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Feedback-Invariant Subspaces in Infinite-Dimensional Systems

Kirsten Morris and Richard Rebarber

***Abstract*— We consider single-input single-output systems on a Hilbert space X, with infinitesimal generator A, bounded control element b, and bounded observation element c. Let $c^\perp$ be the subspace of X perpendicular to c. We consider the problem of finding the largest feedback-invariant subspace of $c^\perp$. If b is in $c^\perp$, and $c \notin D(A^*)$, a largest feedback-invariant subspace does not exist in general.**

I. INTRODUCTION

A subspace V is invariant for a linear system if for all initial conditions in V there exists a control that keeps the state in V for all times. If this is the case, the control can be a constant state feedback. Let V^* be the largest feedback invariant subspace. The zeros of the original system are the eigenvalues of the controlled system restricted to V^*. Furthermore, a disturbance can be decoupled from the output if and only if it lies inside a feedback invariant subspace contained in the kernel of the observation operator [14].

In this paper we consider feedback invariance for single-input single-output infinite-dimensional systems with bounded control and observation. Let X be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let A be the infinitesimal generator of a C_0-semigroup $T(t)$ on X. Let b and c be elements of X. Let $U = Y = \mathbb{C}$ and $u(t) \in U$. We consider the following system in X:

$$\dot{x}(t) = Ax(t) + bu(t) \tag{1.1}$$

with the observation

$$y(t) = Cx(t) := \langle x(t), c \rangle. \tag{1.2}$$

We sometimes refer to this system as (A, b, c). The transfer function is $G(s)$ where $G(s) = \langle R(s, A)b, c \rangle$.

We denote the kernel of C by

$$c^\perp := \{x \in X \mid \langle x, c \rangle = 0\}.$$

When $b \notin c^\perp$, we show that the largest feedback-invariant subspace in $c^\perp$ exists, and is $c^\perp$ itself. We give an explicit representation of a feedback operator K for which $c^\perp$ is $A + bK$-invariant. When $c \notin D(A^*)$, the operator K is not bounded, so semigroup generation of $A + bK$ is not guaranteed.

If $\langle b, c \rangle = 0$ then the theory is quite different. A number of situations may occur, depending on the nature of b and c. In particular, if $c \notin D(A^*)$, then in general no largest feedback-invariant subspace exists. This is in contrast to the finite-dimensional case, where a largest feedback invariant subspace always exists [14]. However, as in the finite-dimensional case, the spectrum of $A + bK$ is identical to the invariant zeros of the system.

This work builds on the results of Curtain and Zwart in the 1980's, see [3], [16], [17], [18]. In [16], [17] there is a standing assumption that (A, b) is such that $A + bK$ is a generator of a C_0-semigroup for any A-bounded K, which is a strong restriction on b. This paper also extends the results in [1], where it is assumed that $b \in D(A)$, $c \in D(A^*)$ and $\langle b, c \rangle \neq 0$. We remove the restrictions $b \in D(A)$ and $c \in D(A^*)$, and, most significantly, also examine the case where $\langle b, c \rangle = 0$.

We should note that even though in most infinite-dimensional systems analysis the assumption that b and c are in X makes the analysis easier, the zeros for partial differential equations with boundary control and observation (which yields unbounded control and observation operators) is often more easily analyzed, see [11].

II. INVARIANCE CONCEPTS

For $\omega \in \mathbb{R}$, let

$$C_\omega = \{z \in \mathbb{C} \mid \mathrm{Re}\ z > \omega\}.$$

Let $R(s, A) = (sI - A)^{-1}$, and let $\omega \in \mathbb{R}$ be such that $\mathbb{C}_\omega$ is a subset of $\rho(A)$. For $\lambda_0 > \omega$, $R(\lambda_0, A)$ exists as a bounded operator from X into X.

Definition 2.1: A subspace Z of X is *feedback invariant* if it is closed and there exists an A-bounded feedback K such that $(A + bK)(Z \cap D(A)) \subset Z$.

The operator K is not specified as unique in the above theorem. However, if $b \notin Z$, and there are two operators K_1 and K_2 that are both (A, b)-invariant on Z, then $b(K_1 x - K_2 x) \in Z$ and so $K_1 x = K_2 x$ for all $x \in Z$.

The following result shows that feedback invariance is equivalent to (A, b)-invariance, which is sometimes easier to work with.

Theorem 2.2: [17, Thm.II.26] A closed subspace Z is feedback-invariant if and only if it is (A, b)-invariant, that is,

$$A(Z \cap D(A)) \subseteq Z + \mathrm{span}\{b\}.$$

Theorem 2.3: If $Z \subseteq c^\perp$ is a feedback-invariant subspace and $b \in Z$ then the system transfer function is identically zero.

