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Synthesis of Output Feedback Controllers for Descriptor Systems Satisfying Closed-Loop Dissipativity

Izumi Masubuchi

Abstract— **This paper is concerned with synthesis of output feedback controllers for descriptor systems to attain dissipativity of the closed-loop system. A necessary and sufficient condition is provided in terms of LMIs for the existence of a controller satisfying dissipativity and admissibility (an internal stability) of the closed-loop system. Unlike previous results, the condition does not depend on the choice of the descriptor realization. The derived LMI condition with a rank constraint for synthesis is a generalization of those for LMI-based H_∞ control of state-space systems to descriptor systems.**

I. Introduction

Descriptor representation of dynamical systems is more general and often more natural than state-space systems (See e.g., [5]). The descriptor form is useful to represent and to handle systems such as mechanical systems, electric circuits, interconnected systems, and so on. Moreover, descriptor representation is utilized in some of recent results on analysis of time-delay systems (e.g., [13]) and gain-scheduled control based on linear parameter-varying systems [9], [10]. These new applications exhibit further importance and usefulness of descriptor systems.

Among basic notions of state-space systems generalized to descriptor systems, dissipativity is one of the most important properties of dynamical systems and plays crucial roles in various problems of analysis and synthesis of control systems, including positive and bounded realness. For state-space systems, Kalman-Popov-Yakuvobich (KYP) Lemma and related results provide characterization of positive or bounded realness in terms of the state-space realization [1], [19], [11]. Also for descriptor systems there have been proposed several criteria for positive or bounded realness [2], [6], [12], [14], [17], [18], [20]. However, most of existing results require a certain assumption or restriction on the realization of descriptor systems, while KYP Lemma for state-space systems is valid independently of the choice of the realization. On the other hand, a new matrix inequality condition is proposed recently that is necessary and sufficient for dissipativity of a descriptor system with any realization [7], [8].

In this paper, we consider output feedback controller synthesis for descriptor systems to attain dissipativity of the closed-loop system. Based on the criterion proposed in [7], [8], a necessary and sufficient condition is derived in terms of LMIs with a rank constraint for existence of a controller satisfying dissipativity and admissibility[1] of the closed-loop system. Unlike previous results, the proposed condition does not depend on the choice of the descriptor realization.

The derivation of the existence criterion is based on the 'variable elimination' methodology, which is a key technique in H_∞ synthesis of state-space systems [3], [4]. However, the LMI of the dissipativity criterion [7], [8] that our results are based on has structure for which output feedback synthesis has never been considered before. Hence we provide methods to handle the new LMI for synthesis of descriptor systems. The proposed LMI condition for synthesis is a generalization of the results for state-space systems [3], [4] to descriptor systems.

Notation. For a matrix X, we denote by X^{-1}, X^{T}, $X^{-\mathsf{T}}$ and X^* the inverse, the transpose, the inverse of the transpose and the conjugate transpose of X, respectively. Let X be a square matrix, $\mathbf{He}X$ stands for $X+X^{\mathsf{T}}$. In addition, $X=(*)^{\mathsf{T}}$ and $X+(*)^{\mathsf{T}}$ mean $X=X^{\mathsf{T}}$ and $X+X^{\mathsf{T}}$, respectively. For a symmetric matrix represented blockwise, such as $X=\begin{bmatrix} X_{11} & X_{12} \\ X_{12}^{\mathsf{T}} & X_{22}\end{bmatrix}$, offdiagonal blocks can be abbreviated with '$*$', as $X=\begin{bmatrix} X_{11} & X_{12} \\ * & X_{22}\end{bmatrix}$. For a matrix $M\in\mathbf{R}^{m\times n}$ with $m>n$, let $M^\perp$ be a matrix satisfying $M^\perp\begin{bmatrix} (M^\perp)^{\mathsf{T}} & M\end{bmatrix}=\begin{bmatrix} I & 0\end{bmatrix}$. For $M\in\mathbf{R}^{m\times n}$ with $m<n$, define $M^\perp:=((M^{\mathsf{T}})^\perp)^{\mathsf{T}}$. The $n\times n$ identity matrix is represented by I_n. The zero matrix of the size $m\times n$ is $0_{m\times n}$.

II. Preliminaries

Consider the following descriptor system:

$$\left\{\begin{array}{rcl} E\dot{x} &=& Ax+Bw, \\ z &=& Cx+Dw, \end{array}\right. \tag{1}$$

where $x\in\mathbf{R}^n$ is the descriptor variable, $w\in\mathbf{R}^m$ is the input and $z\in\mathbf{R}^p$ is the output of the system. Let $E\in\mathbf{R}^{n\times n}$ and $\mathrm{rank}E=r$.

