Proceedings of the
44th IEEE Conference on Decision and Control, and
the European Control Conference 2005
Seville, Spain, December 12-15, 2005

Dynamic Dissipativity Theory for Stability of Nonlinear Feedback Dynamical Systems

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Abstract— **A key result from dissipativity theory states that if two systems are dissipative with respect to compatible supply rates, then the feedback interconnection of the two systems is stable. In this paper, we extend this result to show that the stability result is valid even if the two systems are dynamic dissipative, where we define a system to be dynamic dissipative if the system cascaded with another (static or dynamic) operator is dissipative. The notion of dynamic dissipativity unifies and extends the classical dissipativity theory, Integral Quadratic Constraints (IQCs), and the multiplier theory method from the absolute stability theory.**

I. INTRODUCTION

Dissipativity theory plays a fundamental role in the stability analysis of feedback interconnections of dynamical systems [1–7]. For example, two of the most fundamental results concerning stability of feedback systems are the positivity and small gain theorems [8–12] which are a special case of dissipativity theory. Another fundamental problem in nonlinear systems; namely, the absolute stability problem [13], [14], is also a special case of dissipativity theory. With the exception of few results given in, for example [15], virtually all robust stability analysis results (see [16–18] and numerous references therein) are a special case of the absolute stability theory [19] and its extension; namely, the multiplier theory. In recent papers, dissipativity theory also played a fundamental role in understanding stability analysis of time-delay systems [20].

A key result from dissipativity theory [1] states that if two systems are dissipative with respect to *compatible* supply rates, then the feedback interconnection of the two systems is stable. In this paper, we extend this result to show that the stability result is valid even if the two systems are *dynamic dissipative* with respect to compatible supply rates, where we define a system to be dynamic dissipative if the system cascaded with another (static or dynamic) operator is dissipative. In the case where the cascaded operator is linear and static, the dynamic dissipativity specializes to the notion of classical dissipativity. Alternatively, in the case where the cascaded operator is linear and dynamic, dynamic dissipativity theory is the time domain analog to that of IQCs [21]. Finally the main result of this paper involving stability of feedback interconnections of dynamic dissipative systems extends the multiplier method from the absolute stability theory. The primary objective of this paper is to present a unified theory to study feedback interconnections of nonlinear dynamical systems. In this paper, we discuss two classical problems, one from absolute stability theory and the other from robust stability theory to demonstrate the utility of dynamic dissipativity. It should be emphasized that the concepts presented in this paper can be used to derive many new results (for example, in [22] we derived new sufficient conditions for stability of time-delay systems).

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This research was supported in part by the National Science Foundation under Grant ECS-0551947 and the Air Force Office of Scientific Research under Grant F49620-03-1-0178.

II. MATHEMATICAL PRELIMINARIES

In this section we present the notion of exponential dissipativity. First, we establish standard notation used throughout the paper. Specifically, let $\mathbb{R}$ and $\mathbb{C}$ denote the real and complex numbers, let $\mathbb{R}^n$ denote the set of $n\times 1$ real column vectors, let $\mathbb{R}^{m\times n}$ denote the set of $m\times n$ real matrices, and let $\mathbb{S}^n$ denote the set of $n\times n$ symmetric matrices, let $(\cdot)^{\mathrm{T}}$ and $(\cdot)^*$ denote the transpose and the complex conjugate transpose, respectively, and let I_n or I denote the $n\times n$ identity matrix. Furthermore, we write $\|\cdot\|$ for the Euclidean vector norm, $\sigma_{\max}(\cdot)$ (resp., $\sigma_{\min}(\cdot)$) for the maximum (resp., minimum) singular value, $V'(x)$ for the derivative of V at x, and $M\geq 0$ (resp., $M>0$) to denote the fact that the Hermitian matrix M is nonnegative (resp., positive) definite. Let $G(s)\sim\left[\begin{array}{c|c}A & B\\ \hline C & D\end{array}\right]$ denote a state space realization of a transfer function $G(s)$; that is, $G(s)=C(sI-A)^{-1}B+D$. The notation "$\stackrel{\min}{\sim}$" is used to denote a minimal realization. Finally, let C^0 denote the set of continuous functions and C^n denote the set of functions with n continuous derivatives.

In this paper we consider nonlinear dynamical systems $\mathcal{G}$ of the form

$$\dot{x}(t) = f(x(t))+G(x(t))u(t),\ x(t_0)=x_0,\ t\geq t_0, \quad (1)$$
$$y(t) = h(x(t))+J(x(t))u(t), \quad (2)$$

where $x\in\mathbb{R}^n$, $u\in\mathbb{R}^m$, $y\in\mathbb{R}^l$, $f:\mathbb{R}^n\to\mathbb{R}^n$, $G:\mathbb{R}^n\to\mathbb{R}^{n\times m}$, $h:\mathbb{R}^n\to\mathbb{R}^l$, and $J:\mathbb{R}^n\to\mathbb{R}^{l\times m}$. We assume that $f(\cdot)$, $G(\cdot)$, $h(\cdot)$, and $J(\cdot)$ are continuously differentiable mappings and $f(\cdot)$ has at least one equilibrium so that, without loss of generality, $f(0)=0$ and $h(0)=0$. Furthermore, for the nonlinear dynamical system $\mathcal{G}$ we assume that the required properties for the existence and uniqueness of solutions are satisfied; that is, $u(\cdot)$ satisfies sufficient regularity conditions such that the system (1) has a unique solution forward in time. For the dynamical system $\mathcal{G}$ given by (1) and (2) a function $r:\mathbb{R}^m\times\mathbb{R}^l\to\mathbb{R}$ such that $r(0,0)=0$ is called a *supply rate* [1] if it is locally integrable for all input-output pairs satisfying (1), (2); that is, for all input-output pairs $u\in\mathbb{R}^m$, $y\in\mathbb{R}^l$, satisfying (1), (2), $r(\cdot,\cdot)$ satisfies $\int_{t_1}^{t_2}|r(u(s),y(s))|\mathrm{d}s<\infty$, $t_1,t_2\geq 0$. The following definition introduces the notion of dissipativity and exponential dissipativity.

Definition 2.1 ([1], [2]): A dynamical system $\mathcal{G}$ of the form (1), (2) is *exponentially dissipative with respect to the supply rate* $r(u,y)$ if there exists a constant $\varepsilon>0$, such that the *dissipation inequality*

$$0\leq\int_{t_0}^{t}e^{\varepsilon s}r(u(s),y(s))\mathrm{d}s, \quad (3)$$

is satisfied for all $t \geq t_0$ with $x(t_0) = 0$. A dynamical system $\mathcal{G}$ of the form (1), (2) is *dissipative with respect to the supply rate* $r(u, y)$ if the dissipation inequality (3) is satisfied with $\varepsilon = 0$.

The notion of exponential dissipativity was introduced in [2] to provide a nonlinear (time-domain) analog to strict positive real transfer functions [23]. It can be easily shown that if a nonlinear system is exponentially dissipative, then it is state-strict dissipative [3], [4].

The following definition is needed in this paper.

Definition 2.2 ([3]): A dynamical system $\mathcal{G}$ is *zero-state observable* if for all $u(t) \equiv 0$ and $y(t) \equiv 0$ implies $x(t) \equiv 0$. A dynamical system $\mathcal{G}$ is *completely reachable* if for all $x_{\rm i} \in \mathbb{R}^n$, there exists a finite time $t_{\rm i} \leq 0$ and a square integrable input $u(t)$ defined on $[t_{\rm i}, 0]$ such that the state $x(t)$, $t \geq t_{\rm i}$, can be driven from $x(t_{\rm i}) = 0$ to $x(0) = x_{\rm i}$.

