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Output Feedback Adaptive Controllers with Swapping Identifiers for Two Unstable PDEs with Infinite Relative Degree

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Abstract— **We develop output feedback adaptive controllers for two benchmark parabolic PDEs motivated by a model of thermal instability in solid propellant rockets. Both benchmark plants are unstable, have infinite relative degree, and are controlled from a boundary. One plant has an unknown parameter in the PDE and the other in the boundary condition. Adaptive control of these plants is studied in a companion paper [13] using a Lyapunov method which yields an update law that requires the use of parameter projection and a restriction to low values of adaptation gain. In this paper we show how the swapping identifier design, the most common method used in adaptive control of finite dimensional systems, can be employed for PDEs. This approach does not require parameter projection or restriction on the size of the adaptation gain. The results are illustrated by simulation.**

I. INTRODUCTION

In this paper we consider two parametrically uncertain, unstable, PDE plants controlled from a boundary. While these benchmark plants are simple in appearance, there does not exist an adaptive control design in the literature that is applicable to them due to the fact that they have infinite relative degree. Infinite relative degree arises in many applications where actuators and sensors are on the "opposite sides" of the PDE domains. The two benchmark problems in this paper are motivated by a model of thermal instability in solid propellant rockets. Our control laws are adaptive versions of the explicit boundary control laws developed in [18], [19]. The adaptive observers are infinite dimensional extensions of Kreisselmeier observers [14]. Our identifiers are designed using the swapping approach [14], prevalent in adaptive control of finite dimensional systems of relative degree higher than one. These identifiers remove the need for parameter projection and low adaptation gain present in [13].

Prior literature on adaptive control of infinite-dimensional systems is briefly reviewed in a companion paper [12]. Early works [16] were for plants stabilizable by non-identifier based high gain feedback, under a relative degree one assumption. State-feedback model reference adaptive control (MRAC) was extended to PDEs in [1], [2], [7], [17], [20] but not for the case of boundary control. Efforts in [5], [21] made use of positive realness assumptions where relative degree one is implicit. Stochastic adaptive LQR with least-squares parameter estimation and state feedback was pursued in [6]. Adaptive control of nonlinear PDEs was studied in [15], [10], [11]. Adaptive controllers for nonlinear systems on lattices were designed in [9]. An experimentally validated adaptive boundary controller for a flexible beam was presented in [4].

Although for the sake of clarity we consider two separate benchmark problems, it is possible to design an adaptive controller for a combined problem (Section VI). Another reason for separate consideration is a slightly weaker result for the benchmark plant with the unknown parameter in the boundary condition, due to an inherent difficulty observed in [1], [15].

Throughout the paper we assume well posedness of the closed loop systems in the interest of space and due to the parabolic character of the system which ensures it. An example on how one would handle it is given in [12].

a) Notation.: The spatial $L_2(0,1)$ norm is denoted by $\|\cdot\|$. The temporal norms are denoted by $\mathcal{L}_\infty$ and $\mathcal{L}_2$ for $t \geq 0$. We denote by l_1 a generic function in $\mathcal{L}_\infty \cap \mathcal{L}_2$.

II. BENCHMARK PLANT WITH UNKNOWN PARAMETER IN THE DOMAIN

Consider the following plant

$$u_t(x,t) = u_{xx}(x,t) + g u(0,t)\,, \tag{1}$$
$$u_x(0,t) = 0\,, \tag{2}$$
$$u(1,t) = U(t)\,, \tag{3}$$

where $U(t)$ is a control signal. This system is inspired by a model of thermal instability in solid propellant rockets [3]. For $U(t)=0$ this system is unstable if and only if $g>2$. The plant can be written in the frequency domain as a transfer function from input $u(1)$ to output $u(0)$:

$$u(0,s) = \frac{s}{(s-g)\cosh\sqrt{s}+g}\,u(1,s)\,. \tag{4}$$

We can see that it has no zeros (at $s=0$ the transfer function is $2/(2-g)$) and has infinitely many poles, one of which is unstable and approximately equal to g as $g\to+\infty$. So this is an infinite relative degree system.

The following transformation has been proposed in [18] for the case of known g:

$$w(x,t) = u(x,t) + \int_0^x \sqrt{g}\sinh\sqrt{g}(x-\xi)u(\xi,t)\,d\xi\,, \tag{5}$$

which maps (1)–(2) into an exponentially stable system

$$w_t(x,t) = w_{xx}(x,t)\,, \tag{6}$$
$$w_x(0,t) = 0\,, \tag{7}$$
$$w(1,t) = 0\,. \tag{8}$$

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A stabilizing control law is given by (5) evaluated at $x = 1$:

$$u(1,t) = -\int_0^1 \sqrt{g}\sinh\sqrt{g}(1-\xi)u(\xi,t)\,d\xi\,. \tag{9}$$

Suppose now that we want to stabilize this system when g is unknown. Our main result for this problem is summarized in the following theorem.