Definition 1: (1°) The pencil $sE-A$ is *regular* if $\det(sE-A)$ is not identically zero. (2°) Suppose that $sE-A$ is regular. The *exponential modes* of $sE-A$ are the finite eigenvalues of $sE-A$, namely, $s\in\mathbf{C}$ such that $\det(sE-A)=0$. (3°) Let a vector v_1 satisfy $Ev_1=0$. Then the infinite eigenvalues associated with the generalized eigenvectors v_k satisfying $Ev_k=Av_{k-1}$, $k=2,3,4,\ldots$ are *impulsive modes* of (E,A). (4°) The descriptor system (1) is *impulse-free* if the pencil $sE-A$ is regular and has no impulsive modes. (5°) The pencil $sE-A$ is said to be *admissible* if the pencil $sE-A$ is regular, impulse-free and has no unstable exponential modes.

The author is with Graduate School of Engineering, Hiroshima University, 1-4-1 Kagamiyama, Higashi-Hiroshima 739-8527, Japan. E-mail: msb@hiroshima-u.ac.jp

[1] An internal stability of descriptor systems[6].

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Next, let $S = S^{\mathsf{T}} \in \mathbf{R}^{(m+p)\times(m+p)}$ and consider the following quadratic form of (w, z):

$$s(w,z) = \begin{bmatrix} w \\ z \end{bmatrix}^{\mathsf{T}} S \begin{bmatrix} w \\ z \end{bmatrix}, \tag{2}$$

which defines a *supply rate*.

Definition 2: The descriptor system (1) is said to be *dissipative* with respect to the supply rate $s(\cdot,\cdot)$ if the descriptor system (1) is impulse-free and for any $w \in L_2[0,T]$ it holds that

$$\int_0^T s(w(t), z(t))\mathrm{d}t \le 0, \quad \forall T \ge 0 \tag{3}$$

provided $x(0) = 0$.

The time-domain condition (3) is equivalent to the following condition in the frequency-domain:

$$\begin{bmatrix} I \\ G(j\omega) \end{bmatrix}^{*} S \begin{bmatrix} I \\ G(j\omega) \end{bmatrix} \le 0, \forall \omega \in \mathbf{R} \cup \{\infty\}, \tag{4}$$

where $G(s) = C(sE - A)^{-1}B + D$.

Let the descriptor system (1) be admissible. Then H_∞ norm condition $\|G\|_\infty < \gamma$ is represented via (4) by setting

$$S = \begin{bmatrix} -\gamma^2 I & 0 \\ 0 & I \end{bmatrix}. \tag{5}$$

When $m = p$, the extended strict positive realness (ESPR in short) [20] with admissibility is yielded with

$$S = \begin{bmatrix} 0 & -I \\ -I & 0 \end{bmatrix}. \tag{6}$$

Assumption 1: Denoting

$$S = \begin{bmatrix} S_{11} & S_{12} \\ S_{12}^{\mathsf{T}} & S_{22} \end{bmatrix}, \quad S_{11} \in \mathbf{R}^{m\times m}, \tag{7}$$

we assume that $S_{22} \ge 0$.

Note that the supply rates for these two important specific dissipativity conditions of H_∞ norm and positive realness meet Assumption 1.

There have been proposed several LMI criteria for H_∞ norm condition or positive realness for descriptor systems [2], [6], [12], [14], [17], [18], [20]. e.g., as follows:

Lemma 1: Suppose that $\|D\|_\infty < \gamma$. Then the descriptor system (1) is admissible and its H_∞ norm is less than γ if and only if there exists a matrix $X \in \mathbf{R}^{n\times n}$ that satisfies the following LMI:

$$E^{\mathsf{T}}X = X^{\mathsf{T}}E \ge 0, \tag{8}$$

$$\begin{bmatrix} A^{\mathsf{T}}X + X^{\mathsf{T}}A & X^{\mathsf{T}}B & C^{\mathsf{T}} \\ * & -\gamma I & D^{\mathsf{T}} \\ * & * & -\gamma I \end{bmatrix} < 0. \tag{9}$$

This lemma assumes that $\|D\|_\infty < \gamma$. However, it can be violated by choosing descriptor realization of $G(s)$. For example, consider the following descriptor system:

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 & 0 \\ -3 & -2 & 1 \\ 0 & 0 & -1 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 5 & 1-\kappa \end{bmatrix}, \quad D = \kappa,$$

where κ is a scalar. For every κ,

$$G(s) = C(sE - A)^{-1}B + D = \frac{s^2 + 7s + 5}{s^2 + 2s + 3}.$$

and $\|G\|_\infty = 3.5551$. Nevertheless, the LMI condition (8)–(9) does not hold when descriptor realization is chosen so that $|D| = |\kappa| > \|G\|_\infty$ [7], [8].

This drawback of existing criteria is removed in the recent result for admissibility and dissipativity shown in [7], [8]:

Lemma 2: The following two conditions are equivalent:

(i) The descriptor system (1) is admissible and satisfies

$$\begin{bmatrix} I \\ G(j\omega) \end{bmatrix}^{*} S \begin{bmatrix} I \\ G(j\omega) \end{bmatrix} < 0 \tag{10}$$

for any $\omega \in \mathbf{R} \cup \{\infty\}$.

