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Integral and Second Order Sliding Mode Control of Harmonic Oscillator

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Abstract— **A typical second order Harmonic Oscillator is controlled using both traditional and higher order Sliding Mode Control (SMC). Integral sliding mode control is used to compensate for the bounded disturbances improving the robustness of the controlled Harmonic Oscillator. The advantages of the higher order SMC over the traditional one i.e., control continuity and improved accuracy are shown. A computer simulation is performed to manifest the theoretical analysis.**

I. Introduction

Except for direct digital frequency synthesizers (DDFS), which generate sinusoidal waveform by using an open-loop digital system as a staircase signal [1], the other types of waveform generators are based on a feedback system having two complex-conjugate poles on the imaginary axis. This kind of sine wave generator is called harmonic oscillator. Harmonic oscillators are used in different applications, especially in communication and signal processing.

The main problem in a harmonic oscillator is amplitude stability, which is provided by a nonlinear system like automatic gain control (AGC) [2]. The major drawback of a harmonic oscillator with AGC is distortion appearing in the output waveform due to non-linearity imposed by AGC. Several attempts have been made to overcome this problem, especially in electronics, yet generating a very accurate sine wave with a stabilized frequency by a harmonic oscillator is not an easy task.

Another method to control the amplitude and the frequency of a harmonic oscillator is sliding mode control [3], [4]. This method is considered by several authors, for instance in [5], [6], but none of them has discussed application of integral and high (second) order sliding mode control. In [5] traditional sliding mode control and synchronization of harmonic oscillators is considered in the polar coordinates that may yield implementation problems. In [6] the Van der Pol oscillator is controlled in traditional sliding modes.

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In this paper robustness of sliding mode harmonic oscillators is studied using traditional [3], [4], including integral, and higher order sliding mode (HOSM) control [7]-[9]. Achieving higher accuracy in sliding surface stabilization and possibility of generating continuous control are the advantages of employing HOSM to design harmonic oscillators. Both advantages of HOSM significantly improve the harmonic oscillator performance that is extremely important in applications. To control the harmonic oscillator, traditional, including integral [3], [4] and "super-twist" (second order) sliding mode control (SOSM) [7], [8] are studied, and the effect of a bounded disturbance is investigated analytically and via computer simulations.

The paper is organized as follows: Section II gives the problem formulation. In Section III harmonic oscillator controlled in traditional sliding modes, including integral sliding mode control. In Section IV the harmonic oscillator controlled in second order sliding mode (super-twist algorithm) in combination with integral sliding mode control is studied. Implementation issues are discussed in Section V. A numerical example is considered in Section VI. Conclusions are summarized in Section VII.

II. Problem Formulation

A basic harmonic oscillator has the following dynamical system description [2]

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\omega_0^2 x_1 \\ x_1(0) &= x_{10},\ x_2(0) = x_{20} \end{aligned} \qquad (1)$$

This results in a harmonic solution:

$$\begin{aligned} x_1(t) &= \frac{a_0}{\omega_0}\cos(\omega_0 t + \theta_0) \\ x_2(t) &= -a_0 \sin(\omega_0 t + \theta_0) \end{aligned} \qquad (2)$$

where $a_0 = \sqrt{x_{10}^2\omega_0^2 + x_{20}^2}$ and $\theta_0 = \arctan\left(-\frac{x_{20}}{x_{10}\omega_0}\right)$.

Clearly, the amplitude of the harmonic solution of (1) depends on the initial conditions. The frequency of oscillation ω_0 also depends on the implementation accuracy of the plant (1).

To study the oscillations in a perturbed plant with given amplitude and frequency the following system can be considered

$$\dot{x}_1 = x_2 + f(t) + u_1$$
$$\dot{x}_2 = -(\omega_0 + \widetilde{\omega})^2 x_1 + \psi(t) + u_2 \qquad (3)$$
$$x_1(0) = x_{10,}\ x_2(0) = x_{20}$$

where $f(t)$ and $\psi(t)$ are continuous disturbances to the system, u_1 and u_2 are control functions and $\widetilde{\omega}$ is an uncertainty, which are due to imperfect implementation of the ideal plant (1). Let $f(t)$ and $\varphi(x_1,t) = \psi(t) - \left(2\omega_0\widetilde{\omega} + \widetilde{\omega}^2\right)x_1$ be bounded

$$\left|f(t)\right| \le L_1,\ \ \left|\varphi(x_1,t)\right| \le L_2 + L_3\left|x_1\right| \qquad (4)$$