Theorem 1: Consider the system (1)–(2) with the control

$$u(1,t) = \int_0^1 k(1,\xi,\hat{g})(\hat{g}v(\xi,t)+\eta(\xi,t))\,d\xi\,, \tag{10}$$

$$k(x,\xi,\hat{g}) = \begin{cases} -\sqrt{\hat{g}}\sinh\sqrt{\hat{g}}(x-\xi), & \hat{g}\ge 0,\\ \sqrt{-\hat{g}}\sin\sqrt{-\hat{g}}(x-\xi), & \hat{g}<0, \end{cases} \tag{11}$$

where an update law for $\hat{g}$ is

$$\dot{\hat{g}} = \gamma\frac{(u(0,t)-\hat{g}v(0,t)-\eta(0,t))v(0,t)}{1+v^2(0,t)}\,, \tag{12}$$

and the filters $v(x,t)$, $\eta(x,t)$ are defined as

$$v_t(x,t) = v_{xx}(x,t)+u(0,t)\,, \tag{13}$$
$$v_x(0,t) = 0\,, \tag{14}$$
$$v(1,t) = 0\,, \tag{15}$$
$$\eta_t(x,t) = \eta_{xx}(x,t)\,, \tag{16}$$
$$\eta_x(0,t) = 0\,, \tag{17}$$
$$\eta(1,t) = u(1,t)\,. \tag{18}$$

If the closed loop system (1)–(2), (10)–(18) has a classical solution (u, $\hat{g}$, v, η), then for any $\hat{g}(0)$ and any initial conditions u_0, v_0, $\eta_0 \in L_2(0,1)$, the signals $\hat{g}$, u, v, η are bounded and u is regulated to zero for all $x\in[0,1]$:

$$\lim_{t\to\infty}\max_{x\in[0,1]}|u(x,t)| = 0. \tag{19}$$

□

Note that the control law (10)–(11) is a smooth function of $\hat{g}$ and does not require a-priori knowledge of the bound on $\hat{g}$.

III. Proof of Theorem 1

A. Target system

Introducing the error $e = u - gv - \eta$ we get an exponentially stable system

$$e_t(x,t) = e_{xx}(x,t)\,, \tag{20}$$
$$e_x(0,t) = 0\,, \tag{21}$$
$$e(1,t) = 0\,. \tag{22}$$

The estimate $\hat{e} = u - \hat{g}v - \eta$ satisfies the following PDE

$$\hat{e}_t(x,t) = \hat{e}_{xx}(x,t)+\tilde{g}u(0,t)-\dot{\hat{g}}v(x,t)\,, \tag{23}$$
$$\hat{e}_x(0,t) = 0\,, \tag{24}$$
$$\hat{e}(1,t) = 0\,. \tag{25}$$

The signal $\hat{e}$ can be expressed through e as $\hat{e} = e + \tilde{g}v$.

The transformation

$$\hat{w}(x,t) = \hat{g}v(x,t)+\eta(x,t) - \int_0^x k(x,\xi,\hat{g})(\hat{g}v(\xi,t)+\eta(\xi,t))\,d\xi \tag{26}$$

with $k(x,\xi,\hat{g})$ given by (11) maps (13)–(18) into the following system (Lemma A.1):

$$\hat{w}_t(x,t) = \hat{w}_{xx}(x,t)+\beta(x)\hat{e}(0,t)+\dot{\hat{g}}v + \dot{\hat{g}}\int_0^x \alpha(x-\xi)\left(\hat{g}v(\xi,t)+\hat{w}(\xi,t)\right)\,d\xi, \tag{27}$$
$$\hat{w}_x(0,t) = 0\,, \tag{28}$$
$$\hat{w}(1,t) = 0\,, \tag{29}$$

where

$$\alpha(x) = -\frac{1}{\hat{g}}k(x,0,\hat{g}), \tag{30}$$

$$\beta(x) = k_\xi(x,0,\hat{g}) = \begin{cases} \hat{g}\cosh\sqrt{\hat{g}}x, & \hat{g}\ge 0,\\ \hat{g}\cos\sqrt{-\hat{g}}x, & \hat{g}<0, \end{cases} \tag{31}$$

B. Adaptive law

We take the following equation as a parametric model

$$e(0,t) = u(0,t)-gv(0,t)-\eta(0,t)\,. \tag{32}$$

The estimation error is

$$\hat{e}(0,t) = u(0,t)-\hat{g}v(0,t)-\eta(0,t)\,. \tag{33}$$

We use the gradient update law

$$\dot{\hat{g}} = \gamma\frac{\hat{e}(0,t)v(0,t)}{1+v^2(0,t)}\,. \tag{34}$$

Lemma 2: The adaptive law (34) guarantees the following properties:

$$\frac{\hat{e}(0,t)}{\sqrt{1+v^2(0,t)}} \in \mathcal{L}_2\cap\mathcal{L}_\infty,\quad \tilde{g}\in\mathcal{L}_\infty,\quad \dot{\hat{g}}\in\mathcal{L}_2\cap\mathcal{L}_\infty\,. \tag{35}$$