(ii) There exist matrices $X \in \mathbf{R}^{n\times n}$ and $W \in \mathbf{R}^{n\times m}$ satisfying

$$E^{\mathsf{T}}X = X^{\mathsf{T}}E \ge 0, \quad E^{\mathsf{T}}W = 0, \tag{11}$$

$$M + \begin{bmatrix} X^{\mathsf{T}} \\ W^{\mathsf{T}} \end{bmatrix} \begin{bmatrix} A & B \end{bmatrix} + (*)^{\mathsf{T}} < 0, \tag{12}$$

where

$$M = \begin{bmatrix} 0 & I \\ C & D \end{bmatrix}^{\mathsf{T}} S \begin{bmatrix} 0 & I \\ C & D \end{bmatrix}. \tag{13}$$

Proof: See [7], [8] for the proof. ■

Remark 1: From Assumption 1, the LMI (12) implies $A^{\mathsf{T}}X + X^{\mathsf{T}}A < 0$ and hence X is nonsingular.

By using this lemma, e.g., an H_∞ norm criterion is given independent of the choice of the descriptor realization by the following LMI:

$$E^{\mathsf{T}}X = X^{\mathsf{T}}E \ge 0, \quad E^{\mathsf{T}}W = 0, \tag{14}$$

$$\begin{bmatrix} A^{\mathsf{T}}X + X^{\mathsf{T}}A & A^{\mathsf{T}}W + X^{\mathsf{T}}B & C^{\mathsf{T}} \\ * & B^{\mathsf{T}}W + W^{\mathsf{T}}B - \gamma I & D^{\mathsf{T}} \\ * & * & -\gamma I \end{bmatrix} < 0. \tag{15}$$

III. Main Results

A. Control system in the descriptor form

Based on the new criterion for admissibility and dissipativity of descriptor systems shown in the previous section, we consider synthesis of an output feedback controller to attain admissibility and dissipativity of a control system in the descriptor form. Let us represent the plant as follows:

$$\begin{cases} E\dot{x} &= Ax + B_1 w + B_2 u, \\ z &= C_1 x + D_{11} w + D_{12} u, \\ y &= C_2 x + D_{21} w, \end{cases} \tag{16}$$

where $x \in \mathbf{R}^n$ is the descriptor variable, $w \in \mathbf{R}^{m_1}$ is the external input, $u \in \mathbf{R}^{m_2}$ is the control input, $z \in \mathbf{R}^{p_1}$ is the controlled output and $y \in \mathbf{R}^{p_2}$ is the measured output. Let $E \in \mathbf{R}^{n\times n}$ and $\mathrm{rank}E = r$. We consider the following output feedback controller:

$$\begin{cases} E_c\dot{x}_c &= A_c x_c + B_c y, \\ u &= C_c x_c + D_c y, \end{cases} \tag{17}$$

where $E_c \in \mathbf{R}^{n_c \times n_c}$ with rank$E_c = r_c$ and $x_c \in \mathbf{R}^{n_c}$. Connecting this controller to the plant (16) yields the closed-loop system as follows:

$$\begin{cases} E_{cl}\dot{x}_{cl} &= A_{cl}x_{cl} + B_{cl}w, \\ z &= C_{cl}x_{cl} + D_{cl}w, \end{cases} \tag{18}$$

where $x_{cl}(t) = \begin{bmatrix} x(t)^\mathsf{T} & x_c(t)^\mathsf{T} \end{bmatrix}^\mathsf{T} \in \mathbf{R}^{n_{cl}}$, $n_{cl} = n + n_c$ and

$$E_{cl} = \begin{bmatrix} E & 0 \\ 0 & E_c \end{bmatrix}, \tag{19}$$

$$A_{cl} = \begin{bmatrix} A + B_2 D_c C_2 & B_2 C_c \\ B_c C_2 & A_c \end{bmatrix}, \tag{20}$$

$$B_{cl} = \begin{bmatrix} B_1 + B_2 D_c D_{21} \\ B_c D_{21} \end{bmatrix}, \tag{21}$$

$$C_{cl} = \begin{bmatrix} C_1 + D_{12} D_c C_2 & D_{12} C_c \end{bmatrix}, \tag{22}$$

$$D_{cl} = D_{11} + D_{12} D_c D_{21}. \tag{23}$$

B. Problem statement

Let us consider the supply rate (2) for $S \in \mathbf{R}^{(m_1+p_1)\times(m_1+p_1)}$ and partition (7) with $S_{11} \in \mathbf{R}^{m_1 \times m_1}$. Assume the following without loss of generality.

Assumption 2: 1) $\begin{bmatrix} B_2 \\ D_{21} \end{bmatrix}$ has full column rank.

2) $\begin{bmatrix} C_2 & D_{12} \end{bmatrix}$ has full row rank.

3) $\begin{bmatrix} S_{12} \\ S_{22} \end{bmatrix}$ has full column rank.