Proof: Since Z is feedback-invariant,

$$A(Z \cap D(A)) \subset Z + \mathrm{span}\{b\} \subset Z.$$

This implies that Z is A-invariant. This implies that every $z \in Z$ can be written $z = (sI - A)\xi(s)$ where $\xi(s) \in D(A) \cap$

Kirsten Morris is with Department of Applied Mathematics, University of Waterloo, Waterloo, CANADA. kmorris@uwaterloo.ca

Richard Rebarber is with the Department of Mathematics, University of Nebraska, Lincoln, USA. rrebarbe@math.unl.edu

Z [17, Lem. I.4], and $s \in [r, \infty)$ for some $r \in \mathbb{R}$. Since $b \in Z$, $(sI - A)^{-1} b \in Z$ for all $s \in [r, \infty)$. Since $Z \subset c^{\perp}$, the system transfer function $G(s)$ is zero for $s \in [r, \infty]$. Since G is analytic on $\rho(A)$, it must be identically zero on $\rho(A)$. $\square$

III. NICE CASES

If $b \notin c^{\perp}$, the largest feedback-invariant subspace contained in $c^{\perp}$ is $c^{\perp}$.

Theorem 3.1: [9] Suppose $\langle b, c\rangle \neq 0$. Define

$$Kx = -\frac{\langle Ax, c\rangle}{\langle b, c\rangle}, \qquad D(K) = D(A), \tag{3.3}$$

and define $(A + bK)x = Ax + bKx$ for $x \in D(A + bK) = D(A)$. Then $(A + bK)(c^{\perp} \cap D(A)) \subset c^{\perp}$ and so the largest feedback-invariant subspace in $c^{\perp}$ is $c^{\perp}$ itself.

Definition 3.2: A closed subspace Z of X is *closed-loop invariant* if the closure of $Z \cap D(A)$ in X is Z and there exists an A-bounded feedback K such that $(A+bK)(Z \cap D(A)) \subseteq Z$ and $A + bK$ generates a C_0 semigroup T_K on Z.

The condition that $(A + bK)(Z \cap D(A)) \subset Z$ allows arbitrary elements of $X \setminus D(A)$ to be appended to Z. The additional condition that the closure of $Z \cap D(A)$ is Z eliminates this ambiguity.

In general, $A + bK$ does not generate a C_0-semigroup. In this case $c^{\perp}$ is not closed-loop invariant.

There are many results in the literature that give sufficient conditions for a relatively bounded perturbation of a generator of a C_0-semigroup to be the generator of a C_0-semigroup. For instance, if K is an admissible output element [12, Chap. 5], or if A generates an analytic semigroup [7, Chap. 9, sect. 2.2], then $A + bK$ generates a C_0 semigroup.

Theorem 3.3: [9] In addition to the assumptions of Theorem 3.1, assume that $A + bK$ generates a C_0-semigroup on X. Then it generates a C_0-semigroup on $c^{\perp}$, hence $c^{\perp}$ is closed-loop invariant under $A + bK$.

If $\langle b, c\rangle = 0$, we can still find the largest feedback-invariant subspace in many cases.

We first give a definition of the *relative degree* of (A, b, c), which is a generalization of the standard finite dimensional definition, see for example [5, pg. 99].

Definition 3.4: (A, b, c) is of relative degree $n \in \mathbb{Z}^+$ if

1) the function $(s^n G(s))^{-1}$ is in $H^{\infty}_{\gamma}(\mathbb{C})$ for some $\gamma \in \mathbb{R}$;
2) $\lim_{s\to\infty,\ s\in\mathbb{R}} s^j G(s) = 0$ for $j = 1, 2, \ldots (n - 1)$.

In finite dimensions condition (1) in Definition 3.4 is equivalent to

$$\lim_{s\to\infty,\ s\in\mathbb{R}} s^n G(s) \neq 0.$$

The above definition of relative degree seems to be the most general definition for infinite dimensional systems that guarantees some (limited) regularity of closed loop solutions, see [9].

Define

$$Z_n = c^{\perp} \cap (A^* c)^{\perp} \cap \cdots (A^{*n} c)^{\perp}.$$

The existence of a largest feedback invariant subspace depends on whether $c \in D(A^{*n})$, where $n + 1$ is the relative degree of the system.

Theorem 3.5: [9] Suppose $n \in Z^+$ is such that

$$c \in D(A^{*n}), \qquad b \in Z_{n-1} \tag{3.4}$$

and

$$\langle b, A^{*n} c\rangle \neq 0. \tag{3.5}$$

Then the largest feedback-invariant subspace Z in $c^{\perp}$ is Z_n.

