). We redefined the differential functions as a differential form: $\partial\mathscr{H}/\partial\alpha_q$ is an $(n-q)$-form, $\partial\mathscr{H}/\partial\alpha_p$ is an $(n-p)$-form, $\partial\alpha_q/\partial t$ is a q-form and $\partial\alpha_p/\partial t$ is a p-form. From the definition $e_\bullet\, dx^{\bar{\bullet}} = \partial\mathscr{H}/\partial\alpha_\bullet\, dx^{\bar{\bullet}}$, $f_\bullet\, dx^\bullet = -\partial\alpha_\bullet/\partial t\, dx^\bullet$, (32) can be calculated as follows:

$$\begin{aligned}
d\Omega_V &= (-1)^n \int \Big[e_p\, dx^{\bar{p}} \wedge \left\{f_p\, dx^p - (-1)^r d(e_q\, dx^{\bar{q}})\right\} \\
&\quad + e_q\, dx^{\bar{q}} \wedge \left\{f_q\, dx^q - d(e_p\, dx^{\bar{p}})\right\}\Big] dt \\
&= \int \Big[e_p \wedge \left\{f_p - (-1)^r\, de_q\right\} \\
&\quad + e_q \wedge \left\{f_q - de_p\right\}\Big] dt \wedge d\sigma\,.
\end{aligned} \qquad (34)$$

From the above, if we consider $X_q = \partial/\partial e_q$ and $X_p = \partial/\partial e_p$ as duals of both X_π and X_ϕ, then we have

$$i_{X_q}(d\Omega_V) = 0\,, \quad i_{X_p}(d\Omega_V) = 0\,. \qquad (35)$$

Then the relations

$$f_p\, dx^p = (-1)^r d(e_q\, dx^{\bar{q}})\,, \quad f_q\, dx^q = d(e_p\, dx^{\bar{p}}) \qquad (36)$$

should be satisfied. This means the distributed port-Hamiltonian systems [1]. ∎

This result leads to the next correspondence.

Theorem 4.5: Ω_V is equivalent to Ω in the sense of fundamental forms if $\mathcal{L}$ is a hyper regular.

Proof: Let us consider Hamiltonian density $\mathcal{H}$ such that

$$\begin{aligned}
\mathcal{H}\, d\sigma &= \int_Z \sum_{k=q,p} h_k \cdot g^I_{k,J}\, dx^J_k \wedge (h_k\, dx^I_k) \\
&= \int_Z \sum_{k=q,p} e_k \wedge \alpha_k
\end{aligned} \qquad (37)$$

where $dx^I_k = dx^{i_1} \wedge \cdots \wedge dx^{i_k}$ and $dx^J_k = dx^{j_1} \wedge \cdots \wedge dx^{j_{n-k}}$ are spatial forms such that $d\sigma = dx^I_k \wedge dx^J_k$. Then we have

$$\begin{aligned}
*\Omega_V &= * \int (-1)^u\, dt \wedge e_p \wedge e_q - (-1)^{(n-p)p} \mathcal{H}\, d\bar{\sigma} \\
&= (-1)^{(n-p)p} \int h_p \cdot h_q\, dx^i - \mathcal{H}\, dt
\end{aligned} \qquad (38)$$

where $d\sigma = (-1)^{(n-p)p}\, d\bar{\sigma}$ and $u = (n-q) + (n-q)q + n - p$.

Next, we assume that the system satisfies the canonical structure, then the canonical momentum π_a of ϕ^a is defined as a function of ϕ^a, $\phi^a_{,0}$ and $\phi^a_{,i}$. If $\mathcal{L}$ is a hyper regular, then $\phi^a_{,0}$ can be solved as a function of ϕ^a, $\phi^a_{,i}$ and π_a inversely. Let $\mathcal{H}$ be Hamiltonian density function such that $\mathcal{H} \equiv \phi^a_{,0}\pi_a - \mathcal{L}$. The higher order derivations may appear in solving for $\phi^a_{,0}$. However we take account up to first order and $\mathcal{H}$ is regard as a function of ϕ^a, $\phi^a_{,i}$, π_a and $\pi_{a,i}$ (see (33)).

