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Minimum-Phase Infinite-Dimensional SISO Second-Order Systems

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Abstract— **In general, better performance can be achieved with a controlled minimum-phase system than a controlled non-minimum phase system. We show that a wide class of second-order infinite-dimensional systems with either velocity or position measurements are minimum-phase. The results are illustrated with an example.**

I. Introduction

A stable finite-dimensional system is *minimum phase* or *outer* if and only if its transfer function has no zeros in the right half plane. A more general definition exists for infinite-dimensional systems. For a number of reasons controller design for minimum-phase systems is in general much easier than for non-minimum phase systems. For example, zeros in the right half plane restrict the achievable sensitivity. Also, most adaptive controllers require the system to be minimum-phase.

It is therefore advantageous to establish conditions under which infinite-dimensional systems are minimum-phase. There can be difficulties associated with computing the zeros of an infinite-dimensional system [3], [10]. Furthermore, there are aspects of the dynamics that can lead to non-minimum phase behaviour besides zeros in the right half plane. For example, the transfer function $\exp(-s)$ has no zeros, but is clearly not minimum-phase. Thus, determining minimum-phase behaviour is less straightforward than for finite-dimensional systems.

There are a number of results for first-order systems that guarantee that the transfer function is positive real [4], [6], [5], [11]. In [4] the positive real property, together with exponential stability of the semigroup, and the fact that the system is relative degree one, is shown to imply convergence and stability of an adaptive compensator. These assumptions on the system imply that it is minimum-phase.

In [14], [15] a class of second-order systems

$$\ddot{z}(t) + A_o z(t) + D\dot{z}(t) = B_o u(t), \tag{1}$$

is examined. In these references the damping $D = \frac{1}{2}B_o^* B_o$ and the output

$$y(t) = -B_o^* \dot{z}(t) + u(t),$$

have been studied. We consider more general damping. We also consider both velocity and position measurements. First consider the output

$$y(t) = B_o^* \dot{z}(t). \tag{2}$$

This class of systems has a positive real transfer function. We will show that with this choice of output, and certain assumptions on the damping operator, the system is well-posed and has an outer transfer function. This re-definition of the output is crucial. In one space dimension, the example in [14, sect. 7] has transfer function $\exp(-2s)$, an inner function. With measurements (2) we obtain a transfer function $1 - \exp(-2s)$, an outer function.

Furthermore, we show that a class of similar systems, but with position measurements

$$y(t) = B_o^* z(t) \tag{3}$$

is well-posed. These systems also have an outer transfer function. Our work is motivated by the finite-dimensional results in [9]. Note that these systems do not have positive real transfer functions, even in the finite-dimensional case.

A simple example of position measurements of a flexible beam at the end of this paper illustrates our results.

II. Framework

In common with [14], [15] we make the following assumptions throughout this paper

(A1) The stiffness operator $A_o : D(A_o) \subset H \to H$ is a self-adjoint, positive-definite linear operator on a Hilbert space H with $0 \in \rho(A_o)$. Since A_o is self-adjoint and positive definite, A_o^α is well-defined for $\alpha \geq 0$. A scale of Hilbert spaces H_α is defined as follows: For $\alpha \geq 0$, we define $H_\alpha = [D(A_o^\alpha)]$, and $H_{-\alpha} = H_\alpha^*$. Here the duality is taken with respect to the pivot space H, that is, equivalently $H_{-\alpha}$ is the completion of H with respect to the norm $\|z\|_{H_{-\alpha}} = \|A_o^{-\alpha} z\|_H$. Thus for $\alpha \in \mathbb{R}$, the operator A_o extends (restricts) to $A_o : H_\alpha \to H_{\alpha-1}$. We use the same notation A_o to denote this extension (restriction).

We denote the inner product on H by $\langle \cdot, \cdot \rangle_H$ or $\langle \cdot, \cdot \rangle$, and the duality pairing on $H_{-\alpha} \times H_\alpha$ by $\langle \cdot, \cdot \rangle_{H_{-\alpha} \times H_\alpha}$. Note that for $(z', z) \in H \times H_\alpha$, $\alpha > 0$, we have

$$\langle z', z \rangle_{H_{-\alpha} \times H_\alpha} = \langle z', z \rangle_H.$$

(A2 i) Let U be another Hilbert space. We identify U with its dual. The control operator B_o is a linear and bounded operator from U to $H_{-\frac{1}{2}}$.