We can use this to prove the following:

Theorem 3.6: Suppose $n \in \mathbb{Z}^+ \cup \{0\}$ is such that (A, b, c) is of relative degree $n+1$ and $c \in D(A^{*n})$. Then the largest feedback-invariant subspace Z in $c^{\perp}$ is Z_n.

Closed-loop invariance of Z_n exists under conditions similar to those for the case $\langle b, c\rangle \neq 0$. That is, if Z_n is feedback-invariant under the operator $A+bK_n$, and $A+bK_n$ generates a C_o-semigroup on the original space X, then Z_n is also closed-loop invariant [9].

IV. NOT SO NICE CASE

The following example illustrates that if $\langle b, c\rangle = 0$ and $c \notin D(A^*)$ a largest feedback-invariant subspace as defined in Definition 2.1 might not exist.

Example IV.1. The following example of a controlled delay equation first appeared in Pandolfi [10]:

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) - x_2(t-1) \\ \dot{x}_2(t) &= u(t) \\ y(t) &= x_1(t). \end{aligned} \tag{4.6}$$

The transfer function for this system is

$$G(s) = \frac{1 - e^{-s}}{s^2}. \tag{4.7}$$

The system of equations (4.6) can be written in a standard state-space form (1.1, 1.2), see [4]. Choose the state-space

$$X = R^2 \times L_2(-1, 0) \times L_2(-1, 0).$$

A state-space realization on X is

$$b = \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix}, \qquad c = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}.$$

Define $D(A)$ to be $[r_1, r_2, \phi_1, \phi_2]^T \in X$ such that

$$\phi_1(0) = r_1, \phi_2(0) = r_2, \phi_1 \in H^1(-1, 0), \phi_2 \in H^1(-1, 0).$$

For $[r_1, r_2, \phi_1, \phi_2]^T \in D(A)$,

$$A(r_1, r_2, \phi_1, \phi_2) = \begin{pmatrix} \phi_2(t) - \phi_2(t-1) \\ 0 \\ \dot{\phi}_1 \\ \dot{\phi}_2 \end{pmatrix}.$$

In this example $\langle b, c\rangle = 0$ and $c \notin D(A^*)$. From the transfer function (4.7) we can see that the system has relative degree 2.

Pandolfi [10] showed that the largest feedback-invariant subspace $Z \subset c^{\perp}$, if it exists, is not a delay system. We now show that this system does not have a largest feedback-invariant subspace in $c^\perp$. Define

$$e_k = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \exp(2\pi ikt) \end{bmatrix} \in D(A) \cap c^\perp.$$

For each k the subspace $\mathrm{span}\{e_k\}$ is (A, b)-invariant and hence feedback-invariant (Thm. 2.2). Define

$$V_n = \mathrm{span}_{-n\le k\le n} e_k.$$

Each subspace V_n is feedback-invariant. Define also the union of all finite linear combinations of e_k,

$$V = \bigcup V_n.$$

By well-known properties of the exponentials $\{e^{2\pi ikt}\}_{k=1}^{\infty}$ in $L^2(0,1)$, the closure of $\{\exp(2\pi ikt)\}$ is $L^2(0,1)$. Consider a sequence of elements in V, $[0,1,0,z_n]$ where $z_n(0) = 1$ and $\lim_{n\to\infty} z_n = 0$. This sequence converges to $[0,1,0,0]$ and so we see that the closure of V in X is $\bar{V} = 0 \times R \times 0 \times L_2(-1,0)$. If there is a largest feedback-invariant subspace Z in $c^\perp$, then $Z \supset \bar{V}$. The important point now is that although $b \notin V$, $b \in \bar{V}$. Since b cannot be contained in any feedback invariant subspace (Theorem 2.3), $\bar{V}$ is not feedback-invariant. Hence no largest feedback-invariant subspace exists for this system. □

Assume $\langle b, c\rangle = 0$. Theorem 2.2 implies that any element $x \in D(A)$ of an (A, b)-invariant subspace of $c^\perp$ is contained in the set

$$Z = \{z \in c^\perp \cap D(A) \mid \langle Az, c\rangle = 0\}. \qquad (4.8)$$

The closure of Z is a natural candidate for the largest feedback-invariant subspace of $c^\perp$. When $c \in D(A^*)$, the closure of Z is $Z_1 = c^\perp \cap (A^*c)^\perp$. If $\langle b, A^*c\rangle \neq 0$, this is the largest feedback-invariant subspace in $c^\perp$ (Thm. 3.6). The situation when $c \notin D(A^*)$ is quite different.

Theorem 4.1: If $c \notin D(A^*)$, the set Z is dense in $c^\perp$. Furthermore, $Z \neq c^\perp \cap D(A)$.