Proof: Using a Lyapunov function

$$V = \frac{1}{2}\int_0^1 e^2\,dx + \frac{1}{2\gamma}\tilde{g}^2 \tag{36}$$

we get

$$\begin{aligned}\dot{V} &= -\int_0^1 e_x^2\,dx - \frac{\tilde{g}\hat{e}(0)v(0)}{1+v^2(0)}\\ &\le -\int_0^1 e_x^2\,dx - \frac{\hat{e}^2(0)}{1+v^2(0)} + \frac{e(0)\hat{e}(0)}{1+v^2(0)}\\ &\le -\|e_x\|^2 - \frac{\hat{e}^2(0)}{1+v^2(0)} + \frac{\|e_x\||\hat{e}(0)|}{\sqrt{1+v^2(0)}}\\ &\le -\frac{1}{2}\|e_x\|^2 - \frac{1}{2}\frac{\hat{e}^2(0)}{1+v^2(0)}\end{aligned} \tag{37}$$

This gives the following properties

$$\frac{\hat{e}(0,t)}{\sqrt{1+v^2(0,t)}} \in \mathcal{L}_2,\quad \tilde{g}\in\mathcal{L}_\infty\,. \tag{38}$$

Since

$$\frac{\hat{e}(0,t)}{\sqrt{1+v^2(0,t)}} = \frac{e(0,t)}{\sqrt{1+v^2(0,t)}} + \tilde{g}\frac{v(0,t)}{\sqrt{1+v^2(0,t)}}, \tag{39}$$

$$\dot{\hat{g}} = \gamma\frac{\hat{e}(0,t)}{\sqrt{1+v^2(0,t)}}\frac{v(0,t)}{\sqrt{1+v^2(0,t)}}\,, \tag{40}$$

we get (35). ■

The explicit bound on $\hat{g}$ in terms of initial conditions of all the signals can be obtained from (37):

$$\begin{aligned} \hat{g}^2(t) &\leq 2g^2 + 2\left(\tilde{g}(0)^2 + \gamma \int_0^1 e^2(x,0)\,dx\right) \\ &\leq 2g^2 + 2(g - \hat{g}(0))^2 \\ &\quad + 2\gamma \int_0^1 (u(x,0) - gv(x,0) - \eta(x,0))^2\,dx. \end{aligned} \quad (41)$$

We denote the bound on $\hat{g}$ by g_0. The above properties imply that functions α and β are bounded, let us denote these bounds by α_0 and β_0.

C. Boundedness

The filter v can be rewritten in the following way

$$v_t(x,t) = v_{xx}(x,t) + \hat{w}(0,t) + \hat{e}(0,t)\,, \quad (42)$$
$$v_x(0,t) = 0\,, \quad (43)$$
$$v(1,t) = 0\,. \quad (44)$$

We have two interconnected systems $\hat{w}$, v driven by a signal $\hat{e}(0,t)$ with properties (35). Consider a Lyapunov function

$$V_v = \frac{1}{2}\int_0^1 v^2(x)\,dx + \frac{1}{2}\int_0^1 v_x^2(x)\,dx\,. \quad (45)$$

Using Young's, Poincare's, and Agmon's inequalities we have[1]

$$\begin{aligned} \dot{V}_v &= -\int_0^1 v_x^2\,dx + (\hat{w}(0) + \hat{e}(0))\int_0^1 v\,dx \\ &\quad - \int_0^1 v_{xx}^2\,dx - (\hat{w}(0) + \hat{e}(0))\int_0^1 v_{xx}\,dx \\ &\leq -\|v_x\|^2 + \frac{1}{8}\|v\|^2 + 4\frac{\hat{e}^2(0)}{1+v^2(0)}(1 + \|v_x\|^2) \\ &\quad + 4\|\hat{w}_x\|^2 - \|v_{xx}\|^2 + \frac{1}{2}\|v_{xx}\|^2 + \|\hat{w}_x\|^2 \\ &\quad + \frac{\hat{e}^2(0)}{1+v^2(0)}(1 + \|v_x\|^2) \\ &\leq -\frac{1}{2}\|v_x\|^2 - \frac{1}{2}\|v_{xx}\|^2 + 5\|\hat{w}_x\|^2 + l_1\|v_x\|^2 + l_1, \end{aligned} \quad (46)$$

where l_1 is a generic function of time in $\mathcal{L}_1 \cap \mathcal{L}_\infty$. Using the following Lyapunov function for the $\hat{w}$-system,