Note that when the third assumption is not satisfied one can always redefine the controlled output z so that the same supply rate is defined for the new z with a new S satisfying the assumption.

Now the synthesis problem is stated as follows: *given a plant in the descriptor form (16) and a quadratic supply rate (2), find a controller (17) for which the closed-loop system (18) is admissible and satisfies dissipativity with respect to the supply rate.*

Let $T_{22} \in \mathbf{R}^{q \times p_1}$ be a matrix satisfying $S_{22} = T_{22}^\mathsf{T} T_{22}$. It is easy to see from Lemma 2 that the synthesis problem is solvable if and only if there exist matrices X_{cl} and W_{cl} satisfying

$$E_{cl}^\mathsf{T} X_{cl} = X_{cl}^\mathsf{T} E_{cl} \geq 0, \quad E_{cl}^\mathsf{T} W_{cl} = 0, \tag{24}$$

$$\mathbf{He} U_X + \left[\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 0 & S_{11} & 0 \\ 0 & 0 & -I_q \end{array}\right] < 0, \tag{25}$$

where

$$U_X := \left[\begin{array}{c|c} 0 & X_{cl}^\mathsf{T} \\ \hline S_{12} & W_{cl}^\mathsf{T} \\ T_{22} & 0 \end{array}\right] \left[\begin{array}{c|c} D_{cl} & C_{cl} \\ \hline B_{cl} & A_{cl} \end{array}\right] \left[\begin{array}{c|cc} 0 & I_{m_1} & 0 \\ \hline I_{n_{cl}} & 0 & 0_{n_{cl}\times q} \end{array}\right]. \tag{26}$$

This LMI condition is equivalent to (11)–(12) in Lemma 2 applied to the closed-loop system (18) and is derived via simple manipulations. Since X_{cl} is nonsingular if (25) holds, the condition (24)–(25) is equivalent to the following:

$$E_{cl} Y_{cl}^\mathsf{T} = Y_{cl} E_{cl}^\mathsf{T} \geq 0, \quad E_{cl} Z_{cl}^\mathsf{T} = 0, \tag{27}$$

$$\mathbf{He} U_Y + \left[\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 0 & S_{11} & 0 \\ 0 & 0 & -I_q \end{array}\right] < 0, \tag{28}$$

where

$$U_Y := \left[\begin{array}{c|c} 0 & I_{n_{cl}} \\ \hline S_{12} & 0 \\ T_{22} & 0 \end{array}\right] \left[\begin{array}{c|c} D_{cl} & C_{cl} \\ \hline B_{cl} & A_{cl} \end{array}\right] \left[\begin{array}{c|cc} 0 & I_{m_1} & 0 \\ \hline Y_{cl}^\mathsf{T} & Z_{cl}^\mathsf{T} & 0_{n_{cl}\times q} \end{array}\right]$$

and Y_{cl}, Z_{cl} are set by

$$Y_{cl} = X_{cl}^{-\mathsf{T}}, \quad Z_{cl} = -W_{cl}^\mathsf{T} X_{cl}^{-\mathsf{T}}. \tag{29}$$

The matrix inequalities (24) and (25) are congruent to (27) and (28), respectively. It is easy to see that the equality $E_{cl} Z_{cl}^\mathsf{T} = 0$ is equivalent to $E_{cl}^\mathsf{T} W_{cl} = 0$.

C. Existence condition

In this subsection, we show the existence condition of a controller satisfying admissibility and dissipativity of the closed-loop system. Under Assumptions 1 and 2, define the following matrices from S:

$$M := \begin{bmatrix} \begin{bmatrix} S_{12} \\ T_{22} \end{bmatrix} & \left(\begin{bmatrix} S_{12} \\ T_{22} \end{bmatrix}^\perp \right)^\mathsf{T} \end{bmatrix}, \tag{30}$$

$$\begin{bmatrix} N_1 & N_2 \end{bmatrix} := \begin{bmatrix} I & 0 \end{bmatrix} M^{-\mathsf{T}}. \tag{31}$$

$$\begin{bmatrix} H_{11} & H_{12} \\ H_{12}^\mathsf{T} & H_{22} \end{bmatrix} := M^{-1} \begin{bmatrix} S_{11} & 0 \\ 0 & -I \end{bmatrix} M^{-\mathsf{T}}. \tag{32}$$

Note that $\begin{bmatrix} S_{12} \\ T_{22} \end{bmatrix}$ has full column rank iff so does $\begin{bmatrix} S_{12} \\ S_{22} \end{bmatrix}$.

The following theorem provides an existence condition of a controller that solves the synthesis problem stated in the previous subsection. The criterion is given in terms LMIs and a rank condition.

Theorem 1: The following statements (I) and (II) are equivalent:

(I) There exists a controller (17) for which the closed-loop system (18) is admissible and satisfies dissipativity for the supply rate (2).