Apparently, if x_1 and x_2 are phase variables in (3) then stabilizing equation

$$\omega_0^2 {x_1}^2 + {x_2}^2 - R^2 = 0 \qquad (5)$$

yields a desired harmonic solution

$$x_1(t) = \frac{R}{\omega_0}\cos\left(\omega_0 t + \theta_0\right),\ \ x_2(t) = -R\sin\left(\omega_0 t + \theta_0\right) \qquad (5a)$$

that is robust to the perturbations in harmonic oscillator model (3), (4). The equation (5), which is treated further as a sliding surface, can be easily stabilized by control u_2. A crucial point in stabilizing the desired harmonic solution (5a) is compensation of the disturbance $f(t)$, which is unmatched to control u_2, and, therefore, must be compensated by means of control u_1. Therefore, control u_1 must be designed preserving the system (3) dynamics after compensation of $f(t)$.

So, in this work the robust stabilization of harmonic oscillations in the perturbed plant (3) is addressed via the control functions u_1 and u_2 design such that

- the disturbance $f(t)$ is compensated in a finite time
- the states of the system (3) satisfy eq. (5) in presence of bounded disturbances $f(t)$ and $\varphi(x_1,t)$.

III. Harmonic Oscillator Controlled by Traditional SMC

A.. Integral sliding mode control u_1 design

The integral sliding mode [3] is able to compensate implicitly for the matched bounded disturbances in a finite time or even without any time delay preserving the rest of the system dynamics. Therefore, using this concept for compensating the bounded disturbance $f(t)$ via control u_1 the following integral sliding variable [3] is introduced

$$\sigma_1 = x_1 - \int_o^t x_2 d\tau \qquad (6)$$

The σ_1 – dynamics is derived based on (3) and (6)

$$\dot{\sigma}_1 = f(t) + u_1 \qquad (7)$$

The integral sliding variable σ_1 is stabilized at zero in a finite time via traditional sliding mode control [3], [4]

$$u_1 = -(\rho + L_1)sign\sigma_1,\quad \rho > 0 \qquad (8)$$

Equivalent control that runs system (3) in the sliding mode $\sigma_1 = 0$ is computed as follows:

$$u_{1eq} = -f(t) \qquad (9)$$

and system (3) in the sliding mode $\sigma_1 = 0$ becomes

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\omega_0^2 x_1 + \varphi(x_1,t) + u_2 \qquad (10)$$

Remark 1. If the initial value $x_1(0)$ is known then the initial value of $\int_o^t x_2 d\tau$ can be equated to $-x_1(0)$ that yields $\sigma_1(0) = 0$, and the sliding mode (10) starts without reaching phase. Otherwise the sliding mode starts in a finite time $t_r \le \dfrac{\left|\sigma_1(0)\right|}{\rho}$.

B. Sliding mode control u_2 design

In order to make eq. (5) identity we introduce a siding variable

$$\sigma_2 = \omega_0^2 {x_1}^2 + {x_2}^2 - R^2 \qquad (11)$$

that must be stabilized at zero in a finite time by means of a control function u_2 on the trajectory given by (10).

The dynamics of the sliding quantity σ_2 is derived based on (5) and (10)

$$\dot{\sigma}_2(x) = 2\omega_0^2 x_1 x_2 + 2x_2(-\omega_0^2 x_1 + u_2 + \varphi(x_1,t)) = \\ = 2x_2 u_2 + 2x_2\varphi(x_1,t) \qquad (12)$$

To design SMC that stabilizes σ_2 at zero a Lyapunov function candidate is introduced [10]

$$V(\sigma_2) = 0.5\sigma_2^2 \qquad (13)$$

The existence of the sliding mode is investigated by choosing a control law as

$$u_2 = -\left(L_2 + L_3\left|x_1\right| + \varepsilon\right)sign(x_2)sign(\sigma_2),\ \varepsilon > 0 \qquad (14)$$

It is not difficult to see that

$$\dot{V}(\sigma_2) = \sigma_2\dot{\sigma}_2 \le -2\varepsilon\left|\sigma_2\right|\left|x_2\right| \qquad (15)$$

The conditions (14) and (15) provide convergence for $\sigma_2 = 0$ if $x_2(0) \ne 0$ and $\left|x_2(t)\right| \ne 0$ during consecutive time except for some isolated points.

Theorem. If $x_2(0) \neq 0$ then $x_2(t)$ cannot be identically equal to zero during the consecutive time either in reaching or in sliding phase.

