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The Duality Relation between Maximal Output Admissible Set and Reachable Set

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Abstract— **The concept of positive invariance is central to several problems in control theory, such as constrained control, disturbance rejection and robustness analysis. This paper considers two specific types of positively invariant sets, known as the maximal output admissible set and reachable set, determined for linear continuous-time system. In this paper, the inner and outer approximations of both the maximal output admissible set and reachable set are established by using the forward Euler approximated system and the modified zero order hold system. The main purpose is to show that there exists the duality relation between the maximal output admissible set and reachable set. An application of these two positively invariant sets to characterizing the L^p-induced norm of Hankel operator is discussed.**

I. Introduction

A subset of state space is said to be positively invariant if it has the property that, if it contains the system state at some time, then it will contain the state trajectory originating from such state also in the future for any admissible control function. Because of this significant property, the existence and characterization of positively invariant sets are the fundamental tools in many control analysis and synthesis problems [1], such as constrained control [2], [3], disturbance rejection [4], robustness analysis [5], model predictive control and switching control scheme [6] etc.

In particular, the positively invariant sets known as the maximal output admissible set and reachable set are well-known and extensively applied in various aspects of synthesis and analysis for linear discrete-time systems. The maximal output admissible set is usually defined as the largest set of the initial state which makes resultant output trajectory remain inside given polyhedral region consistently [2], [3] which can be interpreted as a special case of our considerations. On the other hand, the reachable set is the set of all reachable states from the origin under the effect of unknown but bounded exogenous inputs [7], [8].

To characterize such given constraints on state and control variables, the ℓ^∞ norm was typically employed in a number of the previous papers [2], [3], [4], [5], [6], [7], [8], [9]. In contrast with the other existing results, this paper considers the maximal output admissible set and reachable set for linear continuous-time systems and equips the constraints on input and output signals with L^p norm where $p \in [1, \infty]$. Accordingly, there are two crucial issues arising from the transitions respectively from discrete-time to continuous-time and from ℓ^∞ norm to L^p norm. The reason which makes the construction of the positively invariant sets complicated even for simple systems is that there exists constraints applied to the system variables continuously. The characterization of the positively invariant sets for linear continuous-time system naturally requires the intersection of an infinite number of half spaces. Except in the case of L^2 norm, it is impossible to construct the polyhedral invariant sets. Therefore, the problem becomes how to approximate the sets appropriately.

Our previous results presented in [10] show that the inclusion between the positively invariant sets of continuous-time system and its forward Euler approximated discrete-time system holds. In addition, the inclusion monotonically holds in the case of the forward Euler approximated systems, discretized by different sampling periods. The maximal output admissible set for linear continuous-time system always contains the maximal output admissible set of the corresponding forward Euler approximated system as its subset. The improved approximations are consistently obtainable while the smaller sampling periods are applied. On the other hand, the reachable set for a linear continuous-time system is the subset of the reachable set of the corresponding forward Euler approximated discrete-time system. The approximation improves as long as the sampling period decreases.

The contributions of this paper are divided into three parts. First, this paper examines the duality relation between the maximal output admissible set and reachable set. Because the (L^p, ℓ^p) norm constraints those act on the control variables are convex, the Hahn-Banach theorem [11] can be implemented to derive the duality results which are necessary in deriving other significant properties. Second, the outer approximation of the maximal output admissible set and the inner approximation of the reachable set for continuous-time systems are established by utilizing the modified zero order hold systems. Third, a characterization of the L^p-induced norm of Hankel operator based on the positively invariant sets are proposed.

The remainder of this paper is organized as follows: We define the positively invariant sets in consideration and formulate the problem in Section 2. In Section 3, we review the previous results presented in [10] and address the main contributions of this paper in detail. Section 4 shows how the results can be applied to the characterization of the L^p-induced norm of Hankel operator. Section 5 is devoted to the numerical examples of the results proposed in Section 3.

Notations: Let I^- and I^+ denote the set of negative and positive integers. The set of positive real numbers and real numbers are denoted by $\mathbb{R}^+$ and $\mathbb{R}$, respectively. Let $\|f\|_{L^p}$ denote the L^p norm of function $f \in L^p$ and $\|g\|_{\ell^p}$ denote the ℓ^p norm of function $g \in \ell^p$. Let A be a subset in $\mathbb{R}^n$, then $\mathrm{int}(A)$, $\mathrm{cl}(A)$ and $\mathrm{conv}(A)$ denote interior, closure and convex hull of set A. Let $\alpha A = \{\alpha a : a \in A\}$, where $\alpha \in \mathbb{R}$.

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II. Definitions and Problem Formulation

The linear continuous-time system under consideration is expressed by the following state space equation:

$$\Sigma^c : \begin{cases} \dot{x}(t) = Ax(t) + Bw(t), \\ z(t) = Cx(t). \end{cases}$$

The vector signals are defined as follows: $w \in \mathbb{R}$ denotes the exogenous input, $x \in \mathbb{R}^n$ denotes the system state and $z \in \mathbb{R}$ denotes the controlled output.

We make the standing assumptions for the remainder of the paper that Σ^c is asymptotically stable. The pair (A, B) is controllable and the pair (C, A) is observable. For representing the system state x under the effect of an exogenous input w and unforced output of system Σ^c, let us introduce operators $\mathscr{B} : L^p \to \mathbb{R}^n$, $\mathscr{C} : \mathbb{R}^n \to L^p$ as follows:

$$\mathscr{B}w := \int_0^\infty e^{A\xi} Bw(\xi) d\xi, \tag{1}$$

$$\mathscr{C}x(t) := Ce^{At}x. \tag{2}$$

Remind that the dual system of the linear continuous-time system Σ^c can be represented as follows:

$$\Sigma^{c\circ} : \begin{cases} \dot{x}(t) = A^\intercal x(t) + C^\intercal w(t), \\ z(t) = B^\intercal x(t). \end{cases}$$

The state under an exogenous input w and autonomous output of system $\Sigma^{c\circ}$ are expressed by adjoint operators of $\mathscr{C}$ and $\mathscr{B}$, respectively.