(A2 ii) The damping operator $D : H_{\frac{1}{2}} \to H_{-\frac{1}{2}}$ is a self-adjoint linear operator satisfying

$$\langle Dz, z \rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}} \geq 0, \qquad z \in H_{\frac{1}{2}}.$$

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We note that the results of this paper can be extended to a more general situation where $y(t) := MB_o^* z(t)$ or $y(t) := MB_o^* \dot{z}(t)$. Here M is a linear bounded operator from U to another Hilbert space Y.

The position control system is equivalent to the following standard first-order equation

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{4}$$
$$y(t) = C_p x(t) \tag{5}$$

where $A : D(A) \subset H_{\frac{1}{2}} \times H \to H_{\frac{1}{2}} \times H$, $B : U \to H_{\frac{1}{2}} \times H_{-\frac{1}{2}}$ and $C_p : H_{\frac{1}{2}} \times H \to U$ are given by

$$A = \begin{bmatrix} 0 & I \\ -A_o & -D \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ B_o \end{bmatrix}, \quad C_p = \begin{bmatrix} B_o^* & 0 \end{bmatrix},$$

$$D(A) = \left\{ \left[\begin{smallmatrix} z \\ w \end{smallmatrix} \right] \in H_{\frac{1}{2}} \times H_{\frac{1}{2}} \mid A_o z + Dw \in H \right\}.$$

The velocity control system has a first-order form that is identical, except that $y(t) = C_v x(t)$, where $C_v : H_{\frac{1}{2}} \times H_{\frac{1}{2}} \to U$ is given by

$$C_v = \begin{bmatrix} 0 & B_o^* \end{bmatrix}.$$

Properties of the operator A

Since whether the operator generates a semigroup is independent of the control and observation operators, the following result is immediate.

Theorem 2.1: [14, Prop. 5.1] The operator A is the generator of a strongly continuous semigroup $T(t)$ of contractions on the state space $X = H_{\frac{1}{2}} \times H$.

This guarantees that the spectrum of A is contained in the closed right-half plane $\mathrm{Re}(s) \le 0$. For the main result of this paper it is required that there is no spectrum on the imaginary axis. This is implied by the following result.

Proposition 2.2: [15, Prop. 1.4] If there exists a constant $\beta > 0$ such that

$$\langle Dz, z \rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}} \ge \beta \|z\|_H^2, \qquad z \in H_{\frac{1}{2}}, \tag{6}$$

then A generates an exponentially stable semigroup on X.

Let $\sigma(A)$ indicate the spectrum of A. The *continuous spectrum*, $\sigma_c(A)$, is the set of all λ for which $\lambda I - A$ is injective, not surjective, but with range dense in X. The *residual spectrum*, $\sigma_r(A)$ consists of λ for which $\lambda I - A$ is injective, but the range of $\lambda I - A$ is not dense. We have

$$\sigma(A) = \sigma_p(A) \cup \sigma_c(A) \cup \sigma_r(A)$$

where $\sigma_p(A)$ indicates the point spectrum. The *approximate point spectrum*, $\sigma_a(A)$, consists of all λ for which there is a sequence $(x_n)_n$ in $D(A)$ such that

$$\|x_n\|_X = 1 \text{ and } \|(\lambda I - A)x_n\|_X \to 0.$$

(This is different from the definitions used in [15].)

Concerning the spectrum of A on the imaginary axis we know the following.

Theorem 2.3: If $i\eta \in \sigma(A)$, then $-i\eta \in \sigma(A)$, $\eta^2 \in \sigma(A_o)$, and $i\eta \in \sigma_a(A)$.

Moreover if, for non-zero $z \in H_{\frac{1}{2}}$,

$$\langle Dz, z \rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}} > 0, \tag{7}$$

then the operator A has no eigenvalues on the imaginary axis and every $i\eta \in \sigma(A)$ satisfies $i\eta \in \sigma_c(A)$.

