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# Stability and Galerkin approximation in thermoelastic models

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***Abstract*— We consider the coupled partial differential equations which arise in modeling linear thermoelastic structures. We review stability properties of the distributed parameter model, particularly as related to the choice of norm on the state space. We discuss the choice of norm for two models - one with elastic dynamics governed by a wave equation and the other by an Euler-Bernoulli beam equation. We discuss the implications for stability as well as Galerkin approximations.**

## I. INTRODUCTION

A thermoelastic structure is an elastic structure distinguished by a physical mechanism in which mechanical energy is dissipated via conversion to thermal energy. The mathematical model for such a structure usually consists of a pair of coupled partial differential equations, one which models the elastic dynamics of the structure, and the other which models thermal diffusion in the structure. We refer to [1], [2] for references and discussion of modeling issues. In addition a thermomechanical coupling models the exchange between mechanical energy and thermal energy, and it is through this coupling that mechanical energy is dissipated. This type of damping is usually 'weaker' than other type of structural or mechanical damping, and this manifests itself in the rate at which the energy decays. The study of the energy decay rate for various thermoelastic models is an area of ongoing research, and we refer to [3], [4], [5], [6] and the references therein. For thermoelastic models it is typical that different boundary conditions require different techniques to establish stability. One such technique is to construct a new norm on the underlying state space which can be used to establish a dissipative inequality, which in turn implies exponential stablity of the solution semigroup. In [7] we showed how to apply this technique to establish exponential stability for a thermoelastic wave equation with the so-called Dirichlet-Neumann boundary conditions, and in the next section we apply this method to a different set of boundary conditions. We should mention that exponential stability has already been established by other methods for these and other boundary conditions, including the Dirichlet-Dirichlet boundary conditions in [8] and [9], but so far it does not seem that the renorming technique can be successfully applied to the Dirichlet-Dirichlet case. Nonetheless the technique is still useful because the norm which is constructed can often be used to advantage in defining Galerkin approximation schemes. In the third section we consider a thermoelastic Euler-Bernoulli beam, where by introducing a new norm we obtain the space on which the semigroup generator is associated with a coercive sesquilinear form. In the last section we discuss the connection with Galerkin approximations.

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## II. THERMOELASTIC WAVE EQUATION

Let us consider the model

$$\begin{aligned} y_{tt}(t,x) &= y_{xx}(t,x) - \gamma\theta_x(t,x) \\ \theta_t(t,x) &= \theta_{xx}(t,x) - \gamma y_{tx}(t,x), \end{aligned} \qquad (1)$$

with initial conditions

$$y(0,x) = u_0(x),\ y_t(0,x) = v_0(x), \quad \theta(0,x) = \theta_0(x),$$

and boundary conditions

$$y(t,0) = y_x(t,1) = 0, \quad \theta_x(t,0) = \theta(t,1) = 0. \qquad (2)$$

Here $y(t,x)$ represents displacement (longitudinal or transverse, depending upon the application) at time $t$ and position $x$ along the interval $[0,1]$, and $\theta(t,x)$ represents temperature at time $t$ and position $x$. The small positive constant $\gamma$ is the thermomechanical coupling parameter. The energy of this system is given by

$$E(t) = \int_0^1 |y_x(t,x)|^2 + |y_t(t,x)|^2 + |\theta(t,x)|^2\,dx.$$

A natural setting for approximation and control is to reformulate as a Cauchy problem on the energy space. For this model, the energy space is

$$X = H^1_L(0,1) \times L^2(0,1) \times L^2(0,1),$$

where $H^1_L(0,1)$ is the Sobolev space

$$H^1_L(0,1) = \{f \in L^2(0,1) :\ f' \in L^2(0,1),\ f(0) = 0\}.$$

The energy norm is defined by

$$\|(u,v,\theta)\|_X^2 = \int_0^1 |u'(x)|^2 + |v(x)|^2 + |\theta(x)|^2\,dx.$$

Next define the operator $\mathcal{A}$ on the domain

$$\begin{aligned}\operatorname{dom}\mathcal{A} = \ &\big\{(u,v,\theta) \in X :\ u \in H^2(0,1),\ u'(1) = 0,\\ &\theta \in H^2(0,1),\ \theta'(0) = \theta(1) = 0,\\ &v \in H^1_L(0,1)\big\},\end{aligned}$$