Consider the forward Euler approximated system of system Σ^c with sampling period $\tau > 0$:

$$\Sigma_\tau^e : \begin{cases} x(k+1) = (I + \tau A)x(k) + (\tau B)w(k), \\ \quad z(k) = Cx(k). \end{cases}$$

We further assume that the forward Euler approximated system is asymptotically stable i.e. $|\lambda_i(I + \tau A)| < 1$, for all $i \in \{1, \cdots, n\}$. This implies that the sampling period must be chosen within the following range:

$$0 < \tau < \min_{i \in \{1, \cdots, n\}} \left(\frac{-2\Re(\lambda_i(A))}{|\lambda_i(A)|^2} \right) =: \theta.$$

To express the state and output of system Σ_τ^e, define the operators $\mathscr{B}_\tau^e : \ell^p \to \mathbb{R}^n$, $\mathscr{C}_\tau^e : \mathbb{R}^n \to \ell^p$, $\; p \in [1, \infty]$ as

$$\mathscr{B}_\tau^e w := \sum_{k=0}^{\infty} (I + \tau A)^k (\tau B) w(k), \tag{3}$$

$$\mathscr{C}_\tau^e x(k) := C(I + \tau A)^k x. \tag{4}$$

Now let us consider the modified zero order hold system of Σ^c with sampling period $h > 0$ expressed by the followings:

$$\Sigma_h^d : \begin{cases} x(k+1) = e^{Ah}x(k) + E_h Bw(k), \\ \quad z(k) = Ch^{-1}E_h x(k). \end{cases}$$

where, $E(h) := \int_0^h e^{A\xi} d\xi$. To designate the state and output of system Σ_h^d, define the operators $\mathscr{B}_h^d : \ell^p \to \mathbb{R}^n$, $\mathscr{C}_h^d : \mathbb{R}^n \to \ell^p$, $p \in [1, \infty]$ as follows:

$$\mathscr{B}_h^d w := \sum_{k=0}^{\infty} e^{kAh} E_h B w(k), \tag{5}$$

$$\mathscr{C}_h^d x(k) := Ch^{-1} E_h e^{kAh} x. \tag{6}$$

Consider the dual systems of the linear discrete-time systems Σ_τ^e and Σ_h^d determined as follows:

$$\Sigma_\tau^{e\circ} : \begin{cases} x(k+1) = (I + \tau A^\intercal)x(k) + (\tau C^\intercal)w(k), \\ \quad z(k) = B^\intercal x(k). \end{cases}$$

$$\Sigma_h^{d\circ} : \begin{cases} x(k+1) = e^{A^\intercal h}x(k) + E_h^\intercal C^\intercal w(k), \\ \quad z(k) = B^\intercal h^{-1} E_h^\intercal x(k). \end{cases}$$

In fact, the above two systems are not the true dual systems of Σ_τ^e and Σ_h^d, because the positions of the sampling parameters τ, h^{-1} are unaffected. However, for the purposes of this paper we consider these systems to be the dual systems. Note that it is possible to describe the state under exogenous input w and unforced output of systems $\Sigma_\tau^{e\circ}$ and $\Sigma_h^{d\circ}$ by adjoint operators of $\mathscr{C}_\tau^e$, $\mathscr{B}_\tau^e$, $\mathscr{C}_h^d$ and $\mathscr{B}_h^d$, respectively.

Let us consider the zero order hold of linear continuous-time system Σ^c with sampling period $h > 0$:

$$\Sigma_h^s : \begin{cases} x(k+1) = e^{Ah}x(k) + E_h Bw(k), \\ \quad z(k) = Cx(k). \end{cases}$$

It is obvious that it is possible to express the state of system Σ_h^s corresponding to an exogenous input w by utilizing the operator $\mathscr{B}_h^d$ determined in (5). Let us define the linear operators $\mathscr{B}_h^s : \ell^p \to \mathbb{R}^n$ as $\mathscr{B}_h^s w := \mathscr{B}_h^d w$ and $\mathscr{C}_h^s : \mathbb{R}^n \to \ell^p$, $p \in [1, \infty]$ as $\mathscr{C}_h^s x(k) := Ce^{kAh}x$.

First, we recall the following definitions:

Definition 1 *Let us consider the specified set of functions W and the system:*

$$\Sigma : \dot{x}(t) = A_c x(t) + B_c w(t),\; x(k+1) = A_d x(k) + B_d w(k).$$

where, $w \in W$. The set S is a positively invariant set of Σ if $x(0) \in S$ then $x(t) = e^{A_c t}x(0) + \int_0^t e^{A_c(t-\xi)} B_c w(\xi) d\xi \in S$ for all $w \in W$ and all $t \in \mathbb{R}^+$. In the case of discrete-time system, $x(k) = A_d^k x(0) + \sum_{j=0}^{k-1} A_d^{k-j-1} B_d w(j) \in S$ for all $w \in W$ and all $k \in I^+$, respectively.

Now let us consider the two types of positively invariant set known as the maximal output admissible set and reachable set.

A. Maximal Output Admissible Set

Consider linear systems Σ^c, Σ_τ^e, Σ_h^d and Σ_h^s with $B = 0$. The set of initial state such that the corresponding output satisfies given (L^p, ℓ^p) norm constraints is to be characterized by the followings:

Definition 2 *The output admissible set of system Σ^c is the set of initial states of the linear continuous-time system Σ^c such that its corresponding output z satisfies the control specification $\|z\|_{L^p} \leqslant 1$, for fixed $p \in [1, \infty]$.*

The maximal output admissible set is the union of all output admissible sets. In other words, the maximal output admissible set is the largest output admissible set and can be expressed employing previously defined operators as follows:

Definition 3 *The maximal output admissible set for linear continuous-time system Σ^c is defined as*

$$\mathscr{M}_{L^p}(\Sigma^c) := \{x \in \mathbb{R}^n : \|\mathscr{C}x\|_{L^p} \leqslant 1\}. \tag{7}$$

The maximal output admissible set for discrete-time systems Σ_τ^e, Σ_h^s and Σ_h^d are respectively determined by

$$\mathscr{M}_{\ell^p}(\Sigma_\tau^e) := \left\{x \in \mathbb{R}^n : \tau^{\frac{1}{p}} \|\mathscr{C}_\tau^e x\|_{\ell^p} \leqslant 1\right\}, \tag{8}$$

$$\mathscr{M}_{\ell^p}(\Sigma_h^s) := \left\{x \in \mathbb{R}^n : h^{\frac{1}{p}} \|\mathscr{C}_h^s x\|_{\ell^p} \leqslant 1\right\}, \tag{9}$$

$$\mathscr{M}_{\ell^p}(\Sigma_h^d) := \left\{x \in \mathbb{R}^n : h^{\frac{1}{p}} \|\mathscr{C}_h^d x\|_{\ell^p} \leqslant 1\right\}. \tag{10}$$

For linear discrete-time systems, it is worth noting that the constraints are configured by the ℓ^p norm of the output multiplied by $\tau^{1/p}$ or $h^{1/p}$ which are nothing but the L^p-induced norm of the hold operator.