Proof: This result was partly shown in [15]. We give a shorter proof here using Krein spaces. In [15, Proof of Lemma 4.5] it is shown that

$$A^* = JAJ, \qquad \text{with } J = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}.$$

If we define on X the inner product

$$[x, x] := \langle x, Jx \rangle_X, \qquad x \in X,$$

then $(X, [\cdot,\cdot])$ is a Krein space and A is a selfadjoint operator on the Krein space $(X, [\cdot,\cdot])$ (see Bognar [2] for more information). This fact implies immediately, that the spectrum of A is symmetric to the real axis ([2, page 133]), and thus $-i\eta \in \sigma(A)$. Since A generates a bounded C_0-semigroup, we have that $i\eta$ is an element of the boundary of $\sigma(A)$, which proves $i\eta \in \sigma_a(A)$, see Engel and Nagel [7].

We now assume that $\langle Dz, z \rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}}$ is positive for every non-zero $z \in H_{\frac{1}{2}}$. We assume that $i\eta \in \sigma_p(A)$. Then we can find an $x = \left(\begin{smallmatrix} z \\ w \end{smallmatrix} \right)$ such that

$$A \left(\begin{smallmatrix} z \\ w \end{smallmatrix} \right) = i\eta \left(\begin{smallmatrix} z \\ w \end{smallmatrix} \right) \Leftrightarrow w = i\eta z, \; A_o z + Dw = -i\eta w,$$

which is equivalent to $A_o z + i\eta Dz = \eta^2 z$. Taking the inner product with z we get

$$\langle A_o z + i\eta Dz, z \rangle = \eta^2 \|z\|^2.$$

This implies that $i\eta \langle Dz, z \rangle = \eta^2 \|z\|^2 - \|A_o^{\frac{1}{2}} z\|^2$. Since the left side of the equation is an imaginary number and the right side is a real number, we have $\eta = 0$ or $\langle Dz, z \rangle = 0$. In [14] it is shown that $0 \in \rho(A)$ and thus A has no eigenvalues on the imaginary axis. Finally, $i\eta \in \sigma_c(A)$ follows from [2, page 133]. □

Let $\mathbb{C}_\alpha := \{ s \in \mathbb{C} \mid \mathrm{Re}\, s > \alpha \}$ for $\alpha \in \mathbb{R}$. We conclude this subsection with the following theorem, which will be used in the next section.

Theorem 2.4: [14, Prop. 5.3] For every $s \in \rho(A)$,

1) $(sI - A)^{-1}$ is a bounded and invertible map from $H_{\frac{1}{2}} \times H_{-\frac{1}{2}}$ to $H_{\frac{1}{2}} \times H_{\frac{1}{2}}$.
2) The operator $s^2 I + Ds + A_o \in \mathcal{L}(H_{\frac{1}{2}}, H_{-\frac{1}{2}})$ has a bounded inverse $V(s) \in \mathcal{L}(H_{-\frac{1}{2}}, H_{\frac{1}{2}})$.
3) On $H_{\frac{1}{2}} \times H_{-\frac{1}{2}}$, for every non-zero $s \in \rho(A)$,

$$(sI - A)^{-1} = \begin{bmatrix} \frac{1}{s} [I - V(s) A_o] & V(s) \\ -V(s) A_o & sV(s) \end{bmatrix}. \tag{8}$$

The velocity measurement system

We now study the properties of the velocity measurement system, that is, we assume that the output is

$$y(t) = B_o^* \dot{z}(t) = C_v x(t), \tag{9}$$

where $C_v : H_{\frac{1}{2}} \times H_{\frac{1}{2}} \to U$ is given by

$$C_v = \begin{bmatrix} 0 & B_o^* \end{bmatrix}.$$

A linear control system is *well-posed* if the maps from initial condition $x(0)$ and control $u \in \mathcal{L}_2(0, T; U)$ are bounded on any finite-time interval. The operator B is *infinite-time admissible* if the state trajectory corresponding to $x(0)=0$ is uniformly bounded for any $u \in \mathcal{L}_2(0,\infty;U)$. Similarly, the operator C is *infinite-time admissible* if the output $y \in \mathcal{L}_2(0,\infty;U)$ for any initial condition and zero input. If for some $E \in \mathcal{L}(U,U)$, the transfer function satisfies

$$\lim_{s\to+\infty} G_p(s)x = Ex, \quad x \in X,$$

where X is the state space of the system, we say the system is *regular* with feedthrough operator E.