Next, we present a result which shows that exponential dissipativity of a system of the form (1), (2) can be characterized in terms of the system functions $f(\cdot)$, $G(\cdot)$, $h(\cdot)$, and $J(\cdot)$. For the following result we consider the special case of exponentially dissipative systems with quadratic supply rates. Specifically, let $Q \in \mathbb{S}^l$, $R \in \mathbb{S}^m$, and $S \in \mathbb{R}^{l\times m}$ be given and assume $r(u, y) = y^{\rm T}Qy + 2y^{\rm T}Su + u^{\rm T}Ru$. Furthermore, we assume that there exists a function $\kappa : \mathbb{R}^l \to \mathbb{R}^m$ such that $\kappa(0) = 0$, $r(\kappa(y), y) < 0$, $y \neq 0$.

Theorem 2.1 ([2]): Let $Q \in \mathbb{S}^l$, $S \in \mathbb{R}^{l\times m}$, $R \in \mathbb{S}^m$, and let $\mathcal{G}$ be zero-state observable and completely reachable. $\mathcal{G}$ is exponentially dissipative with respect to the quadratic supply rate $r(u, y) = y^{\rm T}Qy + 2y^{\rm T}Su + u^{\rm T}Ru$ if and only if there exist functions $V_{\rm s} : \mathbb{R}^n \to \mathbb{R}$, $\ell : \mathbb{R}^n \to \mathbb{R}^p$, and $\mathcal{W} : \mathbb{R}^n \to \mathbb{R}^{p\times m}$, and a scalar $\varepsilon > 0$ such that $V_{\rm s}(\cdot)$ is continuously differentiable and nonnegative definite, $V_{\rm s}(0) = 0$, and, for all $x \in \mathbb{R}^n$,

$$0 = V_{\rm s}'(x)f(x) + \varepsilon V_{\rm s}(x) - h^{\rm T}(x)Qh(x) + \ell^{\rm T}(x)\ell(x), \quad (4)$$

$$0 = \frac{1}{2}V_{\rm s}'(x)G(x) - h^{\rm T}(x)(QJ(x) + S) + \ell^{\rm T}(x)\mathcal{W}(x), \quad (5)$$

$$0 = R + S^{\rm T}J(x) + J^{\rm T}(x)S + J^{\rm T}(x)QJ(x) - \mathcal{W}^{\rm T}(x)\mathcal{W}(x). \quad (6)$$

Corollary 2.1: Let $Q \in \mathbb{S}^l$, $S \in \mathbb{R}^{l\times m}$, $R \in \mathbb{S}^m$, and let $\mathcal{G}$ be a linear dynamical system with transfer function $G(s) \overset{\min}{\sim} \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$, where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{l\times n}$ and $D \in \mathbb{R}^{l\times m}$. Then, $\mathcal{G}$ is exponentially dissipative with respect to the quadratic supply rate $r(u, y) = y^{\rm T}Qy+2y^{\rm T}Su+u^{\rm T}Ru$ if and only if there exist matrices $P \in \mathbb{S}^n$, $L \in \mathbb{R}^{p\times n}$, and $W \in \mathbb{R}^{p\times m}$, with P nonnegative definite, and a scalar $\varepsilon > 0$ such that

$$0 = A^{\rm T}P + PA + \varepsilon P - C^{\rm T}QC + L^{\rm T}L, \quad (7)$$

$$0 = PB - C^{\rm T}(QD + S) + L^{\rm T}W \quad (8)$$

$$0 = R + S^{\rm T}D + D^{\rm T}S + D^{\rm T}QD - W^{\rm T}W. \quad (9)$$

Remark 2.1: Corollary 2.1 is the extended Kalman-Yakubovich-Popov lemma [23] and Theorem 2.1 presents its nonlinear extensions. In the case where $\varepsilon = 0$, Theorem 2.1 specializes to the extended nonlinear Kalman-Yakubovic -Popov conditions given in [3], [4].

III. Dynamic Dissipative Systems

In this section we introduce the notion of dynamic dissipativity. A system is said to be dynamic dissipative if the system cascaded with another given dynamic system is dissipative. Specifically, we consider a cascade interconnection of the dynamical system $\mathcal{G}$ given by (1), (2) and a system Σ (see Figure 1) given by

$$\dot{\hat{x}}(t) = f_\Sigma(\hat{x}(t)) + G_{\Sigma_u}(\hat{x}(t))u(t) + G_{\Sigma_y}(\hat{x}(t))y(t), \quad \hat{x}(0) = 0, \quad t \geq 0, \quad (10)$$

$$z(t) = h_\Sigma(\hat{x}(t)) + J_{\Sigma_u}(\hat{x}(t))u(t) + J_{\Sigma_y}(\hat{x}(t))y(t), \quad (11)$$

where $\hat{x} \in \mathbb{R}^{\hat{n}}$, $y \in \mathbb{R}^l$, $z \in \mathbb{R}^{\hat{l}}$, $u \in \mathbb{R}^m$, $f_\Sigma : \mathbb{R}^{\hat{n}} \to \mathbb{R}^{\hat{n}}$, $G_{\Sigma_u} : \mathbb{R}^{\hat{n}} \to \mathbb{R}^{\hat{n}\times m}$, $G_{\Sigma_y} : \mathbb{R}^{\hat{n}} \to \mathbb{R}^{\hat{n}\times l}$, $h_\Sigma : \mathbb{R}^{\hat{n}} \to \mathbb{R}^{\hat{l}}$, $J_{\Sigma_u} : \mathbb{R}^{\hat{n}} \to \mathbb{R}^{\hat{l}\times m}$, and $J_{\Sigma_y} : \mathbb{R}^{\hat{n}} \to \mathbb{R}^{\hat{l}\times l}$. We assume that $f_\Sigma(\cdot)$, $G_\Sigma(\cdot)$, $h_\Sigma(\cdot)$, and $J_\Sigma(\cdot)$ are continuously differentiable mappings and $f_\Sigma(0) = 0$ and $h_\Sigma(0) = 0$. The overall cascade system consisting of $\mathcal{G}$ and Σ is given by

$$\dot{\tilde{x}}(t) = \tilde{f}(\tilde{x}(t)) + \tilde{G}(\tilde{x}(t))u(t), \ \tilde{x}(0) = \tilde{x}_0, \ t \geq 0, \quad (12)$$

$$z(t) = \tilde{h}(\tilde{x}(t)) + \tilde{J}(\tilde{x}(t))u(t), \quad (13)$$

where $\tilde{x} = \begin{bmatrix} x \\ \hat{x} \end{bmatrix}$ and

$$\begin{aligned} \tilde{f}(\tilde{x}(t)) &= \begin{bmatrix} f(x(t)) \\ f_\Sigma(\hat{x}(t)) + G_{\Sigma_y}(\hat{x}(t))h(x(t)) \end{bmatrix}, \\ \tilde{G}(\tilde{x}(t)) &= \begin{bmatrix} G(x) \\ G_{\Sigma_u}(\hat{x}(t)) + G_{\Sigma_y}(\hat{x}(t))J(x(t)) \end{bmatrix}, \\ \tilde{h}(\tilde{x}(t)) &= h_\Sigma(\hat{x}(t)) + J_{\Sigma_y}(\hat{x}(t))h(x(t)), \\ \tilde{J}(\tilde{x}(t)) &= J_{\Sigma_u}(\hat{x}(t)) + J_{\Sigma_y}(\hat{x}(t))J(x(t)). \end{aligned}$$