$$V_{\hat{w}} = \frac{1}{2}\int_0^1 \hat{w}^2(x)\,dx \quad (47)$$

we get

$$\begin{aligned} \dot{V}_{\hat{w}} &= -\int_0^1 \hat{w}_x^2\,dx + \hat{e}(0)\int_0^1 \beta\hat{w}\,dx + \dot{\hat{g}}\int_0^1 \hat{w}v\,dx \\ &\quad + \dot{\hat{g}}\int_0^1 \hat{w}(x)\int_0^x \alpha(x-y)(\hat{g}v(y) + \hat{w}(y))\,dy\,dx \\ &\leq -\|\hat{w}_x\|^2 + \frac{c_1}{2}\|\hat{w}\|^2 + \frac{\beta_0^2}{2c_1}\frac{\hat{e}^2(0)}{1+v^2(0)}(1 + \|v_x\|^2) \\ &\quad + \frac{|\dot{\hat{g}}|^2(1+\alpha_0 g_0)^2}{2c_1}\|v\|^2 + c_1\|\hat{w}\|^2 + \frac{|\dot{\hat{g}}|^2\alpha_0^2}{2c_1}\|\hat{w}\|^2 \\ &\leq -(1 - 6c_1)\|\hat{w}_x\|^2 + l_1\|\hat{w}\|^2 + l_1\|v_x\|^2 + l_1\,. \end{aligned} \quad (48)$$

[1]We drop the dependence on time in the proofs to reduce notational burden.

Choosing $c_1 = 1/24$ and using a Lyapunov function $V = V_{\hat{w}} + (1/20)V_v$, we get

$$\begin{aligned} \dot{V} &\leq -\frac{1}{2}\|\hat{w}_x\|^2 - \frac{1}{40}\|v_x\|^2 - \frac{1}{40}\|v_{xx}\|^2 \\ &\quad + l_1\|\hat{w}\|^2 + l_1\|v_x\|^2 + l_1 \\ &\leq -\frac{1}{4}V + l_1 V + l_1 \end{aligned} \quad (49)$$

and by Lemma A.2 we obtain $\|\hat{w}\|$,$\|v\|, \|v_x\| \in \mathcal{L}_2 \cap \mathcal{L}_\infty$. Using these properties we get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\hat{w}_x\|^2 &\leq -\|\hat{w}_{xx}\|^2 + \beta_0|\hat{e}(0)|\|\hat{w}_{xx}\| \\ &\quad + |\dot{\hat{g}}|\|\hat{w}_{xx}\|((1+\alpha_0 g_0)\|v\| + \alpha_0\|\hat{w}\|) \\ &\leq -\frac{1}{8}\|\hat{w}_x\|^2 + l_1, \end{aligned} \quad (50)$$

so that $\|\hat{w}_x\| \in \mathcal{L}_2 \cap \mathcal{L}_\infty$.

D. Regulation

Using the fact that $\|v_x\|$, $\|\hat{w}_x\|$ are bounded we get

$$\left|\frac{d}{dt}(\|v\|^2 + \|\hat{w}\|^2)\right| \leq l_1\|\hat{w}_x\|^2 + l_1\|v_x\|^2 + l_1 < \infty. \quad (51)$$

By Barbalat's lemma $\|\hat{w}\| \to 0$, $\|v\| \to 0$. From (A.2) we have $\|\eta\| \to 0$ and $\|\eta_x\|$ is bounded. Since $u = e + qv + \eta$, we get $\|u\| \to 0$ and $\|u_x\|$ is bounded. Finally, using Agmon's inequality we get

$$\lim_{t\to\infty}\max_{x\in[0,1]}|u(x,t)| \leq \lim_{t\to\infty}(2\|u\|\|u_x\|)^{1/2} = 0\,. \quad (52)$$

IV. Benchmark Plant with Unknown Parameter in the Boundary Condition

Consider the following plant

$$u_t(x,t) = u_{xx}(x,t)\,, \quad (53)$$
$$u_x(0,t) = -qu(0,t)\,, \quad (54)$$
$$u(1,t) = U(t)\,, \quad (55)$$

where $U(t)$ is the control signal. This is an example of a system with a parametric uncertainty in the boundary condition, a hard-to-stabilize case even with full state feedback with in-domain actuation [1]. With $U(1) = 0$ this PDE is unstable if and only if $q > 1$. The plant can be written in the frequency domain as a transfer function from input $u(1)$ to output $u(0)$:

$$u(0,s) = \frac{\sqrt{s}}{\sqrt{s}\cosh\sqrt{s} - q\sinh\sqrt{s}}\,u(1,s)\,. \quad (56)$$

Since this transfer function has infinitely many poles and no zeros (at $s = 0$ the transfer function is $1/(1-q)$), this is an infinite relative degree system. One of the poles is unstable and is approximately equal to q^2 as $q \to +\infty$.