Proof: This will be proven by showing that if Z is not dense in $c^\perp$ then $c \in D(A^*)$. Let $\lambda \in \rho(A)$ and $A_\lambda = A - \lambda I$, so $D(A_\lambda) = D(A)$. $D(A)$ is a Hilbert space with the graph norm, and the graph norm is equivalent to

$$\|x\|_1 := \|A_\lambda x\|. \qquad (4.9)$$

The corresponding inner product on $D(A)$ is

$$\langle x, y\rangle_1 := \langle A_\lambda x, A_\lambda y\rangle. \qquad (4.10)$$

Define $e = (A_\lambda{}^*)^{-1}c \in X$. For $x \in D(A)$, the condition $\langle c, x\rangle = 0$ can be written

$$0 = \langle x, c\rangle = \langle A_\lambda x, e\rangle = \langle A_\lambda x, A_\lambda A_\lambda{}^{-1}e\rangle = \langle x, A_\lambda{}^{-1}e\rangle_1. \qquad (4.11)$$

For $x \in c^\perp \cap D(A_\lambda)$, the condition $\langle Ax, c\rangle = 0$ is equivalent to $\langle A_\lambda x, c\rangle = 0$. Hence for such x we have

$$0 = \langle A_\lambda x, c\rangle = \langle A_\lambda x, A_\lambda A_\lambda{}^{-1}c\rangle = \langle x, A_\lambda{}^{-1}c\rangle_1. \qquad (4.12)$$

We can write Z as

$$\left\{x \in D(A) | \ \langle x, A_\lambda{}^{-1}e\rangle_1 = 0 \text{ and } \langle x, A_\lambda{}^{-1}c\rangle_1 = 0\right\}.$$

We now introduce the notation

$$(y)_1^\perp := \{x \in D(A) \mid \langle x, y\rangle_1 = 0\}.$$

Using this notation,

$$Z = (A_\lambda{}^{-1}e)_1^\perp \cap (A_\lambda{}^{-1}c)_1^\perp.$$

Now suppose that Z is not dense in $c^\perp$ (as a subspace of X). Then there exists $v \in c^\perp$ such that $\langle x, v\rangle = 0$ for all $x \in Z$. Define $w = (A_\lambda{}^*)^{-1}v$. As in (4.11), for $x \in D(A)$, the condition $\langle x, v\rangle = 0$ is equivalent to

$$\langle x, A_\lambda{}^{-1}w\rangle_1 = 0. \qquad (4.13)$$

Hence we see that

$$Z \subseteq (A_\lambda{}^{-1}e)_1^\perp \cap (A_\lambda{}^{-1}w)_1^\perp. \qquad (4.14)$$

Let R be the orthogonal projection from $D(A)$ onto $(A_\lambda{}^{-1}e)_1^\perp$ (using the inner product $\langle\cdot,\cdot\rangle_1$). Then

$$Z = (A_\lambda{}^{-1}e)_1^\perp \cap (RA_\lambda{}^{-1}c)_1^\perp$$

and

$$(A_\lambda{}^{-1}e)_1^\perp \cap (A_\lambda{}^{-1}w)_1^\perp = (A_\lambda{}^{-1}e)_1^\perp \cap (RA_\lambda{}^{-1}w)_1^\perp.$$

Hence (4.14) becomes

$$(A_\lambda{}^{-1}e)_1^\perp \cap (RA_\lambda{}^{-1}c)_1^\perp \subseteq (A_\lambda{}^{-1}e)_1^\perp \cap (RA_\lambda{}^{-1}w)_1^\perp. \qquad (4.15)$$

This implies that there is a scalar γ such that

$$RA_\lambda{}^{-1}c = \gamma RA_\lambda{}^{-1}w.$$

We obtain that

$$A_\lambda{}^{-1}c = \alpha A_\lambda{}^{-1}w + \beta A_\lambda{}^{-1}e.$$

Applying A_λ to both sides of this equation,

$$c = \alpha w + \beta e.$$

Since $w = (A_\lambda{}^*)^{-1}v$ and $e = (A_\lambda{}^*)^{-1}c$, we see that $c \in D(A_\lambda{}^*) = D(A^*)$. Thus, if Z is not dense in $c^\perp$ in $c^\perp$ then $c \in D(A^*)$.

Now assume that $Z = c^\perp \cap D(A)$. Then $(A_\lambda{}^{-1}e)_1^\perp \cap (A_\lambda{}^{-1}c)_1^\perp = (A_\lambda{}^{-1}e)_1^\perp$, so, as above, $c = \beta e$, which would imply that $c \in D(A^*)$. □

Corollary 4.2: Suppose that $q \in X$ and $c \notin D(A^*)$. Then $q^\perp \cap Z$ is dense in $q^\perp \cap c^\perp$. Furthermore, $q^\perp \cap Z \neq q^\perp \cap c^\perp \cap D(A)$.