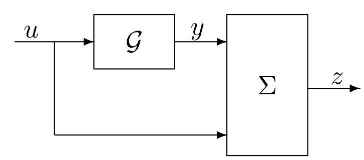

Fig. 1. Interconnection of $\mathcal{G}$ and Σ

Definition 3.1: A dynamical system $\mathcal{G}$ is $(\Sigma, \hat{Q})$-*exponentially dissipative* if the interconnection of $\mathcal{G}$ and Σ given by (12), (13) is exponentially dissipative with respect to the supply rate $z^{\rm T}\hat{Q}z$, where z is the output of the cascaded system and $\hat{Q} \in \mathbb{S}^{\hat{l}}$. Furthermore, $\mathcal{G}$ is $(\Sigma, \hat{Q})$-*dissipative* if the interconnection of $\mathcal{G}$ and Σ is dissipative with respect to the supply rate $z^{\rm T}\hat{Q}z$.

Remark 3.1: Note that $(\Sigma, \hat{Q})$-dissipativity is a time-domain nonlinear analog to IQCs [21] and is also a *dynamic* extension of dissipativity with respect to a quadratic supply rate. To see this, note that if $\mathcal{G}$ is $(\Sigma, \hat{Q})$-dissipative, where Σ is a linear dynamical system given by the transfer function $\hat{G}(s)$, then

$$\int_{-\infty}^{\infty} \begin{bmatrix} U(j\omega) \\ Y(j\omega) \end{bmatrix}^\star \hat{G}^\star(j\omega)\hat{Q}\hat{G}(j\omega) \begin{bmatrix} U(j\omega) \\ Y(j\omega) \end{bmatrix} {\rm d}\omega \geq 0, \quad (14)$$

where $U(s)$ and $Y(s)$, $s \in \mathbb{C}$, are the Laplace transforms of $u(t)$ and $y(t)$, respectively. Alternatively, let $p = l + m$ and let the dynamical system Σ be such that $z = [u^{\rm T} \ y^{\rm T}]^{\rm T}$.

Furthermore, let $\hat{Q} = \begin{bmatrix} R & S^{\mathrm{T}} \\ S & Q \end{bmatrix}$, where $Q \in \mathbb{S}^l$, $S \in \mathbb{R}^{l\times m}$, and $R = R^{\mathrm{T}} \in \mathbb{R}^{m\times m}$. In this case, $\mathcal{G}$ is $(\Sigma, \hat{Q})$-dissipative if and only if $\mathcal{G}$ is dissipative with respect to the quadratic supply rate $r(u, y) = y^{\mathrm{T}}Qy + 2y^{\mathrm{T}}Su + u^{\mathrm{T}}Ru$.

The following theorem is now immediate.

Theorem 3.1: Consider the nonlinear dynamical systems $\mathcal{G}$ and Σ given by (1), (2) and (10), (11) respectively. $\mathcal{G}$ is $(\Sigma, \hat{Q})$-exponentially dissipative (resp., $(\Sigma, \hat{Q})$-dissipative) if and only if there exist functions $\tilde{V}_{\mathrm{s}} : \mathbb{R}^{\tilde{n}} \to \mathbb{R}$, $\tilde{\ell} : \mathbb{R}^{\tilde{n}} \to \mathbb{R}^{\tilde{p}}$, and $\tilde{\mathcal{W}} : \mathbb{R}^{\tilde{n}} \to \mathbb{R}^{\tilde{p}\times m}$ and a scalar $\tilde{\varepsilon} > 0$ (resp., $\tilde{\varepsilon} = 0$), such that $\tilde{V}_{\mathrm{s}}(\cdot)$ is continuously differentiable and positive definite, $\tilde{V}_{\mathrm{s}}(0) = 0$, and, for all $\tilde{x} \in \mathbb{R}^{\tilde{n}}$, where $\tilde{n} = n + \hat{n}$,

$$0 = \tilde{V}_{\mathrm{s}}'(\tilde{x})\tilde{f}(\tilde{x}) + \tilde{\varepsilon}\tilde{V}_{\mathrm{s}}(\tilde{x}) - \tilde{h}^{\mathrm{T}}(\tilde{x})\hat{Q}\tilde{h}(\tilde{x}) + \tilde{\ell}^{\mathrm{T}}(\tilde{x})\tilde{\ell}(\tilde{x}), \quad (15)$$

$$0 = \frac{1}{2}\tilde{V}_{\mathrm{s}}'(\tilde{x})\tilde{G}(\tilde{x}) - \tilde{h}^{\mathrm{T}}(\tilde{x})\hat{Q}\tilde{J}(\tilde{x}) + \tilde{\ell}^{\mathrm{T}}(\tilde{x})\tilde{\mathcal{W}}(\tilde{x}), \quad (16)$$

$$0 = \tilde{J}^{\mathrm{T}}(\tilde{x})\hat{Q}\tilde{J}(\tilde{x}) - \tilde{\mathcal{W}}^{\mathrm{T}}(\tilde{x})\tilde{\mathcal{W}}(\tilde{x}). \quad (17)$$

Remark 3.2: Theorem 3.1 is an extension of the nonlinear Kalman-Yakubovich-Popov conditions given in Theorem 2.1 to dynamic dissipative systems.

The following result provides a necessary and sufficient condition for $(\Sigma, \hat{Q})$-dissipativity of $\mathcal{G}$ in the case where $\mathcal{G}$ and Σ are linear dynamical systems. Specifically, let $\mathcal{G}$ and Σ be given by transfer functions $G(s) \sim \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ and $\hat{G}(s) \sim \left[\begin{array}{c|c} \hat{A} & \hat{B} \\ \hline \hat{C} & \hat{D} \end{array}\right]$, respectively, where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{l\times n}$, $D \in \mathbb{R}^{l\times m}$, $\hat{A} \in \mathbb{R}^{\hat{n}\times\hat{n}}$, $\hat{B} \in \mathbb{R}^{\hat{n}\times(l+m)}$, $\hat{C} \in \mathbb{R}^{p\times\hat{n}}$ and $\hat{D} \in \mathbb{R}^{p\times(l+m)}$. In this case, the interconnection of $\mathcal{G}$ and Σ as shown in Figure 1 is given by the transfer function $\tilde{G}(s) \sim \left[\begin{array}{c|c} \tilde{A} & \tilde{B} \\ \hline \tilde{C} & \tilde{D} \end{array}\right]$, where

$$\tilde{A} = \begin{bmatrix} A & 0 \\ \hat{B}_y C & \hat{A} \end{bmatrix}, \qquad \tilde{B} = \begin{bmatrix} B \\ \hat{B}_y D + \hat{B}_u \end{bmatrix}, \quad (18)$$

$$\tilde{C} = [\hat{D}_y C \ \ \hat{C}], \qquad \tilde{D} = \hat{D}_u + \hat{D}_y D, \quad (19)$$

where $\hat{B}_u \in \mathbb{R}^{\hat{n}\times m}$, $\hat{B}_y \in \mathbb{R}^{\hat{n}\times l}$, $\hat{D}_u \in \mathbb{R}^{p\times m}$, and $\hat{D}_y \in \mathbb{R}^{p\times l}$ are such that $\hat{B} = [\hat{B}_u \ \hat{B}_y]$ and $\hat{D} = [\hat{D}_u \ \hat{D}_y]$.