For the case of known q the transformation

$$w(x,t) = u(x,t) - \int_0^x k(x,\xi)u(\xi,t)\,d\xi \quad (57)$$

was used in [18] to map (53)–(54) into the target system

$$w_t(x,t) = w_{xx}(x,t) - cw(x,t)\,, \quad (58)$$
$$w_x(0,t) = -qw(0,t)\,, \quad w(1,t) = 0\,, \quad (59)$$

which is exponentially stable for $c \geq \max\{q|q|, 0\}$. However, this stability condition cannot be used when q is unknown. Instead, let us use (57) to map (53)–(54) into a different target system,

$$w_t(x,t) = w_{xx}(x,t)\,, \tag{60}$$
$$w_x(0,t) = 0\,, \tag{61}$$
$$w(1,t) = 0\,. \tag{62}$$

It can be shown that the kernel $k(x,\xi)$ must satisfy the following conditions:

$$k_{xx} - k_{\xi\xi} = 0\,, \tag{63}$$
$$k_\xi(x,0) = -qk(x,0)\,, \tag{64}$$
$$k(x,x) = -q\,. \tag{65}$$

The solution to this PDE is

$$k(x,\xi) = -qe^{q(x-\xi)}\,. \tag{66}$$

Suppose now that we want to stabilize the plant (53)–(55) when q is unknown. We have the following result.

Theorem 3: Consider the system (53)–(54) with the control

$$u(1,t) = -\int_0^1 \hat{q}e^{\hat{q}(1-\xi)}(\hat{q}v(\xi,t) + \eta(\xi,t))\,d\xi\,, \tag{67}$$

where the update law for $\hat{q}$ is

$$\dot{\hat{q}} = \gamma\frac{(u(0,t) - \hat{q}v(0,t) - \eta(0,t))v(0,t)}{1 + v^2(0,t)}\,, \tag{68}$$

and the filters $v(x,t)$, $\eta(x,t)$ are defined as

$$v_t(x,t) = v_{xx}(x,t)\,, \tag{69}$$
$$v_x(0,t) = -u(0,t)\,, \tag{70}$$
$$v(1,t) = 0\,, \tag{71}$$
$$\eta_t(x,t) = \eta_{xx}(x,t)\,, \tag{72}$$
$$\eta_x(0,t) = 0\,, \tag{73}$$
$$\eta(1,t) = u(1,t)\,. \tag{74}$$

If the closed loop system (53)–(54), (67)–(74) has a classical solution (u, $\hat{q}$, v, η), then for any $\hat{q}(0)$ and any initial conditions u_0, v_0, $\eta_0 \in L_2(0,1)$, the signals $\hat{q}(t)$, $\|u\|$, $\|v\|$, $\|\eta\|$ are bounded and $\|u\|$ is regulated to zero:

$$\lim_{t\to\infty} \|u\| = 0. \tag{75}$$

In addition, $u(x,t)$ is square integrable in t for all $x \in [0,1]$.

Although the plants considered in Sections II (g-system) and IV (q-system) look quite similar, the adaptive stabilization problem for the latter is substantially harder due to uncertainty in the boundary condition. The proof becomes harder and the end result is a little weaker — L_2 boundedness and regulation instead of pointwise boundedness and regulation.

V. Proof of Theorem 3

A. Target system

Introducing the error $e = u - qv - \eta$ we get an exponentially stable system

$$e_t(x,t) = e_{xx}(x,t)\,, \tag{76}$$
$$e_x(0,t) = 0\,, \tag{77}$$
$$e(1,t) = 0\,. \tag{78}$$

The transformation

$$\hat{w}(x,t) = \hat{q}v(x,t) + \eta(x,t) + \int_0^x \hat{q}e^{\hat{q}(x-\xi)}(\hat{q}v(\xi,t) + \eta(\xi,t))\,d\xi \tag{79}$$

maps (53)–(54), (67) into the following system (Lemma A.1):

$$\hat{w}_t(x,t) = \hat{w}_{xx}(x,t) + \hat{q}^2e^{\hat{q}x}\hat{e}(0,t) + \dot{\hat{q}}v + \dot{\hat{q}}\int_0^x e^{\hat{q}(x-\xi)}(\hat{q}v(\xi,t) + \hat{w}(\xi,t))\,d\xi\,, \tag{80}$$
$$\hat{w}_x(0,t) = -\hat{q}\hat{e}(0,t)\,, \tag{81}$$
$$\hat{w}(1,t) = 0\,. \tag{82}$$