by

$$\mathcal{A}(u,v,\theta) = (v, u'' - \gamma\theta', \theta'' - \gamma v').$$

If we set

$$x(t) = (y(t,x), y_t(t,x), \theta(t,x)),$$

then the system (1)-(2) can be reformulated as the Cauchy problem

$$\begin{aligned}\frac{d}{dt}x(t) &= \mathcal{A}x(t),\\ x(0) &= (u_0, v_0, \theta_0)\end{aligned} \tag{3}$$

evolving on the energy space $X$. The space $X$ is called the energy space because $E(t) = \|x(t)\|_X^2$. We observe that for all $x = (u,v,\theta) \in \operatorname{dom}\mathcal{A}$,

$$\begin{aligned}\operatorname{Re}\langle\mathcal{A}x, x\rangle_X &= \operatorname{Re}\Big\{\int_0^1 (v'\overline{u'} + u''\overline{v}) - \gamma\theta'\overline{v}\\ &\qquad -\gamma v'\overline{\theta} - |\theta'|^2\,dx\Big\}\\ &= \int_0^1 -|\theta'|^2\,dx\\ &\le 0.\end{aligned} \tag{4}$$

It is not difficult to show that $\mathcal{A}$ is the infinitesimal generator of a $C_0$-semigroup $T(t)$ on $X$, but inequality (4) is not enough to imply that $T(t)$ is exponentially stable. Instead we shall choose a norm different from but topologically equivalent to the energy norm, and in which an inequality stronger than (4) can be obtained. To proceed, for positive constants $\alpha_1$, $\alpha_2$, define the norm

$$\begin{aligned}\|(u,v,\theta)\|_e^2 &= \alpha_1\|(u,v,\theta)\|_X^2 + 2\operatorname{Re}\int_0^1 u\overline{v}\,dx\\ &\quad +2\alpha_2\operatorname{Re}\int_0^1 \theta\int_x^1 \overline{v(t)}\,dt\,dx\\ &= \alpha_1\int_0^1 |u'|^2 + |v|^2 + |\theta|^2\,dx + 2\operatorname{Re}\int_0^1 u\overline{v}\,dx\\ &\quad +2\alpha_2\operatorname{Re}\int_0^1 \theta\int_x^1 \overline{v(t)}\,dt\,dx.\end{aligned}$$

In order to check that this is a norm equivalent to the energy norm, recall from the Cauchy-Schwarz inequality that for any $\epsilon > 0$ and any $f, g \in L^2$, we have

$$\pm\operatorname{Re}\int_0^1 f\overline{g}\,dx \le \frac{\epsilon}{2}\int_0^1 |f|^2\,dx + \frac{1}{2\epsilon}\int_0^1 |g|^2\,dx.$$

Also, from the Poincare inequality there exists $K_1 > 0$ such that for any $f \in H^1_L$ we have

$$\int_0^1 |f|^2\,dx \le K_1\int_0^1 |f'|^2\,dx,$$

and similarly, there exists $K_2 > 0$ such that for any $f \in L^2$ we have

$$\int_0^1 \left|\int_x^1 f(t)\,dt\right|^2 dx \le K_2\int_0^1 |f|^2\,dx.$$

These inequalities imply that for all sufficiently large $\alpha_1$, the norms $\|\cdot\|_X$ and $\|\cdot\|_e$ are equivalent. That is, there exist constants $c_1, c_2 > 0$ such that

$$c_1\|x\|_X \le \|x\|_e \le c_2\|x\|_x \quad \text{for all } x \in X.$$

The norm $\|\cdot\|_e$ has a compatible inner product given by

$$\begin{aligned}&\langle(u,v,\theta),(f,g,h)\rangle_e\\ &\quad= \alpha_1\int_0^1 u'\overline{f}' + v\overline{g} + \theta\overline{h}\,dx +\\ &\qquad\int_0^1 v\overline{f} + u\overline{g}\,dx +\\ &\qquad\alpha_2\int_0^1 \theta\int_x^1 \overline{g(t)}\,dt + \int_x^1 v(t)\,dt\,\overline{h}\,dx.\end{aligned}$$