In order to show well-posedness of the control system on the state space X, we assume that the damping operator satisfies the following:

(A3) The damping operator $D : H_{\frac{1}{2}} \to H_{-\frac{1}{2}}$ is a self-adjoint linear operator satisfying

$$\langle Dz, z\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}} \geq \beta \|B_o^* z\|^2, \qquad z \in H_{\frac{1}{2}},$$

for some $\beta > 0$.

Proposition 2.5: If, in addition to the standard assumptions (A1)-(A2), (A3) also holds then

1) The control operator B is infinite-time admissible for the semigroup generated by A.
2) The observation operator C_v is infinite-time admissible for the semigroup generated by A.
3) The system (A, B, C_v) is well-posed.
4) The transfer function of the system (A, B, C_v) is given by $G_v(s) = sB_o^* V(s) B_o$ and satisfies $G_v \in H^\infty(\mathbb{C}_0, \mathcal{L}(U))$.

Proof: The proof of this Proposition uses the approach in [14]. Following the proof of Proposition 5.5 in [14] we obtain, for $u \in H^2(0,\infty;U)$ and $x(0) = z_o, \dot{z}(0) = w_o \in H_{\frac{1}{2}}$ satisfying

$$A_o z_o + D w_o - B_o u(0) \in H,$$

the inequality

$$\frac{1}{2}\frac{d}{dt}\left\|\begin{pmatrix} z(t) \\ \dot{z}(t)\end{pmatrix}\right\|^2 = -\langle D\dot{z}(t), \dot{z}(t)\rangle + \mathrm{Re}\langle B_o u(t), \dot{z}(t)\rangle.$$

Using (A3) and the standard inequality that, for any $\epsilon > 0$,

$$2\mathrm{Re}\langle a, b\rangle \leq \epsilon \|a\|^2 + \frac{1}{\epsilon}\|b\|^2,$$

we obtain

$$\frac{1}{2}\frac{d}{dt}\left\|\begin{pmatrix} z(t) \\ \dot{z}(t)\end{pmatrix}\right\|^2 \leq (-\beta + \frac{\epsilon}{2})\|B_o^* \dot{z}(t)\|^2 + \frac{1}{2\epsilon}\|u(t)\|^2.$$

Choosing $\epsilon < 2\beta$, there are constants $c_1, c_2 > 0$ such that

$$\frac{d}{dt}\left\|\begin{pmatrix} z(t) \\ \dot{z}(t)\end{pmatrix}\right\|^2 \leq c_1\|u(t)\|^2 - c_2\|B_o^* \dot{z}(t)\|^2.$$

Rearranging and writing $y(t) = B_o^* \dot{z}(t)$,

$$c_2\|y(t)\|^2 + \frac{d}{dt}\left\|\begin{pmatrix} z(t) \\ \dot{z}(t)\end{pmatrix}\right\|^2 \leq c_1\|u(t)\|^2. \qquad (10)$$

Integrating this inequality over time and using Theorem (2.1) implies that B and C_v are (infinite-time) admissible and that (A, B, C_v) is a well-posed linear system. This inequality also implies that the control system is L_2-stable and hence the transfer function G_v is in $H^\infty(\mathbb{C}_0, \mathcal{L}(U))$. Using Theorem 1.3 in [14] we see that the corresponding transfer function is given by $G_v(s) = sB_o^* V(s) B_o$. □