Proof: If $q = \lambda c$ for some scalar λ, then $q^\perp \cap Z = Z$ and $q^\perp \cap c^\perp = c^\perp$, and the result follows immediately from Theorem 4.1.

Assume now that q is not parallel to c. Let P be the orthogonal projection of X onto $c^\perp$, and $\tilde{q} = Pq$, so $\tilde{q} \neq 0$. Let $\tilde{X} = \tilde{q}^\perp$, and let Q be the orthogonal projection of X onto $\tilde{q}^\perp$. By construction, $c = Qc \in \tilde{X}$. Let

$$\begin{aligned} \tilde{A} &= QA|_{\tilde{X}}, \ D(\tilde{A}) = D(A) \cap \tilde{X}, \\ \tilde{Z} &= \{x \in D(\tilde{A}) \mid \langle x, c\rangle = 0 \text{ and } \langle \tilde{A}x, c\rangle = 0\}. \end{aligned}$$

We wish to show that $c \notin D(\tilde{A}^*)$. Note that for $x \in \tilde{X}$,

$$\langle \tilde{A}x, c\rangle = \langle \tilde{Q}Ax, c\rangle = \langle Ax, Qc\rangle = \langle Ax, c\rangle. \tag{4.16}$$

Therefore $c \notin D(A^*)$ if the functional $x \to \langle Ax, c\rangle$ is unbounded on $\tilde{X}$. To show this let $b_0 \in D(A) \cap \tilde{X}$ and let Q_0 be the (possibly not orthogonal) projection onto $\tilde{X}$ given by

$$Q_0 x = x - \frac{\langle x, \tilde{q}\rangle}{\langle q_0, \tilde{q}\rangle} q_0.$$

Then $\langle Ax, c\rangle$ is unbounded on $\tilde{X}$ if $\langle AQ_0 x, c\rangle$ is unbounded on X. Since

$$\langle AQ_0 x, c\rangle = \langle Ax, c\rangle - \frac{\langle x, \tilde{q}\rangle}{\langle q_0, \tilde{q}\rangle}\langle Aq_0, c\rangle.$$

The second term on the right is clearly bounded on X, and the first term on the right is unbounded on X since $c \notin D(A^*)$, so $\langle AQ_0 x, c\rangle$ is not a bounded operator on X, hence $c \notin D(\tilde{A}^*)$.

Now we can apply Theorem 4.1 to $\tilde{X}$, $\tilde{A}$, c and $\tilde{Z}$ and conclude that $\tilde{X} \cap \tilde{Z}$ is dense in $\tilde{X} \cap c^\perp$ and $\tilde{X} \cap \tilde{Z} \neq \tilde{X} \cap c^\perp \cap D(A)$.

For $x \in c^\perp$, $\langle x, Pq\rangle = \langle x, q\rangle$ and so

$$\begin{aligned} \tilde{X} \cap c^\perp &= \{x \in X \mid \langle x, c\rangle = 0,\ \langle x, Pq\rangle = 0\} \\ &= \{x \in X \mid \langle x, c\rangle = 0,\ \langle x, q\rangle = 0\} \\ &= q^\perp \cap c^\perp. \end{aligned}$$

Similarly,

$$\tilde{X} \cap \tilde{Z} = \{x \in D(A) \mid \langle x, c\rangle = 0,\ \langle x, q\rangle = 0,\ \langle \tilde{A}x, c\rangle = 0\}. \tag{4.17}$$

This can be written

$$\begin{aligned} \tilde{X} \cap \tilde{Z} &= \{x \in D(A) \mid \langle x, c\rangle = 0, \langle x, q\rangle = 0, \langle Ax, c\rangle = 0\} \\ &= q^\perp \cap Z. \end{aligned}$$

Thus we have shown that $q^\perp \cap Z$ is dense in $q^\perp \cap c^\perp$, and that the two spaces are not equal. □

If $\langle b, c\rangle = 0$, $c \in D(A^*)$, and $\langle b, A^*c\rangle \neq 0$, the largest invariant subspace in $c^\perp$ is $Z_1 = c^\perp \cap (A^*c)^\perp$. Defining $\alpha = \frac{-1}{\langle b, A^*c\rangle}$,

$$\begin{aligned} A + bK &= A + \alpha b\langle Ax, A^*c\rangle, \quad \text{with} \\ D(A + bK) &= \{z \in c^\perp \cap D(A) \mid \langle Az, c\rangle = 0\}, \end{aligned}$$

is Z_1-invariant. In many cases, this operator generates a C_0-semigroup on Z_1. It is tempting to hope, that even if $c \notin D(A^*)$, the operator (with some value of α)

$$\begin{aligned} A + bK &= A + \alpha b\langle A^2 x, c\rangle, \\ D(A + bK) &= \{z \in c^\perp \cap D(A^2) \mid \langle Az, c\rangle = 0\} \end{aligned}$$

is a generator, or has an extension which is a generator. However, we see from the next result that this operator is not closable, so that no extension of it is a generator of a C_0-semigroup.