By multiplying (25) by the n-form

$$d\xi := (-1)^{(n-p)p} \sum_{i=0}^{n} dx^0 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n \qquad (39)$$

from the right-hand side, we have

$$\bar{\Omega}_V = *\Omega_V \wedge d\xi = \int \big(h_p \cdot h_q \, dx^i - \mathcal{H}\, dt\big) d\xi \,. \tag{40}$$

By means of comparing the energy variables (19)–(20), the dual quantities e_p, e_q of Ω_V in the velocity state space are defined as follows:

$$\alpha_p = \frac{\partial \mathcal{L}}{\partial \phi^a_{,0}}, \quad \alpha_q = \phi^a_{,i} \,. \tag{41}$$

Then, by substituting h_p, h_q of (40) with (41), we have

$$\bar{\Omega}_V = \int \left(\frac{\partial \mathcal{L}}{\partial \phi^a_{,0}} \, d\phi^a - \mathcal{E}\, dt \right) d\xi \,. \tag{42}$$

Note that now we are considering on a velocity state space, then the relation $\mathcal{H} \to \mathcal{E}$ is used in (42). Since an annihilation of $*\Omega_V$ defines system equations, $\bar{\Omega}_V$ leads to the same result for (25). ■

Let us conclude this section by giving the next theorem.

Theorem 4.6: If $\mathcal{L}$ is a hyper regular, the correspondence between distributed port-Hamiltonian systems (18) and field equations $\delta \mathscr{L} \equiv 0$ is one-to-one on the quotient space concerning Noether currents.

Proof: The mapping from field equations to distributed port-Hamiltonian systems is presented by *Theorem 4.2*, (18) and (19)–(20). In the proof of *Theorem 4.2*, the invariance of $\mathcal{E}$ holds regardless of the regularity of $\mathcal{L}$. If $\mathcal{L}$ is hyper regular, a Hamiltonian density which corresponds to the Legendre transformation is defined by *Theorem 4.5*. ■

V. Examples

In this section, two examples of modeling of field port-Lagrangian systems are presented.

First example is an equation of elastic thin films. The system has the complex energy variables. Then it is difficult to find such a pair of energy variables with modeling methods based on observation. But the calculation can be carried out by the relation above mentioned automatically. The second example is a potential Boussinesq equation which expresses behavior of waves on shallow waters [13, pp.237]. This shows the relation between higher order Stokes-Dirac structures and boundary conditions of variations.

A. An equation of elastic thin films

We consider a stationary homogeneous elastic thin film on $x = (x^1, x^2) \in Z \subset \mathbb{R}^2$. Then the equation of motion is

$$\rho\, w_{tt} - \mu \sum_{i=1}^{2} \partial_{x^i} \Big(\Phi^{-1} w_{x^i} \Big) = 0 \,, \tag{43}$$

where $\Phi = \big[1 + \sum_{i=1}^2 w_{x^i}^2\big]^{\frac{1}{2}}$ and we assume that the tension μ and the film density ρ are constant. The Lagrangian of the film on the displacement $w(x)$ is $\mathscr{L} = \mathscr{T} - \mathscr{U}$ where

$$\mathscr{T} = \frac{1}{2} \int_Z \rho\, w_t^2 \, dx \,, \quad \mathscr{U} = \int_Z \mu \, (\Phi - 1) \, dx \,. \tag{44}$$

The equation (43) can be given by the variational derivative of $\mathscr{L}$.

$$\delta \mathscr{L} = \int_Z \big(\rho\, w_t \, dw_t - \mu\, \Phi^{-1} w_{x^i} \, dw_{x^i} \big) dx \,. \tag{45}$$

Now we define the following energy variables.

$$\begin{array}{ll} f_p = -\partial_t \, \rho \, w_t \,, & e_p = -w_t \,; \\ f_{q1} = -\partial_t \, w_{x^1} \,, & e_{q1} = -\mu\, \Phi^{-1} w_{x^1} \,; \\ f_{q2} = -\partial_t \, w_{x^2} \,, & e_{q2} = -\mu\, \Phi^{-1} w_{x^2} \,. \end{array} \tag{46}$$