The position measurement system

We now study the properties of the position measurement system

$$y(t) = B_o^* z(t) = C_p x(t), \qquad (11)$$

where $C_p : H_{\frac{1}{2}} \times H \to U$ is given by

$$C_p = \begin{bmatrix} B_o^* & 0 \end{bmatrix}.$$

Proposition 2.6: If, in addition to the standard assumptions (A1)-(A2), (A3) also holds then

1) The observation operator C_p is a bounded operator from X to U and C_p is thus admissible for the semigroup generated by A.
2) The system (A, B, C_p) is regular with feedthrough 0.
3) The transfer function of the system (A, B, C_p) is given by $G_p(s) = B_o^* V(s) B_o$ and satisfies $G_p \in H^\infty(\mathbb{C}_0, \mathcal{L}(U))$.

Proof: The observation operator C_p is a bounded operator on the state space, and thus C_p is an admissible observation operator for the semigroup generated by A. In Proposition 2.5 we showed that B is an infinite-time admissible control operator for the semigroup generated by A. Thus (A, B, C_p) is a well-posed linear system. Using Proposition 2.4 we see that the corresponding transfer function is given by $G_p(s) = B_o^* V(s) B_o$. In Proposition 2.5 we proved that $G_v(s) = sG_p(s)$ is a bounded holomorphic function on the right half plane. Since $\mathbb{C}_0 \subset \rho(A)$, the transfer function $G_p(s)$ is an analytic function on the right half plane, bounded as $|s| \to \infty$, $\mathrm{Re}(s) > 0$. It remains to show that G_p is bounded on the right half plane. The function G_p has an analytic extension to a neighborhood of 0, since $0 \in \rho(A)$ (see [14]). Thus the boundedness of G_p on the right half plane follows from the fact that $sG_p(s)$ is bounded on the right half plane. Since $sG_p(s)$ is uniformly bounded in the right half plane,

$$\lim_{s\to+\infty} G_p(s) = 0,$$

and therefore (A, B, C_p) is a regular linear system with zero feedthrough. □

III. Minimum-Phase Behaviour

Throughout this section we assume that (A1)–(A3) are satisfied. We start with the following useful lemma.

Lemma 3.1: If $B_o \neq 0$, then $s^2 G_p(s) \not\to 0$ as s tends to infinity.

Proof: For $s > 0$ we define $X(s) := A_o^{-\frac{1}{2}}(s^2 I + sD + A_o)A_o^{-\frac{1}{2}}$. Then $X(s)$ is a linear, bounded operator on H (see Proposition 2.4), and it is easy to show that $X(s)$ is positive definite. Thus, there exists for every $s > 0$ a unique positive definite operator $Y(s)$ on H with

$$X(s) = Y(s)^2.$$

For $z \in H$ we have

$$\begin{aligned}\|Y(s)z\|^2 &= \langle z, X(s)z\rangle \\ &\leq \|X(s)\| \, \|z\|^2 \\ &\leq (\|A_o^{-1}\| s^2 + \|A_o^{-\frac{1}{2}} D A_o^{-\frac{1}{2}}\| s + 1)\|z\|^2.\end{aligned}$$

Now

$$\|G_p(s)\| = \sup_{\substack{u,\,v \in U,\\ \|u\| = \|v\| = 1}} |\langle v, B_o^* V(s) B_o u\rangle|,$$

and

$$\begin{aligned}|\langle v, B_o^* V(s) B_o u\rangle| &= |\langle A_o^{-\frac{1}{2}} B_o v, A_o^{\frac{1}{2}} V(s) B_o u\rangle| \\ &= ||\langle A_o^{-\frac{1}{2}} B_o v, A_o^{\frac{1}{2}} V(s) A_o^{\frac{1}{2}} \, A_o^{-\frac{1}{2}} B_o u\rangle| \\ &= |\langle A_o^{-\frac{1}{2}} B_o v, (X(s))^{-1} A_o^{-\frac{1}{2}} B_o u\rangle|.\end{aligned}$$