Corollary 3.1: Consider the dynamical system $\mathcal{G}$ given by the transfer function $G(s) \sim \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$, let $\hat{Q} \in \mathbb{S}^p$, and let Σ be a linear dynamical system given by the transfer function $\hat{G}(s) \sim \left[\begin{array}{c|c} \hat{A} & \hat{B} \\ \hline \hat{C} & \hat{D} \end{array}\right]$. Then, $\mathcal{G}$ is $(\Sigma, \hat{Q})$-exponentially dissipative if and only if there exists a nonnegative-definite matrix $\tilde{P} \in \mathbb{S}^{n+\hat{n}}$ and a scalar $\varepsilon > 0$ such that

$$\begin{bmatrix} \tilde{A}^{\mathrm{T}}\tilde{P} + \tilde{P}\tilde{A} + \varepsilon\tilde{P} & \tilde{P}\tilde{B} \\ \tilde{B}^{\mathrm{T}}\tilde{P} & 0 \end{bmatrix} \le \begin{bmatrix} \tilde{C}^{\mathrm{T}} \\ \tilde{D}^{\mathrm{T}} \end{bmatrix} \hat{Q} \begin{bmatrix} \tilde{C} & \tilde{D} \end{bmatrix}. \quad (20)$$

Furthermore, $\mathcal{G}$ is $(\Sigma, \hat{Q})$-dissipative if and only if there exists a nonnegative-definite matrix $\tilde{P} \in \mathbb{S}^{n+\hat{n}}$ such that (20) holds with $\varepsilon = 0$.

Remark 3.3: Note that, in Corollary 3.1, if $\tilde{G}(s) \overset{\min}{\sim} \left[\begin{array}{c|c} \tilde{A} & \tilde{B} \\ \hline \tilde{C} & \tilde{D} \end{array}\right]$, then $\mathcal{G}$ is $(\Sigma, \hat{Q})$-exponentially dissipative if and only if there exists a positive-definite matrix $\tilde{P}$ such that (20) holds.

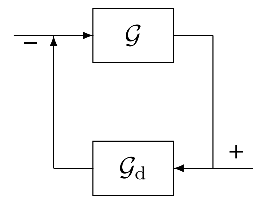

Fig. 2. Feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_{\mathrm{d}}$

Next, we present the main result of this paper on the stability of a negative feedback interconnection of two dynamic dissipative systems. Specifically, consider the negative feedback interconnection of dynamical system $\mathcal{G}$ with a feedback system $\mathcal{G}_{\mathrm{d}}$, which is given by

$$\dot{x}_{\mathrm{d}}(t) = f_{\mathrm{d}}(x_{\mathrm{d}}(t)) + G_{\mathrm{d}}(x_{\mathrm{d}}(t))u_{\mathrm{d}}(t), \quad x_{\mathrm{d}}(0) = x_{\mathrm{d}0}, \quad t \ge 0, \quad (21)$$

$$y_{\mathrm{d}}(t) = h_{\mathrm{d}}(x_{\mathrm{d}}(t)) + J_{\mathrm{d}}(x_{\mathrm{d}}(t))u_{\mathrm{d}}(t), \quad (22)$$

where $x_{\mathrm{d}} \in \mathbb{R}^{n_{\mathrm{d}}}$, $u_{\mathrm{d}} \in \mathbb{R}^{m_{\mathrm{d}}}$, $y_{\mathrm{d}} \in \mathbb{R}^{l_{\mathrm{d}}}$, $f_{\mathrm{d}} : \mathbb{R}^{n_{\mathrm{d}}} \to \mathbb{R}^{n_{\mathrm{d}}}$, $G_{\mathrm{d}} : \mathbb{R}^{n_{\mathrm{d}}} \to \mathbb{R}^{n_{\mathrm{d}}\times m_{\mathrm{d}}}$, $h_{\mathrm{d}} : \mathbb{R}^{n_{\mathrm{d}}} \to \mathbb{R}^{l_{\mathrm{d}}}$, and $J : \mathbb{R}^{n_{\mathrm{d}}} \to \mathbb{R}^{l_{\mathrm{d}}\times m_{\mathrm{d}}}$. Note that with the feedback interconnection given in Figure 2, $u = -y_{\mathrm{d}}$ and $u_{\mathrm{d}} = y$. Furthermore, consider a dynamical system Σ_{d}, given by

$$\dot{\hat{x}}_{\mathrm{d}}(t) = f_{\Sigma}(\hat{x}_{\mathrm{d}}(t)) + G_{\Sigma_y}(\hat{x}_{\mathrm{d}}(t))u_{\mathrm{d}}(t) - G_{\Sigma_u}(\hat{x}_{\mathrm{d}}(t))y_{\mathrm{d}}(t), \quad \hat{x}_{\mathrm{d}}(0) = 0, \quad t \ge 0, \quad (23)$$

$$z_{\mathrm{d}}(t) = h_{\Sigma}(\hat{x}_{\mathrm{d}}(t)) + J_{\Sigma_y}(\hat{x}_{\mathrm{d}}(t))u_{\mathrm{d}}(t) - J_{\Sigma_u}(\hat{x}_{\mathrm{d}}(t))y_{\mathrm{d}}(t), \quad (24)$$

where $\hat{x}_{\mathrm{d}} \in \mathbb{R}^{\hat{n}}$, $y_{\mathrm{d}} \in \mathbb{R}^m$, $u_{\mathrm{d}} \in \mathbb{R}^l$ and $z_{\mathrm{d}} \in \mathbb{R}^{\hat{l}}$. It should be noted that the system Σ_{d} is identical to the system Σ, with $u = -y_{\mathrm{d}}$ and $u_{\mathrm{d}} = y$.

Theorem 3.2: Let $\hat{Q}, \hat{Q}_{\mathrm{d}} \in \mathbb{S}^p$. Consider the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_{\mathrm{d}}$ with input-output pairs (u, y) and $(u_{\mathrm{d}}, y_{\mathrm{d}})$, respectively, and with $u_{\mathrm{d}} = y$ and $u = -y_{\mathrm{d}}$. Assume that $\mathcal{G}$ and $\mathcal{G}_{\mathrm{d}}$ are $(\Sigma, \hat{Q})$-dissipative and $(\Sigma_{\mathrm{d}}, \hat{Q}_{\mathrm{d}})$-dissipative with C^0 storage functions $V_{\mathrm{s}} : \mathbb{R}^n \times \mathbb{R}^{\hat{n}} \to \mathbb{R}$ and $V_{\mathrm{sd}} : \mathbb{R}^{n_{\mathrm{d}}} \times \mathbb{R}^{\hat{n}_{\mathrm{d}}} \to \mathbb{R}$, respectively, such that $V_{\mathrm{s}}(0, 0) = 0$, $V_{\mathrm{sd}}(0, 0) = 0$, and

$$\alpha(\|x\|) \le V_{\mathrm{s}}(x, \hat{x}), \quad (x, \hat{x}) \in \mathbb{R}^n \times \mathbb{R}^{\hat{n}}, \quad (25)$$

$$\alpha_{\mathrm{d}}(\|x_{\mathrm{d}}\|) \le V_{\mathrm{sd}}(x_{\mathrm{d}}, \hat{x}_{\mathrm{d}}), \quad (x_{\mathrm{d}}, \hat{x}_{\mathrm{d}}) \in \mathbb{R}^{n_{\mathrm{d}}} \times \mathbb{R}^{\hat{n}_{\mathrm{d}}}, \quad (26)$$

where $\alpha, \alpha_{\mathrm{d}} : [0, \infty) \to [0, \infty)$ are class $\mathcal{K}_\infty$ functions. Furthermore, assume there exists a scalar $\sigma > 0$ such that $\hat{Q} + \sigma\hat{Q}_{\mathrm{d}} \le 0$. Then the following statements hold:

i) The negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_{\mathrm{d}}$ is Lyapunov stable.

ii) If $\mathcal{G}$ is $(\Sigma, \hat{Q})$-exponentially dissipative, then the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_{\mathrm{d}}$ is Lyapunov stable and for every $x(0) \in \mathbb{R}^n$, $\|x(t)\| \to 0$ as $t \to \infty$.