Now let us check how this choice of norm changes the dissipative inequality (4). For $x = (u,v,\theta) \in \operatorname{dom}\mathcal{A}$, we have

$$\begin{aligned}\operatorname{Re}\langle\mathcal{A}x, x\rangle_e &= \int_0^1 -\alpha_1|\theta'|^2\,dx\\ &\quad+\operatorname{Re}\int_0^1 (u'' - \gamma\theta')\overline{u} + v\overline{v}\,dx\\ &\quad+\alpha_2\operatorname{Re}\int_0^1 (\theta'' - \gamma v')\int_x^1 \overline{v(t)}\,dt\,dx\\ &\quad+\alpha_2\operatorname{Re}\int_0^1\int_x^1 (u'' - \gamma\theta')\,dt\,\overline{\theta}\,dx.\end{aligned}$$

Now several integrations by parts yield

$$\begin{aligned}&\operatorname{Re}\langle\mathcal{A}x, x\rangle_e =\\ &\quad\int_0^1 (-\alpha_1|\theta'|^2 - |u'|^2 - (\alpha_2\gamma - 1)|v|^2)\,dx\\ &\quad+\operatorname{Re}\int_0^1 -\gamma\theta'\overline{u} + \alpha_2\theta'\overline{v} - \alpha_2 u'\overline{\theta} + \alpha_2\gamma|\theta|^2 dx.\end{aligned}$$

In addition to the inequalities mentioned above, the Poincare inequality also imlplies that there exists $K_3 >$

0 such that for any $f \in H^2(0,1)$ satisfying $f'(0) = f(1) = 0$ we have

$$\int_0^1 |f|^2\,dx \le K_3 \int_0^1 |f'|^2\,dx.$$

Thus

$$-\text{Re}\int_0^1 \gamma\theta'\overline{u}dx \le \frac{\gamma^2}{2\epsilon_1}\int_0^1 |\theta'|^2dx + \frac{K_1\epsilon_1}{2}\int_0^1 |u'|^2dx,$$

$$-\text{Re}\int_0^1 \alpha_2\theta'\overline{v}dx \le \frac{\alpha_2^2}{2}\int_0^1 |\theta'|^2dx + \frac{1}{2}\int_0^1 |v|^2dx,$$

$$-\text{Re}\int_0^1 \alpha_2 u'\overline{\theta}dx \le \frac{\epsilon_2}{2}\int_0^1 |u'|^2dx + \frac{\alpha_2^2K_3}{2\epsilon_2}\int_0^1 |\theta'|^2dx,$$

and

$$\alpha_2\gamma\int_0^1 |\theta|^2dx \le \alpha_2\gamma K_3\int_0^1 |\theta'|^2dx.$$

We continue from above to get

$$\begin{aligned}
&\text{Re}\langle\mathcal{A}x, x\rangle_e \le \\
&\quad -[1 - \frac{K_1\epsilon_1}{2} - \frac{\epsilon_2}{2}]\int_0^1 |u'|^2 \\
&\quad -[\alpha_2\gamma - 1 - \frac{1}{2}]\int_0^1 |v|^2dx \\
&\quad -[\alpha_1 - \frac{\gamma^2}{2\epsilon_1} - \frac{\alpha_2^2}{2} - \frac{\alpha_2^2K_3}{2\epsilon_2} - \alpha_2\gamma K_3]\int_0^1 |\theta'|^2dx
\end{aligned}$$

Now choose $\epsilon_1$, $\epsilon_2$ sufficiently small that $1 > (K_1\epsilon_1 + \epsilon_3)/2$, and choose $\alpha_2$ sufficiently large that $\alpha_2\gamma > 3/2$. For these choices of $\alpha_2$, $\epsilon_1$, $\epsilon_2$, choose $\alpha_1$ sufficiently large that $\|\cdot\|_e$ is a norm and such that

$$\alpha_1 > \frac{\gamma^2}{2\epsilon_1} + \frac{\alpha_2^2}{2} + \frac{\alpha_2^2K_3}{2\epsilon_2} + \alpha_2\gamma K_3.$$