Then we obtain the field port-Lagrangian system of (43):

$$\begin{bmatrix} f_p \\ f_{q1} \\ f_{q2} \end{bmatrix} = \begin{bmatrix} 0 & D_{x^1} & D_{x^2} \\ D_{x^1} & 0 & 0 \\ D_{x^2} & 0 & 0 \end{bmatrix} \begin{bmatrix} e_p \\ e_{q1} \\ e_{q2} \end{bmatrix}, \begin{bmatrix} f_b \\ e_b \end{bmatrix} = \begin{bmatrix} e_p \,|_{\partial Z} \\ -(e_{q1} + e_{q2}) \,|_{\partial Z} \end{bmatrix}. \tag{47}$$

B. Potential Boussinesq equation

The potential Boussinesq equation

$$u_{xxtt} + \frac{1}{2} D_x^2 (u_{xx}^2) + u_{xxxxxx} = 0 \tag{48}$$

is obtained by the stationary condition of the action integral

$$\mathscr{L} = \int \left[\frac{1}{2} u_{xt}^2 + \frac{1}{6} u_{xx}^3 - \frac{1}{2} u_{xxx}^2 \right] dx \,. \tag{49}$$

The variational derivative of (49) results the Euler-Lagrange equation. Let us consider two type of the representation of field port-Lagrangian systems.

First, we assume that all boundary conditions of variations are fixed by 0 (that is, Casimir functions [1]). The following energy variables are defined.

$$\begin{array}{ll} f_p = \partial_t u_{xxt} \,, & e_p = -du_t \,; \\ f_{q1} = -\partial_t du \,, & e_{q1} = u_{xxxxxx} \,; \\ f_{q2} = -\partial_t du_x \,, & e_{q2} = -\frac{1}{2} D_x \big(u_{xx}^2 \big) \,. \end{array} \tag{50}$$

Then we have

$$\begin{bmatrix} f_p \\ f_{q1} \\ f_{q2} \end{bmatrix} = \begin{bmatrix} 0 & -I & D_x \\ I & 0 & 0 \\ D_x & 0 & 0 \end{bmatrix} \begin{bmatrix} e_p \\ e_{q1} \\ e_{q2} \end{bmatrix}, \quad \begin{bmatrix} f_b \\ e_b \end{bmatrix} = \begin{bmatrix} e_p |_{\partial Z} \\ -e_{q2} \big|_{\partial Z} \end{bmatrix}. \tag{51}$$

We can see that (51) is the standard form of field port-Lagrangian systems. Note that when the calculation of (51), we used the relation that the following boundary conditions are zero to eliminate the total divergence terms.

$$\begin{aligned} &\int_{\partial Z} u_{xt} \, du_t dx \,, \; \frac{1}{2} \int_{\partial Z} u_{xx}^2 \, du_x dx \,, \int_{\partial Z} u_{xxx} \, du_{xx} dx \,, \\ &- \int_{\partial Z} u_{xxxx} \, du_x dx \,, \; \int_{\partial Z} u_{xxxxx} \, du dx \end{aligned} \tag{52}$$

Next, another expression is considered without the integration by part formula. We define the energy variables

$$\begin{array}{ll} f'_p = \partial_t u_{xxt} \,, & e'_p = -du_t \,; \\ f'_{q1} = -\partial_t du_{xxx} \,, & e'_{q1} = -u_{xxx} \,; \\ f'_{q2} = -\partial_t du_x \,, & e'_{q2} = \frac{1}{2} u_{xx}^2 \end{array} \tag{53}$$

and we have the next higher order representation.