Proof: The proof follows from standard Lyapunov theory and invariant set arguments [23]. See [2], [3] for similar proofs. Specifically, note that $u = -y_{\mathrm{d}}$, $u_{\mathrm{d}} = y$, and since $\hat{x}_0 = \hat{x}_{\mathrm{d}_0} = 0$, $\hat{x}(t) = \hat{x}_{\mathrm{d}}(t)$ and $z_{\mathrm{d}}(t) = z(t)$, $t \ge 0$. Hence, the state of the overall interconnection of $\mathcal{G}$, $\mathcal{G}_\mathrm{d}$, and Σ (see Figure 3) is given by $[x^\mathrm{T},\ x_\mathrm{d}^\mathrm{T},\ \hat{x}^\mathrm{T}]^\mathrm{T}$. Now, consider the Lyapunov function candidate $V(x, x_\mathrm{d}, \hat{x}) = V_\mathrm{s}(x, \hat{x}) + \sigma V_\mathrm{sd}(x_\mathrm{d}, \hat{x})$ and since $\mathcal{G}$ and $\mathcal{G}_\mathrm{d}$ are $(\Sigma, \hat{Q})$-dissipative and $(\Sigma_\mathrm{d}, \hat{Q}_\mathrm{d})$-dissipative, respectively, it follows that

$$\begin{aligned}\dot{V}(x(t), x_\mathrm{d}(t), \hat{x}(t)) &= \dot{V}_\mathrm{s}(x(t), \hat{x}(t)) + \sigma \dot{V}_\mathrm{s}(x_\mathrm{d}(t), \hat{x}(t)) \\ &\leq z^\mathrm{T}(t)(\hat{Q} + \sigma \hat{Q}_\mathrm{d})z(t) \ \leq \ 0.\end{aligned}$$

Now, Lyapunov stability follows from standard arguments.

Next, if $\mathcal{G}$ is $(\Sigma, \hat{Q})$-exponentially dissipative, then it can be shown as above that for every $x_0 \in \mathbb{R}^n$ and $x_{\mathrm{d}0} \in \mathbb{R}^{n_\mathrm{d}}$,

$$\dot{V}(x(t), x_\mathrm{d}(t), \hat{x}(t)) \leq -\varepsilon V_\mathrm{s}(x, \hat{x}) \leq -\varepsilon \alpha(\|x\|) \leq 0,$$

where $\varepsilon > 0$. Hence, $V(x(t), x_\mathrm{d}(t), \hat{x}(t))$, $t \geq 0$, is a monotonically decreasing function and since $V(\cdot,\cdot,\cdot)$ is lower bounded it follows that $c \triangleq \lim_{t\to\infty} V(x(t),\, x_\mathrm{d}(t),\, \hat{x}(t)) \geq 0$ exists. Next it follows from LaSalle's invariant set theorem [23] that the positive limit set $\omega(x_0, x_{\mathrm{d}0}, 0)$ is nonempty and invariant. Thus, $\dot{V}(x(t), x_\mathrm{d}(t), \hat{x}(t)) = 0$, $t \geq 0$, $(x(0), x_\mathrm{d}(0), 0) \in \omega(x_0, x_{\mathrm{d}0}, 0)$, which further implies that $\|x(t)\| = 0$, $t \geq 0$, $(x(0), x_\mathrm{d}(0), 0) \in \omega(x_0, x_{\mathrm{d}0}, 0)$. Hence, $\|x(t)\| \to 0$ as $t \to \infty$. ■

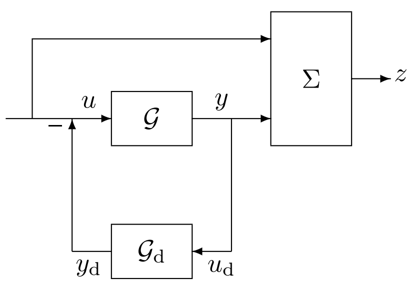

Fig. 3. Interconnection of $\mathcal{G}$, $\mathcal{G}_\mathrm{d}$, and Σ

The following specialization to linear dynamical systems is immediate.

Corollary 3.2: Let $\hat{Q}, \hat{Q}_\mathrm{d} \in \mathbb{S}^p$ and let $\mathcal{G}$ and $\mathcal{G}_\mathrm{d}$ be linear dynamical systems given by transfer functions $\mathcal{G} = G(s) \overset{\min}{\sim} \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] D$ and $\mathcal{G}_\mathrm{d} = G_\mathrm{d}(s) \overset{\min}{\sim} \left[\begin{array}{c|c} A_\mathrm{d} & B_\mathrm{d} \\ \hline C_\mathrm{d} & D_\mathrm{d} \end{array}\right]$, respectively. Consider the feedback system consisting of $\mathcal{G}$ and $\mathcal{G}_\mathrm{d}$ with input-output pairs (u, y) and $(u_\mathrm{d}, y_\mathrm{d})$, respectively, and with $u_\mathrm{d} = y$ and $u = -y_\mathrm{d}$. Assume that there exist a scalar $\varepsilon \geq 0$ and matrices $\tilde{P} \in \mathbb{S}^{\tilde{n}}$ and $\tilde{P}_\mathrm{d} \in \mathbb{S}^{\tilde{n}_\mathrm{d}}$ such that $\tilde{P} \geq \text{block–diag}[\alpha I_n, 0]$, $\tilde{P}_\mathrm{d} \geq \text{block–diag}[\alpha_\mathrm{d} I_{n_\mathrm{d}}, 0]$ where $\alpha, \alpha_\mathrm{d} > 0$, and

$$\left[\begin{array}{cc} \tilde{A}^\mathrm{T}\tilde{P} + \tilde{P}\tilde{A} - \tilde{C}^\mathrm{T}\hat{Q}\tilde{C} + \varepsilon\tilde{P} & \tilde{P}\tilde{B} - \tilde{C}^\mathrm{T}\hat{Q}\tilde{D} \\ \tilde{B}^\mathrm{T}\tilde{P} - \tilde{D}^\mathrm{T}\hat{Q}\tilde{C} & -\tilde{D}^\mathrm{T}\hat{Q}\tilde{D} \end{array}\right] \leq 0, \tag{27}$$