B. Adaptive law properties

This step is almost the same as in Section III for the g-system. We take the following equation as a parametric model

$$e(0,t) = u(0,t) - qv(0,t) - \eta(0,t)\,. \tag{83}$$

The estimation error is

$$\hat{e}(0,t) = u(0,t) - \hat{q}v(0,t) - \eta(0,t)\,. \tag{84}$$

Using the gradient update law

$$\dot{\hat{q}} = \gamma\frac{\hat{e}(0,t)v(0,t)}{1 + v^2(0,t)} \tag{85}$$

we get the following properties (as in Lemma 2)

$$\frac{\hat{e}(0,t)}{\sqrt{1 + v^2(0,t)}} \in \mathcal{L}_2 \cap \mathcal{L}_\infty, \quad \tilde{q} \in \mathcal{L}_\infty, \quad \dot{\hat{q}} \in \mathcal{L}_2 \cap \mathcal{L}_\infty\,. \tag{86}$$

We denote the bound on $\hat{q}$ by q_0.

C. Boundedness

First we rewrite v-filter as

$$v_t(x,t) = v_{xx}(x,t)\,, \tag{87}$$
$$v_x(0,t) = -\hat{w}(0,t) - \hat{e}(0,t)\,, \tag{88}$$
$$v(1,t) = 0\,, \tag{89}$$

We have two interconnected systems for $\hat{w}$ and v driven by the signal $\hat{e}(0,t)$ with properties (86). Consider a Lyapunov function

$$V = \frac{1}{2}\int_0^1 \hat{w}^2(x)\,dx + \frac{1}{2}\int_0^1 v^2(x)\,dx\,. \tag{90}$$

We have

$$\begin{aligned}\dot{V} &= -\hat{w}(0)\hat{w}_x(0) - \int_0^1 \hat{w}_x^2\,dx + \dot{\hat{q}}\int_0^1 \hat{w}(x)v(x)\,dx \\ &\quad + \dot{\hat{q}}\int_0^1 \hat{w}(x)\int_0^x e^{\hat{q}(x-\xi)}(\hat{q}v(\xi)+\hat{w}(\xi))\,d\xi\,dx \\ &\quad + \hat{e}(0)\int_0^1 \hat{q}^2 e^{\hat{q}x}\hat{w}(x)\,dx - v(0)v_x(0) - \int_0^1 v_x^2\,dx \\ &\le -\|\hat{w}_x\|^2 + |\hat{e}(0)|(q_0|\hat{w}(0)| + q_0^2 e^{q_0}\|\hat{w}\|) + c_1\|\hat{w}\|^2 \\ &\quad + \frac{(1+q_0e^{q_0})^2|\dot{\hat{q}}|^2}{2c_1}\|v\|^2 + \frac{e^{2q_0}|\dot{\hat{q}}|^2}{2c_1}\|\hat{w}\|^2 \\ &\quad -\|v_x\|^2 + \frac{1}{2}\|v_x\|^2 + \frac{1}{2}\|\hat{w}_x\|^2 + |v(0)||\hat{e}(0)|\,. \quad (91)\end{aligned}$$

Estimates of particular terms:

$$\begin{aligned}q_0|\hat{e}(0)||\hat{w}(0)| &\le q_0|\hat{w}(0)|\frac{\hat{e}(0)}{\sqrt{1+v^2(0)}}(1+|v(0)|) \\ &\le c_2\|\hat{w}_x\|^2 + \frac{q_0^2}{4c_2}\frac{\hat{e}^2(0)}{1+v^2(0)} \\ &\quad + 2q_0\sqrt{\|\hat{w}\|\|\hat{w}_x\|\|v\|\|v_x\|}\frac{|\hat{e}(0)|}{\sqrt{1+v^2(0)}} \\ &\le c_2\|\hat{w}_x\|^2 + l_1 \\ &\quad + \frac{q_0|\hat{e}(0)|}{\sqrt{1+v^2(0)}}(\|\hat{w}\|\|\hat{w}_x\| + \|v\|\|v_x\|) \\ &\le c_2\|\hat{w}_x\|^2 + c_3\|\hat{w}_x\|^2 + c_4\|v_x\|^2 \\ &\quad + l_1\|v\|^2 + l_1\|\hat{w}\|^2 + l_1, \quad (92)\end{aligned}$$

$$\begin{aligned}q_0^2e^{q_0}|\hat{e}(0)|\|\hat{w}\| &\le q_0^2e^{q_0}\|\hat{w}\|\frac{\hat{e}(0)}{\sqrt{1+v^2(0)}}(1+|v(0)|) \\ &\le c_5\|\hat{w}\|^2 + \frac{q_0^4e^{2q_0}}{4c_5}\frac{\hat{e}^2(0)}{1+v^2(0)} + c_6\|v_x\|^2 \\ &\quad + \frac{q_0^4e^{2q_0}}{4c_6}\frac{\hat{e}^2(0)}{1+v^2(0)}\|\hat{w}\|^2 \\ &\le c_5\|\hat{w}\|^2 + c_6\|v_x\|^2 + l_1\|\hat{w}\|^2 + l_1, \quad (93)\end{aligned}$$