Theorem 4.3: Suppose $b \in X$ and $c \notin D(A^*)$. Then the operator

$$\begin{aligned} A_F x &= Ax + b\langle A^2 x, c\rangle, \\ D(A_F) &= \{x \in c^\perp \cap D(A^2) \mid \langle Ax, c\rangle = 0\} \end{aligned}$$

is not closable.

Proof: Let $\lambda \in \rho(A)$ and $A_\lambda = A - \lambda I$, as above. From Corollary 4.2 we see that $((A_\lambda^{-1})^* c)^\perp \cap Z$ is dense in $((A_\lambda^{-1})^* c)^\perp \cap c^\perp$. Let

$$Tx := \langle A_\lambda x, c\rangle, \qquad D(T) = ((A_\lambda^{-1})^* c)^\perp \cap c^\perp \cap D(A).$$

We will now show that T is not closable. From Corollary 4.2, $((A_\lambda^{-1})^* c)^\perp \cap Z \neq D(T)$. Thus we can choose $f \in D(T)$ such that $f \notin ((A_\lambda^{-1})^* c)^\perp \cap Z$, and there exists $(f_n) \subset ((A_\lambda^{-1})^* c)^\perp \cap Z$ such that $\lim f_n = f$. From the definition of Z, $Tf_n = 0$ for all n. Let $x_n = f - f_n$, so

$$\lim x_n = 0, \text{ and } \lim Tx_n = Tf \neq 0, \tag{4.18}$$

which shows that T is not closable [15, Section II.6, Proposition 2]. It then follows that $I + bT$ with domain $D(T)$ is not closable.

Now note that $y \in D(A_F)$ if and only if $A_\lambda y \in D(T)$, and that for $y \in D(A_F)$

$$A_F y = (I + bT)A_\lambda y + \lambda y,$$

so A_F is closable if and only if $(I + bT)A_\lambda$ is closable. Using the sequence $(x_n) \subset D(T)$ defined above, define $y_n = A_\lambda^{-1} x_n$. Note that $(y_n) \subset D(A_F)$ and

$$\lim y_n = 0 \text{ and } \lim (I + bT)A_\lambda y_n = bTf \neq 0.$$

Hence $(I + bT)A_\lambda$ is not closable, so A_F is not closable. □

Definition 4.4: The *invariant zeros* of (1.1), (1.2) are the set of all λ such that

$$\begin{bmatrix} \lambda I - A & b \\ C & 0 \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{4.19}$$

has a solution for $u \in U$ and non-zero $x \in D(A)$.

One of the important properties of a largest invariant subspace, is the following well-known result. A proof for infinite-dimensional system can be found in, for instance, [9].

Theorem 4.5: Assume a largest feedback-invariant subspace Z of (A, b, c) exists and $G(s)$ is not identically zero, and let K be an operator such that $A + bK$ is Z-invariant. Then the eigenvalues of $(A + bK)|_Z$ are the invariant zeros of the system.

We now show that, for a large class of relative degree 2 systems we can find a feedback K and a subspace of $c^\perp$ that is $(A + bK)$-invariant. In general, such a $A + bK$ is not closable on the original Hilbert space, hence does not generate a C_0-semigroup in the original norm. However, the spectrum of $A + bK$ does yield the invariant zeros. In order to define this space we need to extend $\langle A\cdot, c\rangle$ to a larger set than $D(A)$. Define

$$C_A x = \lim_{s \to \infty, s \in \mathbb{R}} \langle sAR(s, A)x, c\rangle \tag{4.20}$$

with domain

$$D(C_A) = \{x \in X \mid \lim_{s \to \infty, s \in \mathbb{R}} \langle sAR(s, A)x, c\rangle \text{ exists}\}.$$

(This is the same as $(CA)_L$ where the L-extension is given by [13, Defn. 5.6].) It is straightforward to verify that $D(C_A) \supseteq D(A)$. If $x \in D(A)$, then $C_A(x) = \langle Ax, c\rangle$. Also, if $c \in D(A^*)$, then $D(C_A) = X$ and $C_A x = \langle x, A^* c\rangle$.