It follows that

$$\sup_{\substack{u,\,v \in U,\\ \|u\| = \|v\| = 1}} |\langle v, B_o^* V(s) B_o u\rangle| = \|(Y(s))^{-1} A_o^{-\frac{1}{2}} B_o\|^2$$

and so

$$\begin{aligned}&\|s^2 G_p(s)\| \\ &\geq (\|A_o^{-1}\| + \|A_o^{-\frac{1}{2}} D A_o^{-\frac{1}{2}}\| s^{-1} + s^{-2})^{-1} \|A_o^{-\frac{1}{2}} B_o\|^2.\end{aligned}$$

Thus $s^2 G_p(s) \not\to 0$ as s tends to infinity. □

In the following we assume that U is finite-dimensional, that is, $U = \mathbb{C}^m$. For $G : \mathbb{C} \to \mathcal{L}(U)$ the *normal rank* r of G is given by

$$r := \max_{s \in \mathbb{C}_0} \operatorname{rank} G(s).$$

Clearly $r \leq m$. In general $r = m$.

Definition 3.2: Let $G : \mathbb{C} \to \mathcal{L}(U)$. Then $z \in \mathbb{C}_0$ is a *transmission zero* if $G(z)$ has less than normal rank.

Lemma 3.3: Assume that the normal rank of the transfer function G_p is m. Then the transfer functions $G_p(s)$ and $G_v(s)$ have no transmission zeros in the open right half plane.

Proof: It is enough to show the statement for G_p. The proof is similar to that for finite-dimensional second-order systems in [9]. From Theorem (2.1), it follows that $G_p(s)$ is well-defined in the right half plane $\mathbb{C}_0$.

Suppose $s_0 \in \mathbb{C}_0$ is such that there exists $u_o \in U$ such that $G_p(s_0)u_o = 0$ or, from the representation above of $G_p(s)$,

$$B_o^* V(s_0) B_o u_o = 0.$$

Define $z_o = V(s_0)B_o u_o$. If $z_o = 0$ then $B_o u_o = 0$ and $G(s)u_o = 0$ for all s. Unless u_o is also zero, this contradicts the assumption that the normal rank of the G_p is the number of columns. Assume then $z_o \neq 0$. Noting that $z_o \in H_{\frac{1}{2}}$ we can write

$$\begin{aligned}(s_0^2 I + s_0 D + A_o) z_o - B_o u_o &= 0 \\ B_o^* z_o = 0\end{aligned}$$

where the first equation holds in $H_{-\frac{1}{2}}$ and the second in U. Thus,

$$\langle (s_0^2 I + s_0 D + A_o) z_o - B_o u_o, z_o\rangle_{H_{-\frac{1}{2}}, H_{\frac{1}{2}}} = 0.$$

Using $B_o^* z_o = 0$, this becomes

$$\langle (s_0^2 I + s_0 D + A_o) z_o, z_o\rangle_{H_{-\frac{1}{2}}, H_{\frac{1}{2}}} = 0. \tag{12}$$

Decompose s_0 into real and imaginary parts, $s_0 = \sigma + i\eta$ where $\sigma > 0$. The imaginary part of (12) is:

$$\eta \langle [2\sigma I + D] z_o, z_o\rangle = 0.$$

Since $z_o \neq 0$, this is only zero if $\eta = 0$. The real part of (12) is

$$\langle [(\sigma^2 - \eta^2) I + \sigma D + A_o] z_o, z_o\rangle = 0.$$

Since $\eta = 0$ then this equation is not satisfied for any non-zero z_0. Thus, $G(z)u_o = 0$ implies $u_o = 0$ and there are no transmission zero of G with positive real part. □

Next we show that the transfer function G_p has minimum-phase if $U = \mathbb{C}$. It is well-known that every bounded holomorphic function $f : \mathbb{C}_0 \to \mathbb{C}$ can be uniquely factored as $f(s) = j(s)h(s)$, where j is an *inner function*, that is, $|j(s)| \leq 1$ for $s \in \mathbb{C}_0$, and $|j(i\eta)| = 1$ for almost every $\eta \in \mathbb{R}$, and h is an *outer function*, that is,

$$h(s) := \exp\left[\frac{1}{\pi}\int_{-\infty}^{\infty} \log|f(it)| \frac{ts + i}{t + is} \frac{dt}{1 + t^2}\right]. \tag{13}$$

We note that $|h(i\eta)| = |f(i\eta)|$ for a.e. $\eta \in \mathbb{R}$, and that $|h(s)| \geq |f(s)|$ on $\mathbb{C}_0$. Moreover, an outer function has no zeros in the right half plane. For more results on the inner-outer factorization of bounded, holomorphic functions we refer the reader to [13].