Remark 1 *The sets $\mathscr{M}_{L^p}(\Sigma^c)$, $\mathscr{M}_{\ell^p}(\Sigma_\tau^e)$, $\mathscr{M}_{\ell^p}(\Sigma_h^s)$ and $\mathscr{M}_{\ell^p}(\Sigma_h^d)$ are (i) positively invariant for their corresponding systems, (ii) closed, (iii) convex, (iv) balanced, (v) bounded, and (vi) include the origin in their interior.*

B. Reachable set

Next, we consider linear systems Σ^c, Σ_τ^e and Σ_h^d with $C = 0$, $x(0) = 0$ and unknown exogenous input w whose (L^p, ℓ^p) norm is limited.

Definition 4 *The reachable set of systems Σ^c, Σ_τ^e, and Σ_h^d are defined as*

$$\mathscr{R}_{L^p}(\Sigma^c) := \{x \in \mathbb{R}^n : \text{there exists } w \text{ s.t. } \|w\|_{L^p} \leqslant 1 \text{ and } x = \mathscr{B}w\}, \tag{11}$$

$$\mathscr{R}_{\ell^p}(\Sigma_\tau^e) := \{x \in \mathbb{R}^n : \text{there exists } w \text{ s.t. } \tau^{\frac{1}{p}}\|w\|_{\ell^p} \leqslant 1 \text{ and } x = \mathscr{B}_\tau^e w\}, \tag{12}$$

$$\mathscr{R}_{\ell^p}(\Sigma_h^d) := \{x \in \mathbb{R}^n : \text{there exists } w \text{ s.t. } h^{\frac{1}{p}}\|w\|_{\ell^p} \leqslant 1 \text{ and } x = \mathscr{B}_h^d w\}. \tag{13}$$

Remark 2 *The sets $\mathscr{R}_{L^p}(\Sigma^c)$, $\mathscr{R}_{\ell^p}(\Sigma_\tau^e)$ and $\mathscr{R}_{\ell^p}(\Sigma_h^d)$ are (i) invariant for corresponding systems, (ii) convex, (iii) closed except $\mathscr{R}_{L^1}(\Sigma^c)$, (iv) balanced, (v) bounded, and (vi) include the origin in their interior.*

Definition 5 *The polar set of the set $S \subset \mathbb{R}^n$ is defined as*

$$S^\circ := \{y \in \mathbb{R}^n : \langle x, y\rangle \leqslant 1 \text{ for any } x \in S\}.$$

The main problems are to represent the maximal output admissible set by using the polar of reachable set and to express the reachable set in terms of the polar of the maximal output admissible set. Additionally, the approximations of the sets approaching from the opposite directions of those presented in [10] are concerned.

III. The Main Results

In this section, the duality relation between the maximal output admissible set and reachable set is clarified. The (L^p, ℓ^p) norm constraints those act on the control variables are convex, he Hahn-Banach theorem [11] can be implemented in deriving Theorem 1 and Remark 3. An important result of this discovery is that if either one of the maximal output admissible set or reachable set is constructed, the other can be calculated through the duality results. Furthermore, the duality relations are necessary in deriving other significant properties of the sets declared in Theorem 2 which completes the inner and outer approximations of the aforementioned positively invariant sets for linear continuous-time system Σ^c.

Theorem 1 *The following duality relations hold.*

$$(\mathscr{R}_{L^p}(\Sigma^c))^\circ = \mathscr{M}_{L^q}(\Sigma^{c\circ}),\ p \in [1, \infty],$$
$$(\mathscr{M}_{L^p}(\Sigma^c))^\circ = \mathscr{R}_{L^q}(\Sigma^{c\circ}),\ p \in [1, \infty),$$
$$(\mathscr{M}_{L^\infty}(\Sigma^c))^\circ = \mathrm{cl}(\mathscr{R}_{L^1}(\Sigma^{c\circ})).$$

where $1/p + 1/q = 1$.

Proof: Let us consider $x \in \mathscr{M}_{L^q}(\Sigma^{c\circ}) = \{x \in \mathbb{R}^n : \|B^\mathsf{T} e^{A^{\mathsf{T}(\cdot)}} x\|_{L^q} \leqslant 1\}$ and $y \in \mathscr{R}_{L^p}(\Sigma^c)$. We have that,

$$\langle x, y\rangle = \langle x, \mathscr{B}w\rangle = \langle \mathscr{B}^* x, w\rangle \leqslant \|w\|_{L^p} \|B^\mathsf{T} e^{A^{\mathsf{T}(\cdot)}} x\|_{L^q} \leqslant 1.$$

Hence, $\mathscr{M}_{L^q}(\Sigma^{c\circ}) \subset (\mathscr{R}_{L^p}(\Sigma^c))^\circ$. Conversely, suppose that $x \in (\mathscr{R}_{L^p}(\Sigma^c))^\circ \backslash \mathscr{M}_{L^q}(\Sigma^{c\circ})$. Since $\mathscr{M}_{L^q}(\Sigma^{c\circ})$ is closed and convex and $\mathscr{R}_{L^p}(\Sigma^c)$ contains the origin as its interior, by Hahn-Banach theorem, there exists $d > 0$ and $y \in \mathscr{R}_{L^p}(\Sigma^c)$ such that $\langle x, y\rangle > d > \sup_{m \in \mathscr{M}_{L^q}(\Sigma^{c\circ})} \langle m, y\rangle$. However,

$$\sup_{m \in \mathscr{M}_{L^q}(\Sigma^{c\circ})} \langle m, y\rangle = \sup_{\substack{m \in \mathscr{M}_{L^q}(\Sigma^{c\circ}) \\ \|w\|_{L^p} \leqslant 1}} \langle m, \mathscr{B}w\rangle = \sup_{\substack{m \in \mathscr{M}_{L^q}(\Sigma^{c\circ}) \\ \|w\|_{L^p} \leqslant 1}} \langle \mathscr{B}^* m, w\rangle = 1.$$