$$\begin{aligned}|v(0)||\hat{e}(0)| &\le \frac{|v(0)||\hat{e}(0)|}{1+v^2(0)}(1+2\|v\|\|v_x\|) \\ &\le \frac{c_7}{2}\|v_x\|^2 + \frac{1}{2c_7}\frac{\hat{e}^2(0)}{1+v^2(0)} + \frac{c_7}{2}\|v_x\|^2 \\ &\quad + \frac{2}{c_7}\left(\frac{|v(0)||\hat{e}(0)|}{1+v^2(0)}\right)^2\|v\|^2 \\ &\le c_7\|v_x\|^2 + l_1\|v\|^2 + l_1\,. \quad (94)\end{aligned}$$

In the last inequality we used the fact that $\dot{\hat{q}}^2$ is an l_1 function. We have

$$\begin{aligned}\dot{V} &\le -\left(\frac{1}{2} - 4c_1 - c_2 - c_3 - 4c_5\right)\|\hat{w}_x\|^2 + l_1\|\hat{w}\|^2 \\ &\quad -\left(\frac{1}{2} - c_4 - c_6 - c_7\right)\|v_x\|^2 + l_1\|v\|^2 + l_1\,. \quad (95)\end{aligned}$$

Choosing $4c_1 = c_2 = c_3 = 4c_5 = 1/16$, $c_4 = c_6 = c_7 = 1/12$, we get

$$\dot{V} \le -\frac{1}{8}V + l_1 V + l_1 \quad (96)$$

and by Lemma A.2 we obtain $\|\hat{w}\|,\|v\| \in \mathcal{L}_2 \cap \mathcal{L}_\infty$.

D. Regulation

It is easy to see from (96) that $\dot{V}$ is bounded from above. By using an alternative to Barbalat's lemma [15, Lemma 3.1] we get $V \to 0$, that is $\|\hat{w}\| \to 0$, $\|v\| \to 0$. From (A.3) we have $\|\eta\| \to 0$. Since $u = e + qv + \eta$, we get $\|u\| \to 0$.

By integrating (95) we get $\|\hat{w}_x\|,\|v_x\| \in \mathcal{L}_2$, and from (A.3) $\|\eta_x\| \in \mathcal{L}_2$ and therefore $\|u_x\| \in \mathcal{L}_2$. Square integrability in time of $u(x,t)$ for all $x \in [0,1]$ follows from Agmon's inequality.

VI. Plant with Two Unknown Parameters

For the sake of clarity and due to different adaptive regulation properties that can be achieved, we considered two benchmark problems separately. It is also possible to design an adaptive controller for the combined system

$$u_t(x,t) = u_{xx}(x,t) + gu(0,t)\,, \quad (97)$$

$$u_x(0,t) = -qu(0,t)\,, \quad (98)$$

This system is unstable if and only if $2q + g > 2$. The non-adaptive control law can be designed based on the controllers for separate problems by using the method described in [18, Sec. VIII-E]. We state here the result without a proof.

Theorem 4: Consider the plant (97)–(98) with the controller

$$u(1) = \int_0^1 \frac{r_1^2 e^{r_1(1-x)} - r_2^2 e^{r_2(1-x)}}{2\sqrt{\hat{g} + \hat{q}^2/4}}(\hat{g}v + \hat{q}p + \eta)\,dx, \quad (99)$$

where the update laws for $\hat{g}$ and $\hat{q}$ are

$$\dot{\hat{g}} = \gamma_1\frac{\hat{e}(0)v(0,t)}{1+v^2(0,t)+p^2(0,t)}, \quad (100)$$

$$\dot{\hat{q}} = \gamma_2\frac{\hat{e}(0)p(0,t)}{1+v^2(0,t)+p^2(0,t)}, \quad (101)$$

the input filter is

$$\eta_t = \eta_{xx} \quad (102)$$

$$\eta_x(0) = 0 \quad (103)$$

$$\eta(1) = u(1) \quad (104)$$

and the output filters are

$$\begin{aligned} v_t &= v_{xx} + u(0) & \qquad p_t &= p_{xx} \\ v_x(0) &= 0 & p_x(0) &= -u(0) \\ v(1) &= 0 & p(1) &= 0 \end{aligned} \quad (105)$$

with $\hat{e}(0) = u(0) - \hat{g}v(0) - \hat{q}p(0) - \eta(0)$ and

$$r_{1,2} = \frac{\hat{q}}{2} \mp \sqrt{\hat{g} + \frac{\hat{q}^2}{4}}\,. \quad (106)$$

If the closed loop system (97)–(106) has a classical solution $(u,\ \hat{g},\ \hat{q},\ v,\ p,\ \eta)$, then for any $\hat{g}(0)$, $\hat{q}(0)$ and any initial conditions u_0, v_0, p_0, $\eta_0 \in L_2(0,1)$, the signals $\hat{g}(t)$, $\hat{q}(t)$, $\|u\|$, $\|v\|$, $\|p\|$, $\|\eta\|$ are bounded and $\|u\|$ is regulated to zero:

$$\lim_{t\to\infty}\|u\| = 0. \quad (107)$$

In addition, $u(x,t)$ is square integrable in t for all $x \in [0,1]$.