Thus there exists $\omega > 0$ for which

$$\text{Re}\langle\mathcal{A}x, x\rangle_e \le -\omega\|x\|_X^2$$

for all $x \in \text{dom}\,\mathcal{A}$. Since the norms are equivalent, it follows that

$$\text{Re}\langle\mathcal{A}x, x\rangle_e \le -\frac{\omega}{c_2^2}\|x\|_e^2$$

for all $x \in \text{dom}\,\mathcal{A}$, which is the improved dissipative inequality we want. In particular, this implies that

$$\|T(t)\|_e \le e^{-\omega/(c_2^2)t}.$$

To summarize, this new norm implies exponential stability of the thermoelastic system (1)-(2), which is already known from other methods, and allows an estimate of the decay rate $-\omega/c_2^2$, which is not readily available with other methods. More importantly, we shall discuss in section IV how the new norm is used to construct Galerkin approximations and how these approximation schemes compare with those constructed in the energy norm. Before we do that, we consider a thermelastic beam model.

## III. Thermoelastic beam equation

We consider the case in which the elastic structure is an Euler-Bernoulli beam, namely

$$\begin{aligned}
y_{tt}(t,x) + y_{xxxx}(t,x) - \gamma\theta_{xx}(t,x) &= 0 \\
\theta_t(t,x) - \theta_{xx}(t,x) + \gamma y_{txx}(t,x) &= 0.
\end{aligned} \quad (5)$$

Initial conditions are given by

$$y(0,x) = u_0(x),\ y_t(0,x) = v_0(x), \quad \theta(0,x) = \theta_0(x),$$

and various boundary conditions are possible (e.g. clamped, free, hinged, supported, as well as thermal boundary conditions). We restrict consideration to the following simply-supported, fixed temperature boundary conditions:

$$\begin{aligned}
&y(t,0) = y(t,1) = 0, \quad y_{xx}(t,0) = y_{xx}(t,1) = 0, \\
&\theta(t,0) = \theta(t,1) = 0.
\end{aligned} \quad (6)$$

These boundary conditions are useful because the new norm to be constructed, motivated by the results in [10], involves the square root of the fourth derivative operator which appears naturally in the model, and for these boundary conditions the square root operator is also a differential operator (of second order). For other boundary conditions the square root operator is not a differential operator and the norm is less convenient to work with.

In (5)-(6) $y(t,x)$ represents transverse displacement at time $t$ and position $x$ along a beam of length 1 lying on the interval $[0,1]$, and $\theta(t,x)$ represents temperature at time $t$ and position $x$. As in the wave equation model, the small positive constant $\gamma$ is a thermomechanical coupling parameter. The energy of this system is given by

$$E(t) = \int_0^1 |y_{xx}(t,x)|^2 + |y_t(t,x)|^2 + |\theta(t,x)|^2\,dx.$$

It is known (see [1], [5]) that the energy decays exponentially for this model and for other boundary conditions as well.

To proceed, define the energy space

$$X = H^2(0,1)\cap H_0^1(0,1)\times L^2(0,1)\times L^2(0,1), \quad (7)$$

where

$$H_0^1(0,1) = \{f \in H^1(0,1):\ f(0) = f(1) = 0\}. \quad (8)$$

The energy norm is given by

$$\|(u,v,\theta)\|_X^2 = \int_0^1 |u''|^2 + |v|^2 + |\theta|^2\,dx. \quad (9)$$

Next define the operator $\tilde{\mathcal{A}}:\ \text{dom}\,\tilde{\mathcal{A}} \subset X \to X$ on the domain

$$\begin{aligned}
\text{dom}\,\tilde{\mathcal{A}} = \ &\{(u,v,\theta) \in X:\ u \in H^4(0,1),\ u''(0) = 0, \\
&u''(1) = 0,\ v,\theta \in H^2(0,1)\cap H_0^1(0,1)\},
\end{aligned}$$

by

$$\tilde{\mathcal{A}}(u,v,\theta) = (v, -u'''' + \gamma\theta'', \theta'' - \gamma v'').$$