(II) There exist matrices X, Y, W, Z with appropriate sizes satisfying the following LMIs and rank condition:

$$\begin{bmatrix} E^\mathsf{T} & 0 \\ 0 & E \end{bmatrix} \begin{bmatrix} X & I \\ I & Y^\mathsf{T} \end{bmatrix} = (*)^\mathsf{T} \geq 0, \tag{33}$$

$$\operatorname{rank} \begin{bmatrix} E^\mathsf{T} & 0 \\ 0 & E \end{bmatrix} \begin{bmatrix} X & I \\ I & Y^\mathsf{T} \end{bmatrix} \leq r + r_c, \tag{34}$$

$$E^\mathsf{T} W = 0, \quad E Z^\mathsf{T} = 0, \tag{35}$$

$$N_B (L_B + L_B^\mathsf{T} + H_B) N_B^\mathsf{T} < 0, \tag{36}$$

$$N_C^\mathsf{T} (L_C + L_C^\mathsf{T} + H_C) N_C < 0, \tag{37}$$

where

$$N_B := \left[\begin{array}{c:c} N_{B0} & 0 \\ \hdashline 0 & I \end{array}\right],\ N_{B0} := \left[\begin{array}{c} B_2 \\ D_{12} \end{array}\right]^{\perp}, \tag{38}$$

$$N_C := \left[\begin{array}{c:c} N_{C0} & 0 \\ \hdashline 0 & I \end{array}\right],\ N_{C0} := \left[\begin{array}{cc} C_2 & D_{21} \end{array}\right]^{\perp}, \tag{39}$$

$$L_B := \left[\begin{array}{c:c} \left[\begin{array}{c} A \\ C_1 \end{array}\right] Y^{\mathsf T} & \left[\begin{array}{c} B_1 + AZ^{\mathsf T} \\ D_{11} + C_1 Z^{\mathsf T} \end{array}\right] \left[\begin{array}{c:c} N_1 & N_2 \end{array}\right] \\ \hdashline 0 & \left[\begin{array}{c:c} 0 & 0 \end{array}\right] \end{array}\right], \tag{40}$$

$$L_C := \left[\begin{array}{cc:c} X^{\mathsf T} A & X^{\mathsf T} B_1 & 0 \\ S_{12} C_1 + W^{\mathsf T} A & S_{12} D_{11} + W^{\mathsf T} B_1 & 0 \\ \hdashline T_{22} C_1 & T_{22} D_{11} & 0 \end{array}\right], \tag{41}$$

$$H_B := \left[\begin{array}{cc:c} 0 & 0 & 0 \\ 0 & H_{11} & H_{12} \\ \hdashline 0 & H_{12}^{\mathsf T} & H_{22} \end{array}\right], \tag{42}$$

$$H_C := \left[\begin{array}{cc:c} 0 & 0 & 0 \\ 0 & S_{11} & 0 \\ \hdashline 0 & 0 & -I \end{array}\right]. \tag{43}$$

If the conditions (33)–(37) are fulfilled, a controller in the descriptor form (17) with $\mathrm{rank} E_c \leq r_c$ solving the synthesis problem is constructed from a solution X, Y, W, Z. Furthermore, also a controller in the state-space form with order no more than r_c is derived from the solution.

We provide the procedure to obtain a controller satisfying the condition (I) in Subsection III-D, which also proves the sufficiency. The proof of the necessity is based on the matrix elimination lemma applied to LMIs (24)–(25) and (27)–(28). Detail of the necessity proof is omitted.

Remark 2: For matrix S shown in (5) and (6), the matrices M, N_i, H_{ij} are defined as follows:

- S of (5) (for H_∞ norm condition):

$$M = \left[\begin{array}{cc} 0 & I \\ I & 0 \end{array}\right],\quad \left[\begin{array}{c} N_1^{\mathsf T} \\ N_2^{\mathsf T} \end{array}\right] = \left[\begin{array}{c} 0 \\ I \end{array}\right],$$
$$\left[\begin{array}{cc} H_{11} & H_{12} \\ H_{12}^{\mathsf T} & H_{22} \end{array}\right] = \left[\begin{array}{cc} -I & 0 \\ 0 & -\gamma^2 I \end{array}\right].$$

- S of (6) (for ESPR condition):

$$M = \left[\begin{array}{cc} -I & 0 \\ 0 & -I \end{array}\right],\quad \left[\begin{array}{c} N_1^{\mathsf T} \\ N_2^{\mathsf T} \end{array}\right] = \left[\begin{array}{c} -I \\ 0 \end{array}\right],$$
$$\left[\begin{array}{cc} H_{11} & H_{12} \\ H_{12}^{\mathsf T} & H_{22} \end{array}\right] = \left[\begin{array}{cc} 0 & 0 \\ 0 & -I \end{array}\right].$$