$$\left[\begin{array}{cc} \tilde{A_\mathrm{d}}^\mathrm{T}\tilde{P_\mathrm{d}} + \tilde{P_\mathrm{d}}\tilde{A_\mathrm{d}} - \tilde{C_\mathrm{d}}^\mathrm{T}\hat{Q}_\mathrm{d}\tilde{C_\mathrm{d}} & \tilde{P_\mathrm{d}}\tilde{B_\mathrm{d}} - \tilde{C_\mathrm{d}}^\mathrm{T}\hat{Q}_\mathrm{d}\tilde{D_\mathrm{d}} \\ \tilde{B_\mathrm{d}}^\mathrm{T}\tilde{P_\mathrm{d}} - \tilde{D_\mathrm{d}}^\mathrm{T}\hat{Q}_\mathrm{d}\tilde{C_\mathrm{d}} & -\tilde{D_\mathrm{d}}^\mathrm{T}\hat{Q}_\mathrm{d}\tilde{D_\mathrm{d}} \end{array}\right] \leq 0, \tag{28}$$

where

$$\begin{aligned}\tilde{A_\mathrm{d}} &\triangleq \left[\begin{array}{cc} A_\mathrm{d} & 0 \\ -\hat{B}_u C_\mathrm{d} & \hat{A} \end{array}\right], \quad \tilde{B_\mathrm{d}} \triangleq \left[\begin{array}{c} B_\mathrm{d} \\ -\hat{B}_u D_\mathrm{d} + \hat{B}_y \end{array}\right], \\ \tilde{C_\mathrm{d}} &\triangleq [-\hat{D}_u C_\mathrm{d} \quad \hat{C}], \quad \tilde{D_\mathrm{d}} \triangleq \hat{D}_y - \hat{D}_u D_\mathrm{d}.\end{aligned}$$

Furthermore, assume there exists a scalar $\sigma > 0$ such that $\hat{Q} + \sigma\hat{Q}_\mathrm{d} \leq 0$. Then the following statements hold:

i) If $\varepsilon = 0$, then the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_\mathrm{d}$ is Lyapunov stable.

ii) If $\varepsilon > 0$, then the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_\mathrm{d}$ is Lyapunov stable and for every $x(0) \in \mathbb{R}^n$, $\|x(t)\| \to 0$ as $t \to \infty$.

Proof: The proof is a direct consequence of Theorem 3.2 and Corollary 3.1, and hence is omitted. ■

IV. The Absolute Stability Problem

In this section we present applications of dynamic dissipativity theory to the absolute stability problem [12–14], [23]. Specifically, consider the nonlinear dynamical system given by

$$\dot{x}(t) = Ax(t) + B\phi(Cx(t)), \quad x(0) = x_0,\ t \geq 0 \tag{29}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$ and $\phi(\cdot) \in \Phi$ and where $\hat{\Phi} \triangleq \{\phi : \mathbb{R}^m \to \mathbb{R}^m : \phi(0) = 0\}$. The absolute stability problem deals with deriving sufficient conditions on the linear part $G(s) \sim \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right]$ of (29) so that the zero solution to (29) is asymptotically stable for all $\phi(\cdot) \in \Phi$, where $\Phi \subset \hat{\Phi}$ is a given set. The case where Φ contains nonlinear operators that are either passive or nonexpansive (more generally, dissipative) has been thoroughly addressed in the literature based on dissipativity theory [1]. However, if Φ contains operators that are dissipative and satisfy additional conditions such as monotonicity of the functions then classical dissipativity theory is not sufficient. The absolute stability problem for nonlinearities with additional restrictions such as monotonicity has been addressed in the literature using the *multiplier* theory (see [12], for example) wherein an additional dynamical system is introduced in the feedback loop and then the dissipativity theory is applied to the feedback interconnection of a modified plant and modified nonlinearities (see Figure 4). In this section, we show that the multiplier

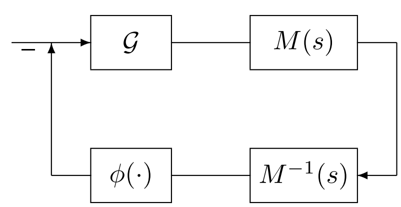

Fig. 4. Feedback interconnection of $\mathcal{G}$ and $\phi(\cdot)$ with multipliers

theory considered in the literature is a special case of dynamic dissipativity theory. Specifically, we show that nonlinearities that are dissipative and satisfy additional restrictions such as monotonicity are all dynamic dissipative. In this paper, we restrict our focus on the absolute stability problem for passive, monotonic nonlinearities. Extensions to other nonlinearities considered in the literature are straightforward and will not be addressed here due to lack of space. Specifically, consider the set of nonlinearities

$$\begin{aligned}\Phi \ \triangleq \ \{\phi : \mathbb{R}^m \to \mathbb{R}^m \ :\ & \phi(y) = [\phi_1(y_1), \ldots, \phi_m(y_m)]^\mathrm{T}, \\ & \phi(0) = 0,\ \phi^\mathrm{T}(y)[\phi(y) - My] \leq 0, \\ & \text{and } \frac{\mathrm{d}\phi_i(y_i)}{\mathrm{d}y_i} \geq 0,\ i = 1, \ldots, m\}, \end{aligned} \tag{30}$$

where, $M \in \mathbb{R}^{m\times m}$, $M = \mathrm{diag}[M_1,\ldots,M_m]$, $M_i > 0$, $i = 1,\ldots,m$. Note that it follows from (30) that the scalar constraint $0 \le y_i\phi_i(y_i) \le M_i y_i^2$, is satisfied for $i = 1,\ldots,m$, and

$$(y_i - \frac{1}{M_i}\phi_i(y_i))\phi_i(y_i) \ge 0, \quad i = 1,\ldots,m. \tag{31}$$

Next, we show that there for every $\phi(\cdot) \in \Phi$, there exists a linear system Σ_{d} and a matrix $\hat{Q}_{\mathrm{d}}$ such that $\phi(\cdot)$ is $(\Sigma_{\mathrm{d}},\hat{Q}_{\mathrm{d}})$-dissipative. For the following result, let Σ_{d} be a linear system given by the transfer function $\hat{G}_{\mathrm{d}}(s) \sim \left[\begin{array}{c|cc}\hat{A}_{\mathrm{d}} & \hat{B}_{\mathrm{d}_u} & -\hat{B}_{\mathrm{d}_y}\\ \hline \hat{C}_{\mathrm{d}} & \hat{D}_{\mathrm{d}_u} & -\hat{D}_{\mathrm{d}_y}\end{array}\right]$, where $\hat{A}_{\mathrm{d}} \in \mathbb{R}^{\tilde{m}\times\tilde{m}}$, $\hat{B}_{\mathrm{d}_u} \in \mathbb{R}^{\tilde{m}\times m}$, $\hat{B}_{\mathrm{d}_y} \in \mathbb{R}^{\tilde{m}\times m}$, $\hat{C}_{\mathrm{d}_u} \in \mathbb{R}^{2m\times\tilde{m}}$, $\hat{D}_{\mathrm{d}_u} \in \mathbb{R}^{2m\times m}$ and $\hat{D}_{\mathrm{d}_y} \in \mathbb{R}^{2m\times m}$ are given by