Therefore, there exists $y \in \mathscr{R}_{L^p}(\Sigma^c)$ such that $\langle x, y\rangle > 1$, contradicting the fact that $x \in (\mathscr{R}_{L^p}(\Sigma^c))^\circ$. This implies $(\mathscr{R}_{L^p}(\Sigma^c))^\circ \subset \mathscr{M}_{L^q}(\Sigma^{c\circ})$. Since for any $p \in (1, \infty]$, $\mathscr{R}_{L^p}(\Sigma^c)$ is closed, convex and $0 \in \mathrm{int}(\mathscr{R}_{L^p}(\Sigma^c))$, then $(\mathscr{R}_{L^p}(\Sigma^c))^{\circ\circ} = \mathscr{R}_{L^p}(\Sigma^c) = (\mathscr{M}_{L^q}(\Sigma^{c\circ}))^\circ$. After some simple rearrangements, We can conclude that $(\mathscr{M}_{L^p}(\Sigma^c))^\circ = \mathscr{R}_{L^q}(\Sigma^{c\circ})$, $p \in [1, \infty)$. Even though $\mathscr{R}_{L^1}(\Sigma^c)$ is not closed, it is straightforward to verify that $(\mathscr{M}_{L^\infty}(\Sigma^c))^\circ = \mathrm{cl}(\mathscr{R}_{L^1}(\Sigma^{c\circ}))$. ■

Remark 3 *In the case of linear discrete-time system, the duality relations are summarized as follows:*

$$(\mathscr{M}_{\ell^p}(\Sigma_\tau^e))^\circ = \mathscr{R}_{\ell^q}(\Sigma_\tau^{e\circ}),\ (\mathscr{R}_{\ell^p}(\Sigma_\tau^e))^\circ = \mathscr{M}_{\ell^q}(\Sigma_\tau^{e\circ}),$$
$$(\mathscr{M}_{\ell^p}(\Sigma_h^d))^\circ = \mathscr{R}_{\ell^q}(\Sigma_h^{d\circ}),\ (\mathscr{R}_{\ell^p}(\Sigma_h^d))^\circ = \mathscr{M}_{\ell^q}(\Sigma_h^{d\circ}).$$

The above results can be verified in the same fashion as described in the proof of Theorem 1. Additionally, it is unnecessary to manipulate the case of the ℓ^1 norm distinctly because reachable sets $\mathscr{R}_{\ell^1}(\Sigma_\tau^e)$ and $\mathscr{R}_{\ell^1}(\Sigma_h^d)$ are closed inherently.

The inner and outer approximation among the maximal output admissible sets and reachable set will be explicitly exhibited in Theorem 2. In addition, the inclusions monotonically hold in cases of discrete-time system, discretized by different sampling periods. The improved inner and outer approximations of the sets are consistently obtainable while the smaller sampling period is applied.

Theorem 2 *Let $p \in [1, \infty]$, $0 < \tau_2 \leqslant \tau_1 < \theta$, $h > 0$ and $N \in I^+$. Then the inclusions*

$$\mathscr{M}_{\ell^p}(\Sigma_{\tau_1}^e) \subset \mathscr{M}_{\ell^p}(\Sigma_{\tau_2}^e) \subset \mathscr{M}_{L^p}(\Sigma^c) \subset \mathscr{M}_{\ell^p}(\Sigma_{h/N}^d) \subset \mathscr{M}_{\ell^p}(\Sigma_h^d), \tag{14}$$

$$\mathscr{R}_{\ell^p}(\Sigma_h^d) \subset \mathscr{R}_{\ell^p}(\Sigma_{h/N}^d) \subset \mathscr{R}_{L^p}(\Sigma^c) \subset \mathscr{R}_{\ell^p}(\Sigma_{\tau_2}^e) \subset \mathscr{R}_{\ell^p}(\Sigma_{\tau_1}^e). \tag{15}$$

hold. $\mathscr{M}_{\ell^p}(\Sigma^e_{\tau_1})$ *is an output admissible set for* $\Sigma^e_{\tau_2}$ *and* Σ^c. *Moreover,*

$$\mathrm{cl}(\bigcup_{\tau>0}\mathscr{M}_{\ell^p}(\Sigma^e_\tau)) = \bigcap_{h>0}\mathscr{M}_{\ell^p}(\Sigma^d_h) = \mathscr{M}_{L^p}(\Sigma^c).$$

are satisfied. On the other hand, the reachable set $\mathscr{R}_{\ell^p}(\Sigma^e_{\tau_1})$ *is an invariant set for systems* $\Sigma^e_{\tau_2}$ *and* Σ^c. *Furthermore,*

$$\mathrm{cl}(\bigcup_{h>0}\mathscr{R}_{\ell^p}(\Sigma^d_h)) = \bigcap_{\tau>0}\mathscr{R}_{\ell^p}(\Sigma^e_\tau) = \mathscr{R}_{L^p}(\Sigma^c),\ p\in(1,\infty],$$
$$\mathrm{cl}(\bigcup_{h>0}\mathscr{R}_{\ell^1}(\Sigma^d_h)) = \bigcap_{\tau>0}\mathscr{R}_{\ell^1}(\Sigma^e_\tau) = \mathrm{cl}(\mathscr{R}_{L^1}(\Sigma^c)).$$

are satisfied.

Proof: As previously assumed in this paper, let sampling periods satisfy $0<\tau_2\leqslant\tau_1<\theta$. For convenience, let us define the ratio of τ_2 and τ_1 as $r:=\tau_2/\tau_1\in(0,1]$. Since $I+\tau_2 A$ can be expressed as

$$I+\tau_2A = (1-r)I + r(I+\tau_1A). \tag{16}$$

When $r=1$, it is obvious that $\mathscr{M}_{\ell^p}(\Sigma^e_{\tau_1})\subset\mathscr{M}_{\ell^p}(\Sigma^e_{\tau_2})$ holds. It was shown in [10] that for fixed $p\in[1,\infty)$ and for any $x\in\mathbb{R}^n$ and $r\in(0,1)$ the next inequality holds.

$$\begin{aligned}&\sum_{k=0}^{\infty}\left|C(I+\tau_2A)^kx\right|^p\tau_2\\ &\quad\leqslant\sum_{k=0}^{\infty}\sum_{j=0}^{k}\binom{k}{j}(1-r)^{k-j}r^j\left|C(I+\tau_1A)^jx\right|^p r\tau_1\\ &\quad=\sum_{k=0}^{\infty}\left|C(I+\tau_1A)^kx\right|^p\tau_1.\end{aligned}$$