Remark 1: If the expression $\hat{g} + \hat{q}^2/4$ becomes negative, $r_{1,2}$ become complex. However, the control gain in (99) remains real and well defined.

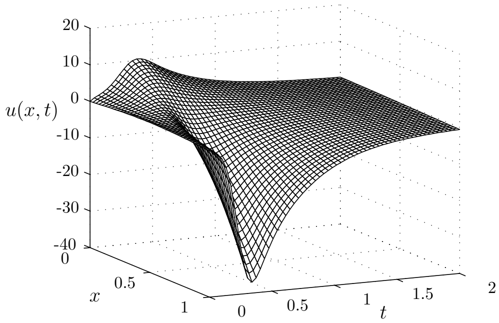

Fig. 1. The state $u(x,t)$ with the adaptive output feedback controller (99).

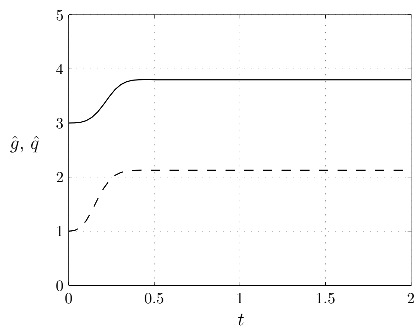

Fig. 2. The parameters $\hat{g}$ (solid) and $\hat{q}$ (dashed). The unknown parameters are set to $g = 4$ and $q = 2$.

VII. Simulations

We present now the results of closed-loop simulations for the system (97)–(98). The plant parameters are set to $g = 4$ and $q = 2$, with these values the unstable eigenvalue ≈ 10. For the update laws we take $\hat{g}(0) = 3$, $\hat{q}(0) = 1$, and $\gamma_1 = \gamma_2 = 15$. The results are shown in Fig. 1-2. We can see that although the instability occurs at the $x = 0$ boundary, the system is successfully regulated to zero by the control from the opposite boundary.

Appendix

Lemma A.1: The transformation (26) maps the system (1)–(2), (10) into (27)–(29). The transformation (79) maps the system (53)–(54), (67) into (80)–(82).

Proof: It is easy to check that boundary conditions (28) and (29) are satisfied. Substituting (26) into (1) we get

$$\begin{aligned} w_t &= w_{xx} + \dot{\hat{g}} v - \dot{\hat{g}} \int_0^x \{(k_{\hat{g}}(x,\xi,\hat{g})\hat{g} + k(x,\xi,\hat{g}))v(\xi) \\ &\quad + k_{\hat{g}}(x,\xi,\hat{g})\eta(\xi)\}\, d\xi + k_\xi(x,0,\hat{g})\hat{e}(0)\,. \end{aligned} \tag{A.1}$$

To express the signal η in terms of v and w we use the inverse transformation to (26):

$$\hat{g}v(x,t) + \eta(x,t) = w(x,t) - \hat{g}\int_0^x (x-\xi)w(\xi,t)\, d\xi\,. \tag{A.2}$$

Changing the order of integration and taking necessary derivatives of $k(x,\xi,\hat{g})$ we come to (27)–(29).

Second part of the lemma is proved in the same way. It is easy to check that (79) satisfies boundary conditions (81) and (82). Substituting (79) into (53) we get (A.1) but with $\hat{g}$ changed to $\hat{q}$ everywhere. To express the signal η in terms of v and w we use the inverse transformation to (79):

$$\hat{q}v(x,t) + \eta(x,t) = w(x,t) - \hat{q}\int_0^x w(\xi,t)\, d\xi\,. \tag{A.3}$$

Changing the order of integration and taking necessary derivatives of $k(x,\xi,\hat{q})$ we come to (80)–(82). ■

Lemma A.2 (Lemma B.6 in [14]): Let v, l_1, and l_2 be real-valued functions defined on R_+, and let c be a positive constant. If l_1 and l_2 are nonnegative and in $\mathcal{L}_1$ and satisfy the differential inequality

$$\dot{v} \le -cv + l_1(t)v + l_2(t), \quad v(0) \ge 0 \tag{A.4}$$

then $v \in \mathcal{L}_\infty \bigcap \mathcal{L}_1$.

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