$$\hat{A}_{\mathrm{d}} = \text{block--diag}[\hat{A}_1,\ldots,\hat{A}_m], \tag{32}$$
$$\hat{B}_{\mathrm{d}_u} = \text{block--diag}[\hat{K}_1,\ldots,\hat{K}_m], \tag{33}$$
$$\hat{B}_{\mathrm{d}_y} = \text{block--diag}[\hat{L}_1,\ldots,\hat{L}_m], \tag{34}$$
$$\hat{C}_{\mathrm{d}} = \begin{bmatrix}\text{block--diag}[\hat{N}_1,\ldots,\hat{N}_m]\\ 0_{m\times\tilde{m}}\end{bmatrix}, \tag{35}$$
$$\hat{D}_{\mathrm{d}_u} = \begin{bmatrix}\mathrm{diag}[q_1,\ldots,q_m]\\ 0_{m\times m}\end{bmatrix}, \tag{36}$$
$$\hat{D}_{\mathrm{d}_y} = \begin{bmatrix}\mathrm{diag}[\frac{1}{M_1}q_1,\ldots,\frac{1}{M_m}q_m]\\ I_m\end{bmatrix}, \tag{37}$$

and where

$$\begin{aligned}
\hat{A}_i &\triangleq \mathrm{diag}[-\eta_{i1},\ldots,-\eta_{im_i}], \quad i = 1,\ldots,m,\\
\hat{K}_i &\triangleq \left[\frac{\alpha_{i1}}{\beta_{i1}},\ldots,\frac{\alpha_{im_i}}{\beta_{im_i}}\right]^{\mathrm{T}}, \quad i = 1,\ldots,m,\\
\hat{L}_i &\triangleq \frac{1}{M_i}\left[\frac{\alpha_{i1}}{\beta_{i1}},\ldots,\frac{\alpha_{im_i}}{\beta_{im_i}}\right]^{\mathrm{T}}, \quad i = 1,\ldots,m,\\
\hat{N}_i &\triangleq [\alpha_{i1},\ldots,\alpha_{im_i}], \quad i = 1,\ldots,m,\\
q_i &\triangleq \alpha_{i0} + \sum_{j=1}^{m_i}\alpha_{ij}, \quad i = 1,\ldots,m,
\end{aligned}$$

and α_{i0}, α_{ij}, β_{ij}, $\eta_{ij} \ge 0$ are such that $\eta_{ij}\beta_{ij} - \alpha_{ij} \ge 0$, $j = i,\ldots,m_i$, $i = 1,\ldots,m$, $m_i > 0$, $i = 1,\ldots,m$ and $\tilde{m} = \sum_{i=1}^{m} m_i$.

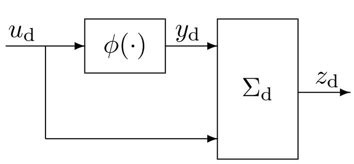

Fig. 5. Interconnection of $\phi(\cdot)$ and Σ_{d}

Proposition 4.1: Let $\phi(\cdot) \in \Phi$ where Φ is given by (30), let Σ_{d} be given by $\hat{G}_{\mathrm{d}}(s) \sim \left[\begin{array}{c|cc}\hat{A}_{\mathrm{d}} & \hat{B}_{\mathrm{d}_u} & -\hat{B}_{\mathrm{d}_y}\\ \hline \hat{C}_{\mathrm{d}} & \hat{D}_{\mathrm{d}_u} & -\hat{D}_{\mathrm{d}_y}\end{array}\right]$, where $\hat{A}_{\mathrm{d}}$, $\hat{B}_{\mathrm{d}_u}$, $\hat{B}_{\mathrm{d}_y}$, $\hat{C}_{\mathrm{d}}$, $\hat{D}_{\mathrm{d}_u}$ and $\hat{D}_{\mathrm{d}_y}$ are given by (32)–(36), and let $\hat{Q}_{\mathrm{d}} \in \mathbb{R}^{2m\times 2m}$ be given by $\hat{Q}_{\mathrm{d}} = \begin{bmatrix} 0 & I_m\\ I_m & 0\end{bmatrix}$. Then $\phi(\cdot)$ is $(\Sigma_{\mathrm{d}},\hat{Q}_{\mathrm{d}})$-dissipative with $(\Sigma_{\mathrm{d}},\hat{Q}_{\mathrm{d}})$-storage function

$$\hat{V}_{\mathrm{sd}}(\hat{x}_{\mathrm{d}}) = 2\sum_{i=1}^{m}\sum_{j=1}^{m_i}\beta_{ij}\int_0^{\hat{x}_{\mathrm{d}_{ij}}}\phi_i(\sigma)\mathrm{d}\sigma, \tag{38}$$

where $\hat{x}_{\mathrm{d}} = [\hat{x}_{\mathrm{d}_i}^{\mathrm{T}},\ldots,\hat{x}_{\mathrm{d}_m}^{\mathrm{T}}]$ and $\hat{x}_{\mathrm{d}_i} = [\hat{x}_{\mathrm{d}_{i1}},\ldots,\hat{x}_{\mathrm{d}_{mi}}]^{\mathrm{T}}$, $i = 1,\ldots,m$.

Proof: Note that the interconnection of Σ_{d} and $\phi(\cdot)$ is given by (see Figure 5)

$$\dot{\hat{x}}_{\mathrm{d}}(t) = \hat{A}_{\mathrm{d}}\hat{x}_{\mathrm{d}}(t) + \hat{B}_{\mathrm{d}_u}u_{\mathrm{d}}(t) - \hat{B}_{\mathrm{d}_y}y_{\mathrm{d}}(t), \quad \hat{x}_{\mathrm{d}}(0) = \hat{x}_{\mathrm{d}0}, \quad t \ge 0, \tag{39}$$
$$\hat{z}_{\mathrm{d}}(t) = \hat{C}_{\mathrm{d}}\hat{x}_{\mathrm{d}}(t) + \hat{D}_{\mathrm{d}_u}u_{\mathrm{d}}(t) - \hat{D}_{\mathrm{d}_y}\hat{y}_{\mathrm{d}}(t). \tag{40}$$

Note that $y_{\mathrm{d}}(t) = \phi(u_{\mathrm{d}}(t))$. In this case, the Lyapunov derivative of $\hat{V}_{\mathrm{sd}}(\cdot)$ along the solutions to (39), (40) is given by

$$\dot{\hat{V}}_{\mathrm{sd}}(\hat{x}_{\mathrm{d}}(t)) = 2\sum_{i=1}^{m}\sum_{j=1}^{m_i}\beta_{ij}\left[-\eta_{ij}\hat{x}_{\mathrm{d}_{ij}}(t) + \frac{\alpha_{ij}}{\beta_{ij}}u_{\mathrm{d}_i}(t) - \frac{1}{M_i}\frac{\alpha_{ij}}{\beta_{ij}}\phi_i(u_{\mathrm{d}_i}(t))\right]\phi_i(\hat{x}_{\mathrm{d}_{ij}}(t))$$

Next, since $\frac{\mathrm{d}\phi_i}{\mathrm{d}y_i} \ge 0$, $i = 1,\ldots,m$, it follows that

$$[\phi_i(y_i(t_1)) - \phi_i(y_i(t_2))][y_i(t_1) - y_i(t_2)] \ge 0, \tag{41}$$

for all $i = 1,\ldots,m$, and $t_1, t_2 \ge 0$. Now, using (31), (41), and the facts $\phi_i(y_i)y_i \ge 0$, $i = 1\ldots,m$, $\eta_{ij}\beta_{ij} - \alpha_{ij} \ge 0$, $j = 1,\ldots,m_i$, $i = 1,\ldots,m$, it can be shown that

$$\dot{\hat{V}}_{\mathrm{sd}}(\hat{x}_{\mathrm{d}}(t)) \le \hat{z}_{\mathrm{d}}^{\mathrm{T}}(t)\hat{Q}_{\mathrm{d}}\hat{z}_{\mathrm{d}}(t), \ t \ge 0, \tag{42}$$

which proves the result. ■

The following result reported in [15] (and references therein) can now be shown to be a direct consequence of dynamic dissipativity theory.