Proposition 4.6: Assume that (A, b, c) has relative degree at least 2. Then $\lim_{s\to\infty} s^2 G(s)$ exists for real s if and only if $b \in D(C_A)$. In this case,

$$\lim_{s\to\infty} s^2 G(s) = C_A b. \tag{4.21}$$

Proof: First note that since the relative degree of the systems is at least 2, $\lim_{s\to\infty} sG(s) = 0$. But,

$$\lim_{s\to\infty} sG(s) = \lim_{s\to\infty} \langle s(sI - A)^{-1} b, c\rangle = \langle b, c\rangle$$

and so $\langle b, c\rangle = 0$. Since

$$s^2 G(s) = \langle s(sI - A)(sI - A)^{-1} b, c\rangle + \langle sA(sI - A)^{-1} b, c\rangle,$$

we obtain

$$\begin{aligned} \lim_{s\to\infty} s^2 G(s) &= \lim_{s\to\infty} s\langle b, c\rangle + \lim_{s\to\infty} \langle sA(sI - A)^{-1} b, c\rangle \\ &= \lim_{s\to\infty} \langle sA(sI - A)^{-1} b, c\rangle. \end{aligned}$$

The result follows. □

Using the operator C_A, the space Z defined above in (4.8) can be extended to

$$Z_A = \{x \in c^\perp \cap D(C_A) | C_A x = 0\}.$$

If $c \in D(A^*)$, then $Z_A = Z_1$.

The following theorem is now straightforward, so we omit the proof.

Theorem 4.7: Assume that a system (A, b, c) has relative degree 2 and $\lim_{s\to\infty} s^2 G(s)$ exists. Define on $c^\perp$

$$A_K x = Ax + bKx, \tag{4.22}$$

where

$$Kx = -\frac{C_A(Ax)}{C_A b} \tag{4.23}$$

with domain

$$D(A_K) = \{x \in D(A) \cap c^\perp | Ax \in D(C_A), C_A x = 0\}.$$

The space Z_A is invariant under A_K.

The operator K in this theorem is in general not A-bounded. If $c \in D(A^*)$, then K is the same $A-$bounded operator defined above. For the general case, we need the extension of $\langle A\cdot, c\rangle$ to C_A in order to define K.

Theorem 4.8: Assume that the system (A, b, c) has relative degree 2 and $\lim_{s\to\infty} s^2 G(s)$ exists. The invariant zeros of (A, b, c) are the eigenvalues of A_K, where A_K is as defined in (4.22, 4.23).

Proof: First assume that λ is an eigenvalue of A_K with eigenvector v. Note that $v \in D(A) \cap c^\perp$, so set $x = v$ and $u = -Kv$ in (4.19) to obtain that λ is an invariant zero of the original system.

Now assume that λ is an invariant zero. That is, there exists $u \in \mathbb{R}$ and $v \neq 0$ such that $v \in c^\perp \cap D(A)$ and

$$\lambda v - Av + bu = 0.$$

We need to first show that $v \in D(A_K)$. First, note that

$$Av = \lambda v - bu.$$

Since $\lim_{s\to\infty} s^2 G(s)$ exists, $b \in D(C_A)$ and since $D(A) \subset D(C_A)$, $Av \in D(C_A)$. Also,

$$\begin{aligned} C_A v &= \langle Av, c\rangle \\ &= \lambda\langle v, c\rangle + u\langle b, c\rangle \\ &= 0 + 0. \end{aligned}$$

Thus, $v \in D(A_K)$. It follows that

$$\begin{bmatrix} \lambda I - A_K & b \\ c & 0 \end{bmatrix} \begin{bmatrix} v \\ Kv + u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Since $b \notin Z_A$, $Kv + u = 0$ and λ is an eigenvalue of A_K on $c^\perp$ with the given domain. □

The following result follows immediately from Theorem 4.3.

Corollary 4.9: Suppose (A, b, c) has relative degree 2 and $\lim_{s\to\infty} s^2 G(s)$ exists. If $c \notin D(A^*)$ then the operator A_K with domain $D(A_K)$ defined in (4.22) is not closable.

It is shown in the next example that, in general, it is not possible to restrict $D(A_K)$ to $D(A^2)$ and obtain the invariant zeros.

Example IV.1 continued. Recall that this controlled delay system has no largest feedback-invariant subspace. A straightforward calculation shows that the invariant zeros of this control system are $i2n\pi$, where n is any integer. We now verify that these are the eigenvalues of A_K on $c^\perp$.