If we set

$$x(t) = (y(t,x), y_t(t,x), \theta(t,x)),$$

then the system (5)-(6) can be reformulated as the Cauchy problem

$$\begin{aligned} \frac{d}{dt}x(t) &= \tilde{\mathcal{A}}x(t), \\ x(0) &= (u_0, v_0, \theta_0) \end{aligned} \tag{10}$$

evolving on the energy space $X$. As was the case for the wave equation model, it is not difficult to show that $\tilde{\mathcal{A}}$ is the infinitesimal generator of a $C_0$-semigroup $\tilde{T}(t)$ on $X$, and satisfies

$$\text{Re}\,\langle \tilde{\mathcal{A}}x, x\rangle_X \le 0 \tag{11}$$

for all $x = (u,v,\theta) \in \text{dom}\,\tilde{\mathcal{A}}$. We shall construct a new norm to improve the dissipative inequality (11), but our result will be even stronger than the one obtained in the previous section. That is, we will find a space $V$ compactly embedded in $X$ which satisfies $\text{dom}\,\tilde{\mathcal{A}} \subset V$ and

$$\text{Re}\,\langle \tilde{\mathcal{A}}x, x\rangle_e \le -\omega\|x\|_V^2 \tag{12}$$

for all $x = (u,v,\theta) \in \text{dom}\,\tilde{\mathcal{A}}$. This implies not only that the solution semigroup $\tilde{T}(t)$ is exponentially stable, but also that it is analytic, and that $\tilde{\mathcal{A}}$ is m-sectorial and associated with a coercive sesquilinear form. In particular, for positive constants $\alpha_1$, $\alpha_2$, define on $X$ the norm

$$\begin{aligned} \|(u,v,\theta)\|_e^2 &= \alpha_1\|(u,v,\theta)\|_X^2 - 2\,\text{Re}\int_0^1 u''\,\overline{v}\,dx \\ &\quad -2\alpha_2\,\text{Re}\int_0^1 \theta\,\overline{v}\,dx \\ &= \alpha_1\int_0^1 |u''|^2 + |v|^2 + |\theta|^2\,dx - 2\,\text{Re}\int_0^1 u''\,\overline{v}\,dx \\ &\quad -2\alpha_2\,\text{Re}\int_0^1 \theta\,\overline{v}\,dx. \end{aligned}$$

Arguments similar to those in the previous section show that for $\alpha_1$ sufficiently large this is a norm is equivalent to the energy norm on $X$. Next define the space

$$\begin{aligned} V = \{&(u,v,\theta) \in X :\ u \in H^3(0,1), \\ &u''(0) = u''(1) = 0,\ v,\theta \in H_0^1(0,1)\}, \end{aligned}$$

with norm

$$\|(u,v,\theta)\|_V^2 = \int_0^1 |u'''|^2 + |v'|^2 + |\theta'|^2\,dx.$$

Clearly $\text{dom}\,\tilde{\mathcal{A}} \subset V \subset X$ and the embedding is compact. For any $x = (u,v,\theta) \in \text{dom}\,\tilde{\mathcal{A}}$, a straightforward calculation similar to the one in the previous section yields

$$\begin{aligned} &\text{Re}\,\langle \tilde{\mathcal{A}}x, x\rangle_e = \\ &\quad \int_0^1 -\alpha_1|\theta'|^2 - |u'''|^2 - \alpha_2|v'|^2\,dx \\ &\quad +\text{Re}\int_0^1 |v'|^2 + \gamma\theta'\overline{u'''}\,dx \\ &\quad +\text{Re}\int_0^1 \alpha_2\theta'\overline{v'} - \alpha_2 u'''\overline{\theta'} + \alpha_2\gamma|\theta'|^2\,dx. \end{aligned}$$