Hereby we see that for H_∞ synthesis (with S of (5)), the inequalities (36) and (37) become

$$N_B \left[\begin{array}{cc:c} AY^{\mathsf T} + Y^{\mathsf T} A & YC_1^{\mathsf T} & B_1 + AZ^{\mathsf T} \\ C_1 Y^{\mathsf T} & -I & D_{11} + C_1 Z^{\mathsf T} \\ \hdashline B_1^{\mathsf T} + ZA^{\mathsf T} & D_1 1^{\mathsf T} + ZC_1^{\mathsf T} & -\gamma^2 I \end{array}\right] N_B^{\mathsf T} < 0, \tag{44}$$

$$N_C^{\mathsf T} \left[\begin{array}{c:c} \mathbf{He}\left[\begin{array}{cc} X^{\mathsf T} A & X^{\mathsf T} B_1 \\ W^{\mathsf T} A & W^{\mathsf T} B_1 \end{array}\right] - \left[\begin{array}{cc} 0 & 0 \\ 0 & -\gamma^2 I \end{array}\right] & \begin{array}{c} * \\ * \end{array} \\ \hdashline \left[\begin{array}{cc} C_1 & D_{11} \end{array}\right] & -I \end{array}\right] N_C < 0, \tag{45}$$

respectively. For synthesis of a controller satisfying closed-loop ESPR (with S of (6)), we can easily see that the conditions (36) and (37) are equivalent to

$$N_{B0} \mathbf{He} \left[\begin{array}{cc} AY^{\mathsf T} & -(B_1 + AZ^{\mathsf T}) \\ C_1 Y^{\mathsf T} & -(D_{11} + C_1 Z^{\mathsf T}) \end{array}\right] N_{B0}^{\mathsf T} < 0, \tag{46}$$

$$N_{C0}^{\mathsf T} \mathbf{He} \left[\begin{array}{cc} X^{\mathsf T} A & X^{\mathsf T} B_1 \\ -C_1 + W^{\mathsf T} A & -D_{11} + W^{\mathsf T} B_1 \end{array}\right] N_{C0} < 0, \tag{47}$$

respectively.

Remark 3: The inequalities of existence condition for H_∞ synthesis are similar to those in previous results for descriptor systems. The largest difference is that our inequality condition has variables W and Z. With these variables the LMI condition (33)–(37) is equivalent to the solvability of the synthesis problem without depending on the realization of the plant. Actually, denoting by γ^* the infimum of H_∞ norm from w to z that the closed-loop system can attain via feedback (17), we see that (44) and (45) never hold if W and Z are set zero and the realization of (16) is such that $\|D_{11}\|_\infty > \gamma^*$; see Section IV.

As in the state-space case, the order of the dynamic part of the controller corresponds to the rank condition (34). This makes the whole inequality condition nonconvex. When the order r_c is not required to be less than r, the rank condition (34) is removed and the other inequalities are convex.

Lastly, when $E = I$, the existence condition (33)–(37) coincides with that for state-space systems [3], [4], where (35) implies that W and Z vanish for $E = I$.

D. Construction of a controller

Here we show a procedure to derive a controller satisfying admissibility and dissipativity of the closed-loop system when the conditions (33)–(37) hold. Without loss of generality, we assume that

$$E = \left[\begin{array}{cc} I_r & 0 \\ 0 & 0_{s\times s} \end{array}\right],\quad r + s = n. \tag{48}$$

We seek a controller with setting E_c as follows:

$$E_c = \left[\begin{array}{cc} I_{r_c} & 0 \\ 0 & 0_{s_c\times s_c} \end{array}\right],\quad r_c + s_c = n_c. \tag{49}$$

(Step 1.) According to the above block form, the variables X, Y, W, Z satisfying the equality conditions (33) and (35) are represented as:

$$\left\{ \begin{array}{l} \begin{bmatrix} X_{p11} & 0 \\ X_{p21} & X_{p22} \end{bmatrix} := X, \quad \begin{bmatrix} Y_{p11} & Y_{p12} \\ 0 & Y_{p22} \end{bmatrix} := Y, \\ \begin{bmatrix} 0_{r\times m_1} \\ W_{p2} \end{bmatrix} := W, \quad \begin{bmatrix} 0_{m_1\times r} & Z_{p2} \end{bmatrix} := Z \end{array} \right. \tag{50}$$

with $X_{p11}, Y_{p11} \in \mathbf{R}^{r\times r}$ being symmetric. Then we see that the submatrices X_{p22} and Y_{p22} can always be chosen to be nonsingular [6]; though solving (33)–(37) results singular X_{p22} and/or Y_{p22}, one can replace them with $X_{p22} + \varepsilon I$ and/or $Y_{p22} + \varepsilon I$, respectively, without violating the conditions (33)–(37), where ε is a scalar to be chosen appropriately. This is due to the fact that X_{p22} and Y_{p22} appear only in strict inequalities (36), (37), and hence they remain to hold in spite of small perturbation of X_{p22} and Y_{p22}, and that there exists an interval $(0, \bar{\varepsilon})$ such that $\forall \varepsilon \in (0, \bar{\varepsilon})$, $\det(X_{p22} + \varepsilon I) \det(Y_{p22} + \varepsilon I) \neq 0$.