Thus, we can conclude that $\tau_2^{1/p}\|\mathscr{C}^e_{\tau_2}x\|_{\ell^p}\leqslant\tau_1^{1/p}\|\mathscr{C}^e_{\tau_1}x\|_{\ell^p}$ holds. In the case of $p=\infty$, it is also easy to show that $\|\mathscr{C}^e_{\tau_2}x\|_{\ell^\infty}\leqslant\|\mathscr{C}^e_{\tau_1}x\|_{\ell^\infty}$. Because we can take sampling period $\tau>0$ arbitrarily small, for $p\in[1,\infty]$, the relations $\|\mathscr{C}x\|_{L^p}\leqslant\tau^{1/p}\|\mathscr{C}^e_\tau x\|_{\ell^p}$ are inherited. Next, the outer approximation of $\mathscr{M}_{L^p}(\Sigma^c)$ will be considered. For fixed $p\in[1,\infty)$ and any $N\in I^+$, the next inequality is satisfied.

$$\begin{aligned}&(Nh)\sum_{k=0}^{\infty}\left|C(Nh)^{-1}\left(\int_0^{Nh}e^{A\xi}d\xi\right)e^{kANh}x\right|^p\\ &\quad=(Nh)^{1-p}\sum_{k=0}^{\infty}\left|C\left(\sum_{j=0}^{N-1}e^{jAh}\int_0^h e^{A\xi}d\xi\right)e^{kANh}x\right|^p\\ &\quad\leqslant h\sum_{k=0}^{\infty}\left|Ch^{-1}\left(\int_0^h e^{A\xi}d\xi\right)e^{kAh}x\right|^p.\end{aligned}$$

Hence, $h^{\frac{1}{p}}\|\mathscr{C}^d_hx\|_{\ell^p}\leqslant(h/N)^{\frac{1}{p}}\|\mathscr{C}^d_{h/N}x\|_{\ell^p}$ is verified. It is possible to show that $\|\mathscr{C}^d_hx\|_{\ell^\infty}\leqslant\|\mathscr{C}^d_{h/N}x\|_{\ell^\infty}$ is satisfied. As a result, for any $p\in[1,\infty]$ the inclusions (14) hold. Due to (16), if a state $\xi\in\mathscr{M}_{\ell^p}(\Sigma^e_{\tau_1})$, then for any sampling period τ_2 such that $0<\tau_2\leqslant\tau_1<\theta$, $(I+\tau_2A)\xi\in\mathscr{M}_{\ell^p}(\Sigma^e_{\tau_1})$ holds. This implies $\mathscr{M}_{\ell^p}(\Sigma^e_{\tau_1})$ is the positively invariant set for systems $\Sigma^e_{\tau_2}$ and Σ^c. Now we will show that if the sampling period decreases, the approximated set converges to the maximal output admissible set of continuous-time system, i.e. $\mathrm{cl}(\bigcup_{\tau>0}\mathscr{M}_{\ell^p}(\Sigma^e_\tau)) = \bigcap_{h>0}\mathscr{M}_{\ell^p}(\Sigma^d_h) = \mathscr{M}_{L^p}(\Sigma^c)$. Since $\mathscr{M}_{L^p}(\Sigma^c)$, $\mathscr{M}_{\ell^p}(\Sigma^e_\tau)$ are closed, for any $\tau>0$, we have that $\mathscr{M}_{\ell^p}(\Sigma^e_\tau)\subset\mathscr{M}_{L^p}(\Sigma^c)$. This implies $\mathrm{cl}(\bigcup_{\tau>0}\mathscr{M}_{\ell^p}(\Sigma^e_\tau))\subset\mathscr{M}_{L^p}(\Sigma^c)$. To prove $\mathscr{M}_{L^p}(\Sigma^c)\subset\mathrm{cl}(\bigcup_{\tau>0}\mathscr{M}_{\ell^p}(\Sigma^e_\tau))$, the facts $\lim_{\tau\to0}\tau^{1/p}\|\mathscr{C}^e_\tau x\|_{\ell^p}=\|\mathscr{C}x\|_{L^p}$ and $\mathscr{M}_{L^p}(\Sigma^c)=\mathrm{cl}(\{x\in\mathbb{R}^n|\|\mathscr{C}x\|_{L^p}<1\})$ are employed. In cases of the outer approximation, for each $h>0$, $\mathscr{M}_{L^p}(\Sigma^c)\subset\mathscr{M}_{\ell^p}(\Sigma^d_h)$ is satisfied. Therefore, $\mathscr{M}_{L^p}(\Sigma^c)\subset\bigcap_{h>0}\mathscr{M}_{\ell^p}(\Sigma^d_h)$ is derived. In order to prove the inclusion $\bigcap_{h>0}\mathscr{M}_{\ell^p}(\Sigma^d_h)\subset\mathscr{M}_{L^p}(\Sigma^c)$, the following three facts are required. For any $h>0$, it can be shown that $h^{1/p}\|\mathscr{C}^d_hx\|_{\ell^p}\leqslant\|\mathscr{C}x\|_{L^p}$ and $\lim_{h\to0}h^{1/p}\|\mathscr{C}^d_hx\|_{\ell^p}=\|\mathscr{C}x\|_{L^p}$ and there exists $\bar{h}>0$ such that $\bar{h}^{1/p}\|\mathscr{C}^d_{\bar{h}}x\|_{\ell^p}>1$ if $\|\mathscr{C}x\|_{L^p}>1$. These facts lead to the desired results. Next, let us prove the inclusions (15). By combining the inclusions among the maximal output admissible set and the property of polar set that $S_1\subset S_2$ then, $S_2^\circ\subset S_1^\circ$, we can observe that if $0<\tau_2\leqslant\tau_1<\theta$, $h>0$ and $N\in I^+$, the inclusions $(\mathscr{M}_{\ell^p}(\Sigma^d_h))^\circ\subset(\mathscr{M}_{\ell^p}(\Sigma^d_{h/N}))^\circ\subset(\mathscr{M}_{L^p}(\Sigma^c))^\circ\subset(\mathscr{M}_{\ell^p}(\Sigma^e_{\tau_2}))^\circ\subset(\mathscr{M}_{\ell^p}(\Sigma^e_{\tau_1}))^\circ$ are immediately derived. Finally, when we consider the results presented in Theorem 1 and Remark 3, $\mathscr{R}_{\ell^p}(\Sigma^d_h)\subset\mathscr{R}_{\ell^p}(\Sigma^d_{h/N})\subset\mathscr{R}_{L^p}(\Sigma^c)\subset\mathscr{R}_{\ell^p}(\Sigma^e_{\tau_2})\subset\mathscr{R}_{\ell^p}(\Sigma^e_{\tau_1})$, $p\in(1,\infty]$ are satisfied. While $p=1$, it is immediate that $\mathscr{R}_{\ell^1}(\Sigma^d_h)\subset\mathscr{R}_{\ell^1}(\Sigma^d_{h/N})\subset\mathrm{cl}(\mathscr{R}_{L^1}(\Sigma^c))\subset\mathscr{R}_{\ell^1}(\Sigma^e_{\tau_2})\subset\mathscr{R}_{\ell^1}(\Sigma^e_{\tau_1})$ holds. By concerning the above inclusions, the fact $\mathscr{R}_{\ell^1}(\Sigma^d_h)\subset\mathscr{R}_{\ell^1}(\Sigma^d_{h/N})\subset\mathscr{R}_{L^1}(\Sigma^c)\subset\mathscr{R}_{\ell^1}(\Sigma^e_{\tau_2})\subset\mathscr{R}_{\ell^1}(\Sigma^e_{\tau_1})$ is quite straightforward to understand. From the inclusions (15), it is trivial that $\mathscr{R}_{\ell^p}(\Sigma^e_{\tau_1})$ is an invariant set for systems Σ^c and $\Sigma^e_{\tau_2}$. Next, the convergent property of the reachable set when sampling period is decreased will be examined. It can be shown that for any set $A_\tau\subset\mathbb{R}^n$, $(\cup_{\tau>0}A_\tau)^\circ=\cap_{\tau>0}A_\tau^\circ$. By using this fact together with $\mathrm{cl}(\cup_{\tau>0}\mathscr{M}_{\ell^p}(\Sigma^e_\tau))=\mathscr{M}_{L^p}(\Sigma^c)$, $\cap_{\tau>0}\mathscr{R}_{\ell^p}(\Sigma^e_\tau)=\mathscr{R}_{L^p}(\Sigma^c)$, $p\in(1,\infty]$ and $\cap_{\tau>0}\mathscr{R}_{\ell^1}(\Sigma^e_\tau)=\mathrm{cl}(\mathscr{R}_{L^1}(\Sigma^c))$ follow. Next, consider a closed, convex set containing the origin as its interior $A_h\subset\mathbb{R}^n$, it can be shown that $(\cap_{h>0}A_h)^\circ=\mathrm{cl}(\mathrm{conv}(\cup_{h>0}A_h^\circ))$. By applying Theorem 1, Remark 3, $\mathscr{R}_{\ell^p}(\Sigma^d_h)\subset\mathscr{R}_{\ell^p}(\Sigma^d_{h/N})$ and $\cap_{h>0}\mathscr{M}_{\ell^p}(\Sigma^d_h)=\mathscr{M}_{L^p}(\Sigma^c)$, $\mathrm{cl}(\bigcup_{h>0}\mathscr{R}_{\ell^p}(\Sigma^d_h))=\mathscr{R}_{L^p}(\Sigma^c)$, $p\in(1,\infty]$ and $\mathrm{cl}(\bigcup_{h>0}\mathscr{R}_{\ell^1}(\Sigma^d_h))=\mathrm{cl}(\mathscr{R}_{L^1}(\Sigma^c))$ can be ascertained. ■