Therefore

$$\begin{aligned} &\text{Re}\,\langle \tilde{\mathcal{A}}x, x\rangle_e \le \\ &\quad -(\alpha_1 - \frac{\gamma}{2\epsilon_1} - \frac{\alpha_2}{2} - \frac{\alpha_2}{2\epsilon_2} - \alpha_2\gamma)\int_0^1 |\theta'|^2\,dx \\ &\quad -(1 - \frac{\gamma}{2}\epsilon_1 - \frac{\alpha_2}{2}\epsilon_2)\int_0^1 |u'''|^2\,dx \\ &\quad -(\frac{\alpha_2}{2} - 1)\int_0^1 |v'|^2\,dx. \end{aligned}$$

The positive numbers $\alpha_1$, $\alpha_2$, $\epsilon_1$, $\epsilon_2$ can be chosen so that there exists $\omega > 0$ such that

$$\text{Re}\,\langle \tilde{\mathcal{A}}x, x\rangle_e \le -\omega\|x\|_V^2 \tag{13}$$

for all $x = (u,v,\theta) \in \text{dom}\,\tilde{\mathcal{A}}$. In particular, we can define the sesquilinear form $\sigma_1 :\ V \times V \to \mathbb{C}$ by

$$\begin{aligned} &\sigma_1((u,v,\theta),(f,g,h)) = \\ &\quad \int_0^1 \alpha_1[v'\overline{f'''} - u'''\overline{g'} + \sigma\theta'\overline{g'} + \theta'\overline{h'} - \gamma v'\overline{h'}]\,dx \\ &\quad -\int_0^1 [v'\overline{g'} - u'''\overline{f'''} + \gamma\theta'\overline{f'''}]\,dx \\ &\quad -\int_0^1 \alpha_2[\theta'\overline{g'} - \gamma v'\overline{g'} - u'''\overline{h'} + \gamma\theta'\overline{h'}]\,dx, \end{aligned}$$

for all $(u,v,\theta),(f,g,h) \in V$. The form $\sigma_1$ is $V$-bounded and has the property that

$$\sigma_1(x,y) = \langle -\tilde{\mathcal{A}}x, y\rangle_e \tag{14}$$

for all $x \in \text{dom}\,\tilde{\mathcal{A}}$ and $y \in V$. The above inequality verifies that $\sigma_1$ is $V$-coercive. It follows from results in [11] that $\tilde{T}(t)$ is analytic. We turn in the next section to a discussion of Galerkin approximation schemes and the implication of a new norm.

## IV. GALERKIN APPROXIMATIONS

A standard approach to constructing Galerkin approximation schemes for models like (1)-(2) or (5)-(6) is to start with a weak form of the equations. For example, the weak form of the thermoelastic beam model (5)-(6) is given by

$$\begin{aligned} \int_0^1 y_{tt}(t,x)f(x) + y_{xx}(t,x)f''(x) + \\ \gamma\theta_x(t,x)f'(x)\,dx = 0, \end{aligned} \tag{15}$$

$$\int_0^1 \theta_t(t,x)g(x) + \theta_x(t,x)g'(x) - \gamma y_{tx}(t,x)g'(x)\,dx = 0, \qquad (16)$$

for all $f \in H^2(0,1) \cap H_0^1(0,1)$, $g \in H_0^1(0,1)$. For any set of finite element basis functions $\{a_i(x)\}_{i=1}^n \subset H^2 \cap H_0^1(0,1)$, $\{b_i(x)\}_{i=1}^n \subset H_0^1(0,1)$, one defines the finite element solution $y^n(t,x) = \sum_{i=1}^n y_i(t)a_i(x)$ and $\theta^n(t,x) = \sum_{i=1}^n \theta_i(t)b_i(x)$. Then $y^n(t,x)$ and $\theta^n(t,x)$ are determined from (15)-16), and in particular $y(t) = [y_1(t),\ldots,y_n(t)]$ and $\theta(t) = [\theta_1(t),\ldots,\theta_n(t)]$ satisfy

$$\begin{aligned} M_1\ddot{y}(t) + K_1 y(t) + \gamma D_1\theta(t) &= 0,\\ M_2\dot{\theta}(t) + K_2\theta(t) - \gamma D_2\dot{y}(t) &= 0. \end{aligned} \qquad (17)$$