(Step 2.) Let us consider the condition (33) and (34). The above block structure of E implies that the condition (33)–(34) is equivalent to:

$$X_{p11} = X_{p11}^{\mathsf{T}}, \quad Y_{p11} = Y_{p11}^{\mathsf{T}} \tag{51}$$

$$\begin{bmatrix} X_{p11} & I \\ I & Y_{p11} \end{bmatrix} \geq 0, \tag{52}$$

$$\operatorname{rank} \begin{bmatrix} X_{p11} & I \\ I & Y_{p11} \end{bmatrix} \leq r + r_c. \tag{53}$$

As in the state-space case, we see that X_{p11}, Y_{p11} and $X_{p11} - Y_{p11}^{-1}$ is positive semidefinite and $\operatorname{rank}(X_{p11} - Y_{p11}^{-1}) = r_c$. Let $L \in \mathbf{R}^{r\times r_c}$ be a full rank decomposition such that $X_{p11} - Y_{p11}^{-1} = LL^{\mathsf{T}}$. Then the equality

$$\begin{bmatrix} X_{p11} & X_{pc11} \\ X_{cp11} & X_{c11} \end{bmatrix} \begin{bmatrix} Y_{p11} & Y_{pc11} \\ Y_{cp11} & Y_{c11} \end{bmatrix} = \begin{bmatrix} I_r & 0 \\ 0 & I_{r_c} \end{bmatrix} \tag{54}$$

holds for

$$X_{pc11} = X_{cp11}^{\mathsf{T}} = L, \quad X_{c11} = I_{r_c}, \tag{55}$$

$$Y_{pc11} = Y_{cp11}^{\mathsf{T}} = -Y_{p11}L, \quad Y_{c11} = I_{r_c} + L^{\mathsf{T}} Y_{p11} L. \tag{56}$$

These matrices form a controller in the next step.

(Step 3.) The synthesis problem is equivalent to solving the matrix inequality (24)–(25), or equivalently (27)–(28), shown in Subsection III-B. It is proven in the following section that when the condition (33)–(37) holds there exists a controller (17) with $s_c = s$ in (49) satisfying (24)–(25) and (27)–(28) for the following (X_{cl}, W_{cl}) and (Y_{cl}, Z_{cl}), respectively:

$$X_{cl} = \left[\begin{array}{cc|cc} X_{p11} & 0 & X_{pc11} & 0 \\ X_{p21} & X_{p22} & X_{pc21} & X_{pc22} \\ \hline X_{cp11} & 0 & X_{c11} & 0 \\ X_{cp21} & X_{cp22} & X_{c21} & X_{c22} \end{array}\right], \tag{57}$$

$$W_{cl} = \left[\begin{array}{cc|cc} 0 & W_{p2}^{\mathsf{T}} & 0 & W_{c2}^{\mathsf{T}} \end{array}\right]^{\mathsf{T}}, \tag{58}$$

$$Y_{cl} = \left[\begin{array}{cc|cc} Y_{p11} & Y_{p12} & Y_{pc11} & Y_{pc12} \\ 0 & Y_{p22} & 0 & Y_{pc22} \\ \hline Y_{cp11} & Y_{cp12} & Y_{c11} & Y_{c12} \\ 0 & Y_{cp22} & 0 & Y_{c22} \end{array}\right], \tag{59}$$

$$Z_{cl} = \left[\begin{array}{cc|cc} 0 & Z_{p2} & 0 & Z_{c2} \end{array}\right], \tag{60}$$

where

$$\left. \begin{array}{l} X_{pc21} = 0, \\ X_{pc22} = I_s, \\ X_{cp21} = -(Y_{p22}^{\mathsf{T}} X_{p21} + Y_{p12}^{\mathsf{T}} \Delta_Y^{-\mathsf{T}}), \\ X_{cp22} = I_s - Y_{p22}^{\mathsf{T}} X_{p22}, \\ X_{c21} = Y_{p12}^{\mathsf{T}} \Delta_Y^{-\mathsf{T}} Y_{cp11}^{\mathsf{T}} Y_{c11}^{-\mathsf{T}}, \\ X_{c22} = -Y_{p22}^{\mathsf{T}}, \\ Y_{cp12}^{\mathsf{T}} = 0, \\ Y_{cp22}^{\mathsf{T}} = I_s, \\ Y_{pc12}^{\mathsf{T}} = -(X_{p21} Y_{p11}^{\mathsf{T}} + X_{p22} Y_{p12}^{\mathsf{T}}), \\ Y_{pc22}^{\mathsf{T}} = I_s - X_{p22} Y_{p22}^{\mathsf{T}}, \\ Y_{c12}^{\mathsf{T}} = -X_{p21} Y_{cp11}^{\mathsf{T}}, \\ Y_{c22}^{\mathsf{T}} = -X_{p22}, \\ W_{c2} = -(Z_{p2} + Y_{p22}^{\mathsf{T}} W_{p2}), \\ Z_{c2}^{\mathsf{T}} = -(X_{p22} Z_{p2}^{\mathsf{T}} + W_{p2}) \end{array} \right\} \tag{61}$$

with $\Delta_Y = Y_{p11} - Y_{pc11} Y_{c11}^{-1} Y_{cp11} (> 0)$. The other submatrices have been defined above in (50), (55) and (56).