Remark 4 *The outer approximations of* $\mathscr{M}_{L^2}(\Sigma^c)$ *and* $\mathscr{M}_{L^\infty}(\Sigma^c)$ *can also be represented by* $\mathscr{M}_{\ell^2}(\Sigma^s_h)$ *and* $\mathscr{M}_{\ell^\infty}(\Sigma^s_h)$. *Let* $0<h_1\leqslant h_2$. *Then*

$$\mathscr{M}_{\ell^2}(\Sigma^s_{h_1})\subset\mathscr{M}_{\ell^2}(\Sigma^s_{h_2})\subset\mathscr{M}_{L^2}(\Sigma^c),$$
$$\mathscr{M}_{L^\infty}(\Sigma^c)\subset\mathscr{M}_{\ell^\infty}(\Sigma^s_{h/N})\subset\mathscr{M}_{\ell^\infty}(\Sigma^s_h),\ N\in I^+.$$

IV. The Characterization of the L^p-Induced Norm of Hankel Operator

Let $G(s)=C(sI-A)^{-1}B$ be the transfer function of a strictly proper stable linear continuous-time system. Then Hankel operator $H:L^p(-\infty,0]\to L^p[0,\infty)$, $p\in[1,\infty]$ is represented by the following:

$$Hw(t):=Ce^{At}\int_{-\infty}^0 e^{-A\xi}Bw(\xi)d\xi. \tag{17}$$

For the sake of brevity, let us denote $L^p(-\infty,0]$ and $L^p[0,\infty)$ by L^p_- and L^p_+, respectively. The L^p-induced norm of Hankel operator is defined as

$$\|H\|_p := \sup_{w\in L^p_-} \frac{\|Hw\|_{L^p_+}}{\|w\|_{L^p_-}}. \tag{18}$$

Next, consider the linear discrete time-invariant systems Σ^e_τ, Σ^d_h Hankel operators with relation to the above systems $H^e_\tau : \ell^p_- \to \ell^p_+$, $H^d_h : \ell^p_- \to \ell^p_+$ are defined as follows:

$$H^e_\tau w(k) := C(I+\tau A)^k \sum_{j=-\infty}^{0} (I+\tau A)^{-j}(\tau B)w(j),$$

$$H^d_h w(k) := Ch^{-1}E(h)e^{kAh} \sum_{j=-\infty}^{0} e^{-jAh}E(h)Bw(j).$$

where ℓ^p_- and ℓ^p_+ are shorthand notations for $\ell^p(I^- \cup \{0\})$ and $\ell^p(I^+ \cup \{0\})$, respectively. Similarly, their ℓ^p-induced norms are defined as follows:

$$\|H^e_\tau\|_p := \sup_{w\in\ell^p_-} \frac{\|H^e_\tau w\|_{\ell^p_+}}{\|w\|_{\ell^p_-}},\ \ \|H^d_h\|_p := \sup_{w\in\ell^p_-} \frac{\|H^d_h w\|_{\ell^p_+}}{\|w\|_{\ell^p_-}}$$