We summarize some results on inner functions. Let $\{\beta_n\}_{n \in \mathbb{N}}$ be a sequence of points in $\mathbb{C}_0$ satisfying the *Blaschke condition*

$$\sum_{n=1}^{\infty} \frac{\operatorname{Re}\beta_n}{1 + |\beta_n|^2} < \infty. \tag{14}$$

Then the *Blaschke product* Θ corresponding to the sequence $\{\beta_n\}_{n \in \mathbb{N}}$ is given by

$$\Theta(s) = \prod_{n \in \mathbb{N}} \frac{|1 - \beta_n^2|}{1 - \beta_n^2} \frac{s - \beta_n}{s + \overline{\beta}_n}, \tag{15}$$

where $\frac{|1-\beta_n^2|}{1-\beta_n^2}$ is assumed to be 1 if $\beta_n = 1$. The function Θ is in $H_\infty(\mathbb{C}_0)$ and the zeros of Θ are precisely the points β_n, each zero having multiplicity equal to the number of times it occurs in the sequence. Moreover, $|\Theta(s)| \leq 1$ for all s with positive real part, and $|\Theta(i\eta)| = 1$ for almost all real η's. Thus every Blaschke product is an inner function. However, not every inner function can be written as a Blaschke product. Another class of inner functions are the singular functions. A *singular function* is a holomorphic function $S : \mathbb{C}_0 \to \mathbb{C}$ which can be written as

$$S(s) = e^{-\rho s} \exp\left[-\int_{\mathbb{R}} \frac{ts + i}{t + is} d\mu(t)\right], \tag{16}$$

where μ is a finite singular positive measure on $\mathbb{R}$ and ρ is a non-negative real number. Every inner function f can be written as

$$f(s) = e^{ia}\Theta(s)S(s), \tag{17}$$

where a is a real number, Θ is a Blaschke product and S is a singular function.

Theorem 3.4: Assume that the second-order system (1) satisfies assumptions (A1)-(A3) with $B_o \neq 0$. If in addition, the resolvent of A contains the imaginary axis, then $G_p(s)$ and $G_v(s)$ are outer functions times a constant of modulus one.

Proof: It is enough to show the result for $G_p(s)$. Above we have seen that $G_p(s)$ can be factored as $G_p(s) = e^{ia}\Theta(s)S(s)h(s)$, where a is a real number, $\Theta(s)$ is a Blaschke product, $S(s)$ is a singular function and $h(s)$ is an outer function. In Lemma 3.3 it was shown that G has no zeros in the right half plane, and so the Blaschke product is only 1. Lemma 3.1 implies that $\lim_{s\to\infty} s^2 G_p(s) \neq 0$, and thus the constant ρ in (16) is zero. It remains only to show that the measure μ in (16) is zero. However, since the resolvent set of A contains the imaginary axis, it is possible to extend the resolvent to an open subset of $\mathbb{C}$ containing the closed right half plane. Thus also the transfer function G_p has a continuation to a holomorphic function on Ω. (For more details concerning the continuation of transfer functions we refer the reader to [16].) Since the transfer function has this analytic continuation [13, page 142] implies that $\mu = 0$. There is therefore no singular part to the transfer function. This shows that the transfer function of G_p is an outer function times a constant of modulus 1. □

In particular, if A generates an exponentially stable C_0-semigroup and $B_0 \neq 0$ then the transfer function is an outer function times a constant of modulus one.