We can calculate C_A from its definition to be

$$C_A x = r_2 - \lim_{s\to\infty} se^{-s} \int_{-1}^{0} e^{-s\tau} \phi_2(\tau) d\tau.$$

Denote the limiting value of

$$\lim_{s\to\infty} se^{-s} \int_{-1}^{0} e^{-s\tau} \psi(\tau) d\tau$$

by $E_{-1}\psi$, when this limit exists. (If the value of ψ at -1 exists, $E_{-1}\psi = \psi(-1)$.) Then

$$\begin{aligned} D(C_A) &= \{[r_1, r_2, \phi_1, \phi_2]^T \in X; E_{-1}\phi_2 \text{ defined}\} \\ &\supset \{[r_1, r_2, \phi_1, \phi_2]^T \in X; \phi_2 \in H_1(-1, 0)\}. \end{aligned}$$

We have $C_A b = 1$ and $A_K = A + bK$, where

$$Kx = -C_A(Ax) = E_{-1}\dot{\phi_2}, \tag{4.24}$$

with $D(A_K)$

$$\{(0, r_2, \phi_1, \phi_2); \phi_1(0) = 0, \phi_2(0) = \phi_2(-1) = r_2, \\ \phi_1 \in H_1(-1, 0), \phi_2 \in H_1(-1, 0), E_{-1}\dot{\phi_2} \text{ defined}\}.$$

When $A_K x = \lambda x$, $x \in D(A_K)$, we obtain

$$\begin{aligned} 0 &= 0 \\ E_{-1}\dot{\phi_2} &= \lambda r_2 \\ \dot{\phi_1} &= \lambda\phi_1 \\ \dot{\phi_2} &= \lambda\phi_2. \end{aligned}$$

This system of equations has a non-trivial solution in $D(A_K)$ for $\lambda = i2n\pi$ with

$$x = \begin{bmatrix} 0 \\ r_2 \\ 0 \\ r_2 e^{i2n\pi t} \end{bmatrix}.$$

Thus, the invariant zeros of this system are $i2n\pi$. These are exactly the invariant zeros. Suppose we restrict the domain $D(A_K)$ to the more obvious

$$D(A_K) = \{x \in D(A) \cap c^\perp | Ax \in D(A), \langle Ax, c\rangle = 0\}.$$

This yields that A_K is invariant on Z as defined in (4.8). For this example, $D(A_K)$ is

$$\{(0, r_2, \phi_1, \phi_2); \phi_1(0) = 0, \phi_2(0) = \phi_2(-1) = r_2,$$
$$\phi_1 \in H_2(-1, 0), \phi_2 \in H_2(-1, 0), \dot{\phi_1}(0) = 0, \dot{\phi_2}(0) = 0\}.$$

However, with this choice of domain, A_K does not have any eigenvalues. □

The feedback (4.24) matches that obtained in [18] by direct calculation on the delay differential equation. However, not only do we now have a general definition of the appropriate feedback, we have an rigorous definition of its domain.

Example IV.2 We give here a system (A, b, c) for which there is no largest feedback invariant subspace of $c^\perp$. Let X be the Hilbert space ℓ^2, with index set $\mathbb{N}$. Let $h = [1, 1, 1, \ldots]$, $\vec{0} = [0, 0, 0, 0 \ldots]^T$ and $D = \mathrm{diag}\{\lambda_2, \lambda_3, \lambda_4 \ldots\}$, where $\lambda_j = -j$ for $j = 2, 3, \ldots$. Define

$$A = \begin{bmatrix} -1 & h \\ \vec{0} & D \end{bmatrix}, \quad c = [1, 0, 0, 0, \ldots]^T,$$

and, for any fixed integer $N > 2$,

$$b = [0, b_2, b_3 \ldots b_N, 0, 0, \ldots]^T, \text{ where } \sum_{j=2}^{N} b_j \neq 0.$$

It is easy to verify that $\langle b, c\rangle = 0$ and $c \notin D(A^*)$. Also, since $b \in D(A)$, $C_A b = \langle Ab, c\rangle = \sum_{j=2}^{N} b_j \neq 0$. For positive integers $n > N$, define the subspace of X

$$V_n = \{[0, x_2, \ldots, x_n, 0, \ldots]^T;\ x_j = 0 \text{ if } j > n,\ \sum_{k=2}^{n} x_k = 0\};$$

For $x \in V_n$, define

$$K_n x = \frac{1}{C_A b} \sum_{j=2}^{n} j x_j.$$

It is easy to verify that V_n is $A + bK_n$-invariant. Define

$$V = \cup_{n\in\mathbb{N}} V_n.$$

Any largest feedback-invariant subspace must contain V. It is clear that V is dense in

$$Z = \{[x_j]_{j\in\mathbb{N}} \in D(A) \mid x_1 = 0,\ \sum_{j\in\mathbb{N}} x_j = 0\}.$$

Since Z can also be written as (4.8), Theorem (4.1) implies that V is dense in $c^\perp$. However, $b \in c^\perp$ and so, from Theorem 2.3 the closure of V is not feedback invariant. Hence, no largest feedback-invariant subspace exists. □

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