Theorem 3 *The following equations hold.*

$$\|H\|_p = \inf\{\alpha :\ \mathscr{R}_{L^p}(\Sigma^c) \subset \alpha\mathscr{M}_{L^p}(\Sigma^c)\}, \tag{19}$$

$$\|H^d_h\|_p = \inf\{\alpha :\ \mathscr{R}_{\ell^p}(\Sigma^d_h) \subset \alpha\mathscr{M}_{\ell^p}(\Sigma^d_h)\}, \tag{20}$$

$$\|H^e_\tau\|_p = \inf\{\alpha :\ \mathscr{R}_{\ell^p}(\Sigma^e_\tau) \subset \alpha\mathscr{M}_{\ell^p}(\Sigma^e_\tau)\}. \tag{21}$$

Proof: By the definition, it is obvious that $\|H\|_p \leqslant \alpha$ for all $\alpha > 0$ such that $\mathscr{R}_{L^p}(\Sigma^c) \subset \alpha\mathscr{M}_{L^p}(\Sigma^c)$. In the converse direction, for all $\varepsilon > 0$, there exists $w \in L^p_-$ such that $\|Hw\|_{L^p_+}/\|w\|_{L^p_-} \geqslant \inf\{\alpha :\ \mathscr{R}_{L^p}(\Sigma^c) \subset \alpha\mathscr{M}_{L^p}(\Sigma^c)\} - \varepsilon$, since $\mathscr{R}_{L^p}(\Sigma^c)$ is a reachable set. The linear discrete-time cases can be proved in the same manner. ■

Remark 5 *The following equations can be derived from Theorem 1 and Theorem 3.*

$$\begin{aligned}\|H\|_p &= \inf\{\alpha :\ \alpha^{-1}(\mathscr{M}_{L^p}(\Sigma^c))^\circ \subset (\mathscr{R}_{L^p}(\Sigma^c))^\circ\}\\ &= \inf\{\alpha :\ \mathscr{R}_{L^q}(\Sigma^{c\circ}) \subset \alpha\mathscr{M}_{L^q}(\Sigma^{c\circ})\}\\ &= \|H^*\|_{L^q}.\end{aligned}$$

where $H^* : L^q_+ \to L^q_-$ is the adjoint operator of Hankel operator H and can be expressed by

$$H^* z(t) = B^\mathsf{T} e^{A^\mathsf{T} t}\int_{-\infty}^{0} e^{-A^\mathsf{T}\xi}C^\mathsf{T} z(\xi)d\xi\ ;\quad t\in(0,\infty).$$

This shows that we can represent the L^p-induced norm of Hankel operator $\|H\|_p$ in terms of $\mathscr{R}_{L^q}(\Sigma^{c\circ})$ and $\mathscr{M}_{L^q}(\Sigma^{c\circ})$.

Remark 6 *By utilizing Theorem 3 and simple statement that if $A_1 \subset A_2$ and $B_2 \subset B_1$ then $\alpha^o_2 \leqslant \alpha^o_1$ holds, where $\alpha^o_i := \inf\{\alpha :\ B_i \subset \alpha A_i\}$, we can prove that the following inequalities are satisfied for fixed $p \in [1,\infty]$.*

$$\|H^d_h\|_p \leqslant \|H^d_{h/N}\|_p \leqslant \|H\|_p \leqslant \|H^e_{\tau_2}\|_p \leqslant \|H^e_{\tau_1}\|_p.$$

These magnitude relations of the induced norms still hold, even when exogenous inputs w and controlled outputs z are taken from different L^p spaces. This follows from the fact that the inclusions with relation to the maximal output admissible set and the inclusions with relation to the reachable set hold individually for any $p \in [1,\infty]$.

V. Numerical Examples

Consider the following data to illustrate the inclusions between the maximal output admissible sets. The computational results are given in Fig.1, Fig.2 and Fig.3.

$$A = \begin{bmatrix} -1.3602 & 0.1253 \\ -1.6656 & -0.6390 \end{bmatrix},\quad C = [1 \quad 3]. \tag{22}$$

In Fig.1, Hankel singular value was utilized to construct the polyhedral set that achieves the ℓ^1 norm constraint [12]. In this set, when we construct a positively invariant set for system Σ^e_τ by a standard algorithm proposed in [3], the inner approximation of sets $\mathscr{M}_{\ell^1}(\Sigma^e_\tau)$, $\mathscr{M}_{L^1}(\Sigma^c)$ which is an output admissible set for systems Σ^e_τ and Σ^c can be constructed. Even though the outer approximation is quite difficult to construct directly, according to Theorem 2 and Remark 3, the outer approximation can be obtained by calculating the inner approximation of $\mathscr{R}_{\ell^\infty}(\Sigma^{d\circ}_h)$ using the algorithm proposed in [9]. Then we can compute the polar of $\mathscr{R}_{\ell^\infty}(\Sigma^{d\circ}_h)$ to derive the outer approximation of $\mathscr{M}_{L^1}(\Sigma^c)$.

It is well-known that we can employ the observability grammian of the corresponding systems to construct $\mathscr{M}_{L^2}(\Sigma^c)$, $\mathscr{M}_{\ell^2}(\Sigma^e_\tau)$ and $\mathscr{M}_{\ell^2}(\Sigma^d_h)$ shown in Fig.2.

Fig.3 is obtained by applying the algorithm introduced in [3]. The detail that should be emphasized is that there are two methods to construct the outer approximation from Theorem 2 and Remark 4. Experimentally, $\mathscr{M}_{\ell^\infty}(\Sigma^s_h)$ converges to $\mathscr{M}_{L^\infty}(\Sigma^c)$ more rapidly than $\mathscr{M}_{\ell^\infty}(\Sigma^d_h)$.

The following parameters are given for verifying that the inclusions between the reachable sets presented in Section 3 hold. The results are shown in Fig.4, Fig.5, and Fig.6.

$$A = \begin{bmatrix} -0.3721 & -0.0479 \\ 0.3878 & -0.5598 \end{bmatrix},\quad B = \begin{bmatrix} 0.3710 \\ 0.2749 \end{bmatrix}. \tag{23}$$

In general, it is quite hard to construct the reachable set with relation to (L^1, ℓ^1) norm directly. According to Theorem 1, Theorem 2, Remark 3 and Remark 4, we can still derive the reachable set from computing the polar sets of the sets $\mathscr{M}_{\ell^\infty}(\Sigma^{s\circ}_h) := \left\{x \in \mathbb{R}^n : \|B^\mathsf{T} e^{A^\mathsf{T} h(\cdot)}x\|_{\ell^\infty} \leqslant 1\right\}$, $\mathscr{M}_{\ell^\infty}(\Sigma^{d\circ}_h)$ and $\mathscr{M}_{\ell^\infty}(\Sigma^{e\circ}_\tau)$. Hence, the inner approximations $\mathscr{R}_{\ell^1}(\Sigma^d_h)$, $(\mathscr{M}_{\ell^\infty}(\Sigma^{s\circ}_h))^\circ$ of $\mathscr{R}_{L^1}(\Sigma^c)$ and the outer approximation $\mathscr{R}_{\ell^1}(\Sigma^e_\tau)$ of $\mathscr{R}_{L^1}(\Sigma^c)$ are computable as in Fig.4.