The $n \times n$ mass, damping and stiffness matrices are defined by $M_1(i,j) = \int_0^1 a_i a_j\,dx$, $K_1(i,j) = \int_0^1 a_i'' a_j''\,dx$, $D_1(i,j) = \int_0^1 b_i' a_j'\,dx$, $M_2(i,j) = \int_0^1 b_i b_j\,dx$, $K_2(i,j) = \int_0^1 b_i' b_j'\,dx$, $D_2 = D_1^T$. If we then in a standard way define $z(t) = [y(t), \dot{y}(t), \theta(t)]$, the matrix representation for the finite element approximation becomes

$$\frac{d}{dt}z(t) = A^N z(t) \qquad (18)$$

$$= \begin{bmatrix} 0 & I & 0 \\ -M_1^{-1}K_1 & 0 & -\gamma M_1^{-1}D_1 \\ 0 & \gamma M_2^{-1}D_2 & -M_2^{-1}K_2 \end{bmatrix} z(t).$$

It is possible to define approximation schemes directly from the abstract formulations (3) and (10). For (10) for example, define the space $V_2 = H^2(0,1) \cap H_0^1(0,1) \times H^2(0,1) \cap H_0^1(0,1) \times H_0^1(0,1)$ with norm $\|(u,v,\theta)\|_{V_2}^2 = \int_0^1 |u''|^2 + |v''|^2 + |\theta'|^2\,dx$, and the sesquilinear form $\sigma_2 : V_2 \times V_2 \to \mathbb{C}$ by

$$\sigma_2((u,v,\theta),(f,g,h)) = \int_0^1 -v''\overline{f''} + u''\overline{g''} + \gamma\theta'\overline{g'} + \theta'\overline{h'} - \gamma v'\overline{h'}\,dx. \qquad (19)$$

We note that $\tilde{\mathcal{A}}$ is related to $\sigma_2$ via the energy norm, since

$$\sigma_2(x,y) = \langle -\tilde{\mathcal{A}}x, y\rangle \qquad (20)$$

for all $x \in \operatorname{dom}\tilde{\mathcal{A}}$ and $y \in V_2$, but the space $V_2$ is not compactly embedded in $X$, and the form $\sigma_2$ is not coercive. Nevertheless $\sigma_2$ and $V_2$ can still be used to construct a Galerkin approximation scheme, which in fact is precisely the same as (18). In particular, for any finite-dimensional subspace $V^N \subset V_2$, $V^N = \operatorname{span}\{e_i\}_{i=1}^N$, the form $\sigma_2$ defines an operator $\mathcal{A}^N : V^N \to V^N$ by the relationship

$$-\langle \mathcal{A}^N x, y\rangle_Z = \sigma_2(x,y)$$

for all $x, y \in V^N$. The matrix representation of $\mathcal{A}^N$ is given by $\mathcal{M}^{-1}\mathcal{Q}^T$, where the matrices $\mathcal{M}$ and $\mathcal{Q}$ are defined by $\mathcal{M}(i,j) = \langle e_i, e_j\rangle_X$ and $\mathcal{Q}(i,j) = -\sigma_2(e_i,e_j)$. Let us in particular define the basis functions $e_i$ as follows. For $i = 1,\ldots,n$, define $e_i = (a_i,0,0)$, $e_{n+i} = (0,a_i,0)$, $e_{2n+i} = (0,0,b_i)$. Then

$$\mathcal{M}(i,j) = \langle e_i, e_j\rangle_X = \begin{bmatrix} K_1 & 0 & 0 \\ 0 & M_1 & 0 \\ 0 & 0 & M_1 \end{bmatrix}$$

and

$$\mathcal{Q}(i,j) = -\sigma_2(e_i,e_j) = \begin{bmatrix} 0 & -K_1 & 0 \\ K_1 & 0 & \gamma D_2 \\ 0 & -\gamma D_1 & -K_2 \end{bmatrix}.$$