$$\begin{bmatrix} f'_p \\ f'_{q1} \\ f'_{q2} \end{bmatrix} = \begin{bmatrix} 0 & D_x^3 & -D_x^2 \\ D_x^3 & 0 & 0 \\ D_x^2 & 0 & 0 \end{bmatrix} \begin{bmatrix} e'_p \\ e'_{q1} \\ e'_{q2} \end{bmatrix},$$

$$f'_b = \begin{bmatrix} D_x^2 e'_p|_{\partial Z} \\ D_x e'_p|_{\partial Z} \\ e'_p|_{\partial Z} \\ D_x e'_p|_{\partial Z} \\ e'_p|_{\partial Z} \end{bmatrix}, \; e'_b = \begin{bmatrix} -e'_{q1}|_{\partial Z} \\ D_x e'_{q1}|_{\partial Z} \\ -D_x^2 e'_{q1}|_{\partial Z} \\ -e'_{q2}|_{\partial Z} \\ D_x e'_{q2}|_{\partial Z} \end{bmatrix}. \qquad (54)$$

This is not as the same system essentially as it looks, but there exists the next relation. If we substitute the zero boundary conditions of (52) to the boundary ports of (54), then $f_b = f'_b, e_b = e'_b$. Then we can see that (54) is the representation with free boundary conditions of variations (that is, conservation laws [1], [9]).

VI. Conclusions

This paper presented the relationship of distributed port-Hamiltonian formulations to general field equations based on the Euler-Lagrange formalism of the variational complex. This result suggests that a lot of systems calculated by variational problems can be rewritten as distributed port-Hamiltonian systems systematically.

The study of the system construction procedure in the case of an unknown Lagrangian can be considered as future works. An interesting topic may be drawn by further analysis of this formulation in terms of symmetry.

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Appendix A
Proof of *Theorem 3.1*

Lemma A.1: For any functions $\phi, \eta \in \mathscr{A}$, the following relations holds.

$$\sum_{i=1}^{n}(-1)^i D_x(D_x^{n-i}\phi \wedge D_x^{i-1}\eta) + D_x^n\phi \wedge \eta - (-1)^n \phi \wedge D_x^n \eta = 0\,. \qquad (55)$$

Proof of Theorem 3.1: The statement follows from the proof of the original [1] and [7].

First $\mathbb{D} \subset \mathbb{D}^\perp$ is showed. Indeed, if we substitute (13) for the right side of (12) with *Lemma A.1*, we see that the first term is

$$\int_Z \big[-e_p^1 \wedge (-D_x)^n e_q^2 + e_q^1 \wedge D_x^n e_p^2 - e_p^2 \wedge (-D_x)^n e_q^1 + e_q^2 \wedge D_x^n e_p^1\big]\, dx$$
$$= \int_Z -\Big[\sum_{i=1}^{n}(-1)^i D_x(D_x^{n-i} e_p^1 \wedge D_x^{i-1} e_q^2) + \sum_{i=1}^{n}(-1)^i D_x(D_x^{n-i} e_p^2 \wedge D_x^{i-1} e_q^1)\Big]\, dx\,, \qquad (56)$$

the second term is

$$\int_{\partial Z}\big(f_b^1 \wedge e_b^2 + f_b^2 \wedge e_b^1\big)\, dx \qquad (57)$$

and the sum of these terms is equal to zero. Then $\mathbb{D} \subset \mathbb{D}^\perp$.

Next, we will show $\mathbb{D}^\perp \subset \mathbb{D}$. Let us consider a condition of $(f^1, e^1) \in \mathbb{D}^\perp$ such that (12) is zero for all $(f^2, e^2) \in \mathbb{D}$. Then we consider

$$\int_Z \big[-e_p^1 \wedge (-D_x)^n e_q^2 + e_q^1 \wedge D_x^n e_p^2 + e_p^2 \wedge f_p^1 + e_q^2 \wedge f_q^1\big]\, dx + \int_{\partial Z}\Big[\sum_{i=1}^{n}\big(f_b^1 \wedge (-1)^i D_x^{i-1} e_q^2 + D_x^{n-i} e_p^2 \wedge e_b^1\big)\Big]\, dx = 0 \qquad (58)$$

Then if (58) holds for every $(f^2, e^2) \in \mathbb{D}$, then each wedge products have to be zero. Using *Lemma A.1* again, we can see that $(f^1, e^1) \in \mathbb{D}$ also. Namely, this implies that $\mathbb{D}^\perp \subset \mathbb{D}$. As a result we have $\mathbb{D} = \mathbb{D}^\perp$. ■