The controllability grammian was employed to exhibit $\mathscr{R}_{L^2}(\Sigma^c)$, $\mathscr{R}_{\ell^2}(\Sigma^e_\tau)$ and $\mathscr{R}_{\ell^2}(\Sigma^d_h)$ in Fig.5.

By applying the algorithm proposed in [7], we can obtain the inner approximation of $\mathscr{R}_{\ell^\infty}(\Sigma^d_h)$ which is also the inner approximation of $\mathscr{R}_{L^\infty}(\Sigma^c)$, and the outer approximation of $\mathscr{R}_{\ell^\infty}(\Sigma^e_\tau)$ which is the outer approximation of $\mathscr{R}_{L^\infty}(\Sigma^c)$ as shown in Fig.6. It is worth noting that the outer approximation $\mathscr{R}_{L^\infty}(\Sigma^c)$ which is an invariant set for linear continuous-time system Σ^c can be also effectively constructed by calculating the polar set of $\mathscr{M}_{\ell^1}(\Sigma^{e\circ}_\tau)$.

The legends of the figures are given in the following format. The first element denotes the system in consideration. The second parameter determines the sampling period used in discrete-time systems. The third argument, if it exists, prescribes whether the inner (i) or the outer approximation (o) of the sets characterized by the first two parameters is represented.

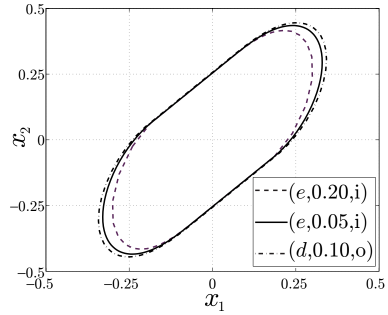

Fig.1: The inner and outer approximations of $\mathscr{M}_{L^1}(\Sigma^c)$

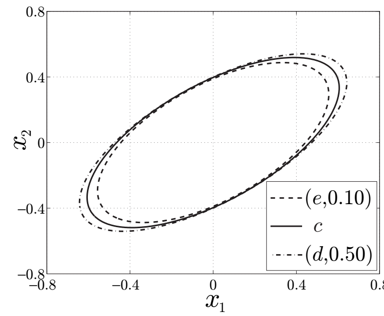

Fig.2: $\mathscr{M}_{L^2}(\Sigma^c)$, $\mathscr{M}_{\ell^2}(\Sigma^e_\tau)$ and $\mathscr{M}_{\ell^2}(\Sigma^d_h)$

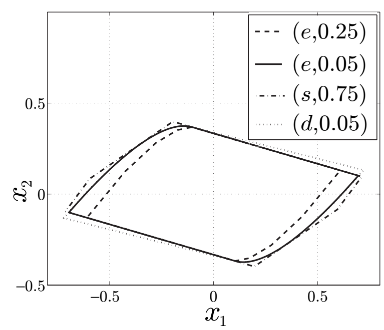

Fig.3: $\mathscr{M}_{\ell^\infty}(\Sigma^e_\tau)$, $\mathscr{M}_{\ell^\infty}(\Sigma^s_h)$ and $\mathscr{M}_{\ell^\infty}(\Sigma^d_h)$

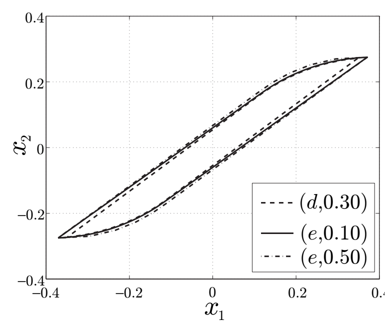

Fig.4: $\mathscr{R}_{\ell^1}(\Sigma^d_h)$ and $\mathscr{R}_{\ell^1}(\Sigma^e_\tau)$

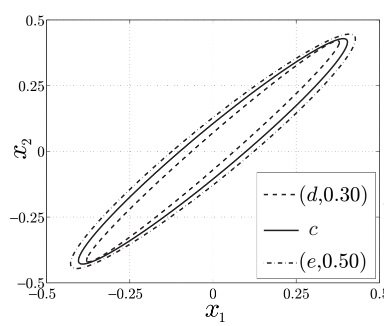

Fig.5: $\mathscr{R}_{L^2}(\Sigma^c)$, $\mathscr{R}_{\ell^2}(\Sigma^e_\tau)$ and $\mathscr{R}_{\ell^2}(\Sigma^d_h)$

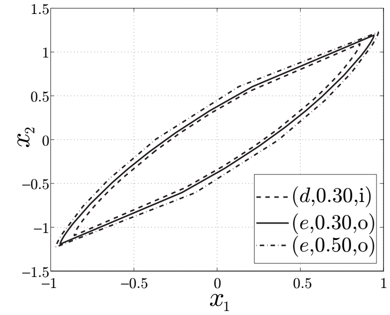

Fig.6: The inner and outer approximations of $\mathscr{R}_{L^\infty}(\Sigma^c)$

VI. Conclusions

This paper has considered the maximal admissible set and reachable set for linear continuous-time system. We show that the inner and outer approximations are obtained by employing the forward Euler approximated discrete-time system and the modified zero order hold system. The inclusions between the positively invariant sets of the continuous-time system and the positively invariant sets of the discrete-time systems hold. The approximated sets approach to the positively invariant set of continuous-time system monotonically within arbitrary accuracy. The duality relations between the maximal output admissible set and reachable set for each system are verified. This duality results provide us an alternative computational method for the construction of the positively invariant sets. Furthermore, the duality relations are necessary in deriving other significant results. Finally, it was prescribed that the L^p-induced norm of Hankel operator is the smallest scaling parameter such that the reachable set is contained in the scaled maximal output admissible set.

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