IV. Example

Consider a Euler-Bernoulli beam of unit length pinned at each end and let $z(r,t)$ denote the deflection of the beam from its rigid body motion at time t and position r. Use of the Kelvin-Voigt damping model leads to the following description of the beam vibrations:

$$\frac{\partial^2 z}{\partial t^2} + \frac{\partial^2}{\partial r^2}\left[E\frac{\partial^2 z}{\partial r^2} + C_d\frac{\partial^3 z}{\partial r^2 \partial t}\right] = 0, \qquad 0 < r < 1.$$

Here E and C_d are positive physical constants. The beam is pinned at each end so

$$\begin{aligned} z(0,t) &= 0, \ \left[E\tfrac{\partial^2 z}{\partial r^2} + C_d\tfrac{\partial^3 z}{\partial r^2\partial t}\right]_{r=0} &= 0, \\ z(1,t) &= 0, \ \left[E\tfrac{\partial^2 z}{\partial r^2} + C_d\tfrac{\partial^3 z}{\partial r^2\partial t}\right]_{r=1} &= 0. \end{aligned}$$

A force u is applied at some point ξ, $0 < \xi < 1$, with position measurement at the same point:

$$\begin{aligned} \left[E\frac{\partial^3 z}{\partial r^3} + C_d\frac{\partial^4 z}{\partial r^3 \partial t}\right]_{r=\xi} &= u(t), \\ z(\xi,t) &= y(t). \end{aligned}$$

Let $x(t) = (z(\cdot,t), \dot{z}(\cdot,t))$.

We will put this control system into the framework of this paper. Here H is $\mathcal{L}_2(0,1)$ and $A_o = E\frac{d^4}{dr^4}$ with $D(A_o)$ given by

$$\left\{ z \in \mathcal{H}^4(0,1) : z(0) = \frac{d^2 z}{dr^2}(0) = z(1) = \frac{d^2 z}{dr^2}(1) = 0 \right\}.$$

Also,

$$H_{\frac{1}{2}} = \left\{ z \in \mathcal{H}^2(0,1) : z(0) = z(1) = 0 \right\}$$

with inner product $\langle z, v\rangle_{\frac{1}{2}} = E\langle z'', v''\rangle$ and $X = H_{\frac{1}{2}} \times L^2(0,1)$. The damping operator $D : H_{\frac{1}{2}} \to H_{-\frac{1}{2}}$ is defined by

$$\langle Dz, \phi\rangle_{H_{-\frac{1}{2}}, H_{\frac{1}{2}}} = \frac{C_d}{E}\langle z, \phi\rangle_{\frac{1}{2}}$$

for $z, \phi \in H_{\frac{1}{2}}$. The operator $B_o = \delta_\xi$ and $C_o = B_o^* z = z(\xi)$. Sobolev's Inequality implies that evaluation at a point is bounded on $H_{\frac{1}{2}}$ and so the output operator C_o is bounded from $H_{\frac{1}{2}}$ to $\mathbb{R}$. Assumptions (A1)-(A2) are satisfied. Notice that the damping and control in this example is not included in the special class covered in [14], [15].

The inequality

$$\langle Dz, z\rangle_{-\frac{1}{2},\frac{1}{2}} \geq \frac{C_d}{E}|z|^2_{\frac{1}{2}} \geq \alpha |z|^2$$

for positive constant α, implies the well-known result that A generates an exponentially stable analytic semigroup on X. Furthermore, for $z \in H_{\frac{1}{2}}$,

$$\begin{aligned} \langle Dz, z\rangle &= \frac{C_d}{E}|z|^2_{\frac{1}{2}} \\ &\geq \beta |z(\xi)|^2 \\ &= \beta |B_o^* z|^2 \end{aligned}$$

for some $\beta > 0$ by Sobolev's Inequality. Thus (A3) is also satisfied, implying well-posedness of the control system (Prop. 2.6)). Theorem 3.4 implies that the transfer function is an outer function.

If the position measurement is replaced by velocity measurement, the same conclusions hold.

For this simple example, the conclusion of no zeros in the right half plane could be seen by analysis of the transfer function, although this is not straightforward. Determining that there is no part $e^{-\rho s}$ is more difficult. However, the main advantage of the results in this paper is that they can be applied to vibrations on general domains. They also apply to wave problems such as [14, sect. 7],[15, sect. 5.2].

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