(Step 4.) It is guaranteed that the LMIs (24)–(25) and (27)–(28) hold for some A_c, B_c, C_c, D_c with E_c, (X_{cl}, W_{cl}) and (Y_{cl}, Z_{cl}) as determined in Step 3 (See the proof of the sufficiency). Therefore we can seek A_c, B_c, C_c, D_c by solving an ***LMI*** (25) or (28) regarding A_c, B_c, C_c, D_c as the decision variable with E_c, (X_{cl}, W_{cl}) and (Y_{cl}, Z_{cl}) fixed. We can also derive A_c, B_c, C_c, D_c via the Parrott's Theorem.

(Step 5.) According to E_c in (49), denote

$$A_c = \begin{bmatrix} A_{c11} & A_{c12} \\ A_{c21} & A_{c22} \end{bmatrix}, \quad A_{c11} \in \mathbf{R}^{r_c\times r_c}. \tag{62}$$

If A_{c22} is nonsingular, the obtained controller in the descriptor form is reduced to a state-space one by eliminating the static part of the descriptor variable x_c. Otherwise one can perturb A_{c22} such that it becomes nonsingular and still satisfies the LMI (25). Such perturbation is always available for the same reason as stated in Step 1. Thus we obtain a proper (impulse-free) controller satisfying the closed-loop admissibility and dissipativity.

IV. Numerical Examples

Let us consider the plant (16) with $E = \operatorname{diag}\{I_5, 0_{2\times 2}\}$ and the following coefficient matrices:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -2 & -5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -3 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^{\mathsf{T}},$$

$$B_2 = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}^{\mathsf{T}},$$

$$C_1 = \begin{bmatrix} 0 & 0 & 0 & 2 & 1 & 1-\kappa & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$C_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$D_{11} = \begin{bmatrix} \kappa & 0 \\ 0 & 0 \end{bmatrix}, \quad D_{12} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad D_{21} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}^{\mathsf{T}},$$

where κ is a scalar. This system has an identical transfer function independent of κ as shown below:

$$\left[\begin{array}{c} z \\ \hline y \end{array}\right] = \left[\begin{array}{cc|c} \frac{s^2+3s+5}{s^2+2s+3} & 0 & \frac{s^3+8s^2+14s+4}{s^4+7s^3+15s^2+19s+6} \\ 0 & 0 & 1 \\ \hline \frac{s^2}{s^2+2s+3} & 1 & \frac{s(s^4+8s^3+21s^2+21s+6)}{s^4+7s^3+15s^2+19s+6} \end{array}\right] \left[\begin{array}{c} w \\ \hline u \end{array}\right].$$

Thus the optimal H_∞ norm of the controlled system should be the same in spite of different values of κ.

We solved the LMI condition in Theorem 1 without the rank constraint for integer κ from -10 to 10 and obtained the same optimal value $\gamma^* = 1.7064$ along with state-space controllers satisfying the H_∞ norm condition with closed-loop admissibility. On the other hand, solving LMIs with $W = 0$ and $Z = 0$, which corresponds to the case of using conventional H_∞ norm conditions, yields values of optimal γ larger than γ^* when $\|D\|_\infty > \gamma^*$. These results of optimal γ for each κ via the conventional and proposed methods are plotted in Fig. 1.

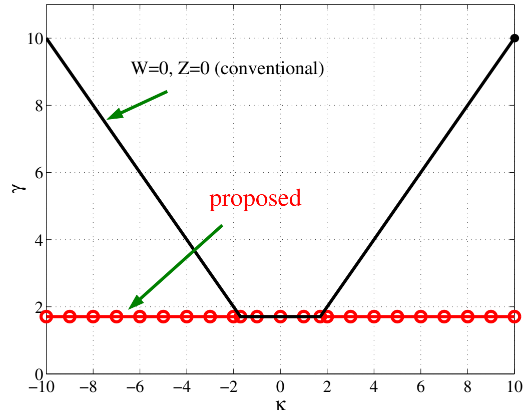

Fig. 1. Optimal γ v.s. κ

V. Conclusions

In this paper, we considered a synthesis problem of output feedback controllers for descriptor systems to attain closed-loop dissipativity and admissibility. We provided a necessary and sufficient condition for the existence of such a controller, based on the recent result on the dissipativity analysis of descriptor systems [7], [8]. The proposed LMI condition is a generalization of the widely-known results for state-space systems [3], [4], and it is inherit from [7], [8] that the LMI condition does not depend on the choice of realization of the plant in the descriptor form.

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