Theorem 4.1: Consider the nonlinear dynamical system given by (29) and where $\phi \in \Phi$, which is given by (30). Assume there exists a nonnegative matrix $\tilde{P}$ and scalars ε, $\eta > 0$ such that $\tilde{P} \ge \text{block--diag}[\eta I_n,\ 0_{\hat{n}\times\hat{n}}, 0_{\hat{n}\times\hat{n}}]$ and (20) holds where

$$\tilde{A} = \begin{bmatrix} A & 0\\ \hat{B}_{\mathrm{d}_u}C & \hat{A}_{\mathrm{d}}\end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} B\\ \hat{B}_{\mathrm{d}_y}\end{bmatrix} \tag{43}$$
$$\tilde{C} = [\hat{D}_{\mathrm{d}_u}C \ \ \hat{C}_{\mathrm{d}}], \quad \tilde{D} = \hat{D}_{\mathrm{d}_y}, \tag{44}$$

where $\hat{A}_{\mathrm{d}}$, $\hat{B}_{\mathrm{d}_u}$, $\hat{B}_{\mathrm{d}_y}$, $\hat{C}_{\mathrm{d}}$, $\hat{D}_{\mathrm{d}_u}$ and $\hat{D}_{\mathrm{d}_y}$ are given by (32)–(36). Then the nonlinear dynamical system (29) is asymptotically stable for all $\phi \in \Phi$.

Proof: The proof is a direct consequence of Corollary 3.1 and Proposition 4.1 and hence is omitted. ■

V. Stability Analysis of Linear Systems with Structured Uncertainties

In this section, we apply the dynamic dissipativity theory to the problem of robust stability analysis of linear systems perturbed by a structured uncertainty. Robust stability analysis with structured uncertainty has been extensively studied in the literature ([15–18], [24], [25] and references within). Specifically, consider the linear, uncertain, time-invariant dynamical system $\mathcal{G}$ given by

$$\dot{x}(t) = (A + B\Delta C)x(t), \qquad x(0) = x_0, \qquad t \geq 0, \quad (45)$$

where $x(t) \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, and $C \in \mathbb{R}^{m\times n}$ are known matrices, $\Delta \in \mathbb{R}^{m\times m}$ is an unknown matrix that belongs to the set

$$\begin{aligned} \boldsymbol{\Delta} \quad \stackrel{\triangle}{=} \quad & \{\Delta \in \mathbb{R}^{m\times m}:\ \sigma_{\max}(\Delta) \leq 1, \\ & \Delta = \text{block--diag}[\delta_1 I_{m_1}, \cdots, \delta_k I_{m_k}, \Delta_{k+1}, \cdots, \Delta_p], \\ & \Delta_i \in \mathbb{R}^{m_{k+i}\times m_{k+i}}\}, \qquad (46) \end{aligned}$$

where m_i, $i = 1, \ldots, k+p$, are such that $\sum_{i=1}^{k+p} = m$. Note that (45) may be written as a negative feedback interconnection of the linear dynamical system $\mathcal{G}$ given by the transfer function $\mathcal{G} = G(s) \sim \left[\begin{array}{c|c} A & -B \\ \hline C & 0 \end{array}\right]$ and the linear uncertainty operator $\mathcal{G}_\text{d} = \Delta(\cdot)$ (see Figure 2).

Next, we present a result to show that every $\Delta \in \boldsymbol{\Delta}$ is dynamic dissipative. For this result, let Σ_d be a linear dynamical system given by the transfer function $\hat{G}_\text{d}(s) \sim \left[\begin{array}{c|c} \hat{A}_\text{d} & \hat{B}_\text{d} \\ \hline \hat{C}_\text{d} & \hat{D}_\text{d} \end{array}\right]$ where $\hat{A}_\text{d} = \begin{bmatrix} A_1 & 0 \\ 0 & A_1 \end{bmatrix}$, $\hat{B}_\text{d} = \begin{bmatrix} 0 & -B_1 \\ B_1 & 0 \end{bmatrix}$, $\hat{C}_\text{d} = \begin{bmatrix} C_1 & 0 \\ 0 & C_1 \end{bmatrix}$, and $\hat{D}_\text{d} = \begin{bmatrix} 0 & -D_1 \\ D_1 & 0 \end{bmatrix}$ and where $A_1 \in \mathbb{R}^{\hat{n}\times\hat{n}}$, $B_1 \in \mathbb{R}^{\hat{n}\times m}$, $C_1 \in \mathbb{R}^{m\times\hat{n}}$, and $D_1 \in \mathbb{R}^{m\times m}$ are such that A_1 is Hurwitz, $D_1\Delta = \Delta D_1$, $\Delta \in \boldsymbol{\Delta}$, and $M_1(s)\Delta = \Delta M_1(s)$, $\Delta \in \boldsymbol{\Delta}$, with $M_1(s) \stackrel{\triangle}{=} C_1(sI - A_1)^{-1}B_1$.

The following results are now immediate. The proofs are omitted due to space limitations.

Lemma 5.1: Let $\Delta \in \boldsymbol{\Delta}$, $\hat{Q}_\text{d} = \text{block--diag}[-I, I]$ and Σ_d be a linear dynamical system given by the transfer function $\hat{G}_\text{d}(s) \sim \left[\begin{array}{c|c} \hat{A}_\text{d} & \hat{B}_\text{d} \\ \hline \hat{C}_\text{d} & \hat{D}_\text{d} \end{array}\right]$. Then Δ is $(\Sigma_\text{d}, \hat{Q}_\text{d})$-dissipative.

Theorem 5.1: Consider the linear uncertain dynamical system given by (45) where $\Delta \in \boldsymbol{\Delta}$ and where $\boldsymbol{\Delta}$ is given by (46). Let $\hat{Q}_\text{d} = \text{block--diag}[-I, I]$. Assume there exists a nonnegative-definite matrix $\tilde{P}$ and scalars $\varepsilon, \eta > 0$ such that (20) holds and $\tilde{P} \geq \text{block--diag}[\eta I_n,\ 0_{\hat{n}\times\hat{n}},\ 0_{\hat{n}\times\hat{n}}]$, where

$$\tilde{A} = \begin{bmatrix} A & 0 & 0 \\ 0 & A_1 & 0 \\ B_1 C & 0 & A_1 \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} B \\ B_1 \\ 0 \end{bmatrix}, \quad (47)$$

$$\tilde{C} = \begin{bmatrix} 0 & C_1 & 0 \\ D_1 C & 0 & C_1 \end{bmatrix}, \quad \tilde{D} = \begin{bmatrix} D_1 \\ 0 \end{bmatrix}. \quad (48)$$

Then the linear uncertain dynamical system given by (45) is asymptotically stable for all $\Delta \in \boldsymbol{\Delta}$.

Remark 5.1: Theorem 5.1 provides an LMI condition for checking whether the complex structured singular value upper bound [24] is less than unity. This idea has been explored and extended to control design as well (see [26] for example). However, dynamic dissipativity theory provides additionally a method to construct Lyapunov-type functionals for proving stability.

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