We have arrived at the same matrix representation as the 'usual' one in (18), since $A^N = \mathcal{M}^{-1}\mathcal{Q}^T$. It is natural to investigate what is the effect, if any, of using the new norms $\|\cdot\|_e$ instead of the energy norms in the construction. For the beam equation, this means using the space $V$, the sesquilinear form $\sigma_1$, and the norm $\|\cdot\|_e$ instead of the space $V_2$, the sesquilinear form $\sigma_2$, and the energy norm. It may be necessary to choose different basis functions, since the finite dimensional spaces $V^N$ must be subspaces of $V$ instead of $V_2$. If $\mathcal{A}^N : V^N \to V^N$ is the finite dimensional operator constructed using the energy norm, then the semigroup $T^N(t)$ converges to $\tilde{T}(t)$ in the energy norm $\|\cdot\|_X$. However if $\tilde{\mathcal{A}}^N : V^N \to V^N$ is the finite dimensional operator constructed using the norm $\|\cdot\|_e$, then the semigroup $\tilde{T}^N(t)$ converges to $\tilde{T}(t)$ in the stronger norm $\|\cdot\|_V$. This is one very significant advantage of the norm $\|\cdot\|_e$ over the energy norm $\|\cdot\|_X$. We point out that if one uses the 'usual' approach and starts with the weak form of the equations (15), it is impossible to obtain the approximations $\tilde{\mathcal{A}}^N$, because it is impossible to obtain the space $V$ without the new norm.

Substantially the same idea is used to construct a Galerkin finite element approximation for the thermoelastic wave equation (1)-(2). For the wave equation (with other boundary conditions than those considered here) it was shown in [7] that the eigenvalues of the finite dimensional operators $\mathcal{A}^N$ constructed using the energy norm are not uniformly bounded away from the imaginary axis, a failure which is corrected with the operators constructed with the new norm. The same behavior occurs with the boundary conditions considered here, and we illustrate this in Fig. 1 and Fig. 2. (We note that due to scaling, some negative real eigenvalues have been left off of the plots). It is still unclear all of the implications of using Galerkin approximations constructed with norms other than the energy norm, especially when the approximation schemes are applied to optimal control problems, and especially when the new norm yields a coercive estimate (the case for the beam equation but not the wave equation). This issue is open for investigation.

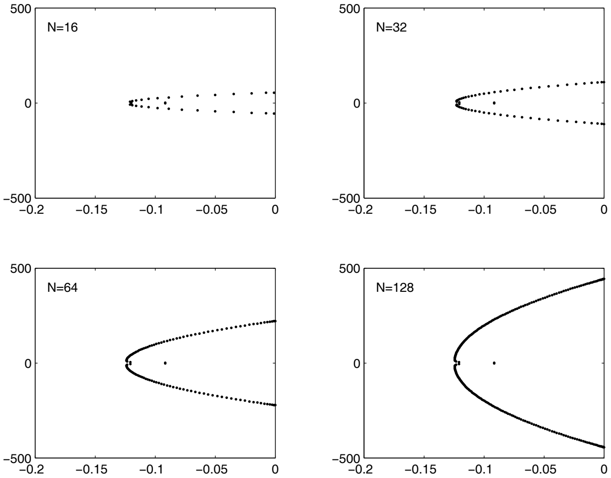


Fig. 1. Eigenvalues of $\mathcal{A}^N$, energy norm

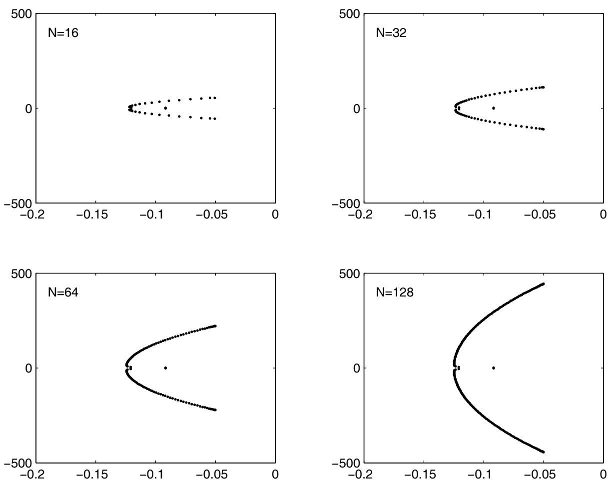


Fig. 2. Eigenvalues of $\mathcal{A}^N$, new norm

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