Proof. First of all, consider a reaching phase with $x_1(0) \neq 0$, $x_2(0) \neq 0$ and $\sigma_2 \neq 0$, then the system (10)-(12) dynamics is described as follows:

$$\begin{aligned} &\dot{x}_1 = x_2 \\ &\dot{x}_2 = -\omega_0^2 x_1 + \varphi(x_1, t) - \\ &-\left(L_2 + L_3|x_1| + \varepsilon\right) sign(x_2) sign(\sigma_2) \end{aligned} \quad (16)$$

No high frequency switching in control is expected because $\sigma_2 \neq 0$. Obviously, if $x_1(0) \neq 0$ and $x_2(0) \neq 0$ then a solution $x_2(t)$ of (16) cannot be identically equal to zero $\forall t \in [0, t_r]$.

Next, considering a sliding phase in (10)-(12) on $\sigma_2 = 0$ $\forall t \geq t_r$, it can be proved that $|x_2(t)| \neq 0$ except for some isolated points. Indeed, the system (10) dynamics in the sliding mode $\sigma_2 = 0$ can be described $\forall t \geq t_r$ by substituting $\sigma_2 = 0$ and equivalent control $u_{2eq} = -\varphi(x_1, t)$ [11], defined by $\dot{\sigma}_2 = 0$, in (10). This is

$$\begin{aligned} &\dot{x}_1 = x_2 \\ &\dot{x}_2 = -\omega_0^2 x_1 \quad , \\ &\omega_0^2 x_1^2 + x_2^2 - R^2 = 0 \end{aligned} \quad (17)$$

Apparently, a solution $x_2(t)$ of (17) cannot be identically equal to zero $\forall t \geq t_r$. □

Remark 2. Consequently, integral sliding mode control u_1 in (6), (8) altogether with traditional sliding mode control u_2 in (11), (14) robustly enforce harmonic oscillations (5a) with the given frequency ω_0 and the amplitude $a_0 = R$ in the perturbed harmonic oscillator (3) in presence of bounded disturbances (4).

Remark 3. Consider an unperturbed case, when $x_1(0) = x_2(0) = 0$, $f(t) = \varphi(x_1, t) \equiv 0$ and $sign(z) \approx \frac{z}{|z| + \delta}$ in (14) with an arbitrary small $\delta > 0$. In this case $x_1(0) = x_2(0) = 0$ will be an equilibrium point of (10). Therefore, it can be immediately concluded $\dot{V}(\sigma_2) \equiv 0$ based on (15). Consequently, the equilibrium point $\sigma_2 = 0$ of (12) will not be asymptotically stable. In other words, the sliding mode $\sigma_2 = 0$ will not exist in the unperturbed harmonic oscillator (10) controlled by the control function (11) and (14) if $x_1(0) = x_2(0) = 0$. This case must be avoided in practical implementation of the SMC controlled Harmonic Oscillator.

IV. Harmonic Oscillator Controlled by Second Order (Super-Twist) SMC

The second order sliding mode control [7], [8], in particular a super-twisting algorithm, is used to robustly stabilize σ_1 and σ_2 at zero in a finite time for the perturbed harmonic oscillator (3). This kind of control, simultaneously improves the stabilization accuracy of both sliding variables σ_1 and σ_2.

It is well known that a solution $z(t)$ of the following differential equation

$$\dot{z} + a|z|^{1/2} sign(z) + b\int sign(z) d\tau = \xi(t) \quad (18)$$

and its derivative $\dot{z}(t)$ converge to zero in a finite time if $a \geq 0.5\sqrt{C}$ and $b \geq 4C$ with $|\xi(t)| \leq C$ [7], [8].

Substituting following super twist control

$$u_1 = -\alpha_1 |\sigma_1|^{1/2} sign(\sigma_1) + \beta_1 \int sign(\sigma_1) d\tau \quad (19)$$

into (7) the σ_1 − compensated dynamics is achieved

$$\dot{\sigma}_1 + \alpha_1 |\sigma_1|^{1/2} sign(\sigma_1) + \beta_1 \int sign(\sigma_1) d\tau = f(t) \quad (20)$$

In order to obtain a finite time convergence $\sigma_1 \to 0$ and $\dot{\sigma}_1 \to 0$ the parameters of the super-twist controller (19) must satisfy the following inequalities $\alpha_1 \geq 0.5\sqrt{L_1}$ and $\beta_1 \geq 4L_1$.

Substituting following super-twist like control [7,8]

$$\begin{aligned} u_2 = &-sign(x_2)[\alpha_2 |\sigma_2|^{1/2} sign(\sigma_2) + \\ &+ \beta_2 \int sign(\sigma_2) d\tau] \end{aligned} \quad (21)$$

into (12), the σ_2 − compensated dynamics is achieved

$$\begin{aligned} &\dot{\sigma}_2 + 2\alpha_2 |x_2| |\sigma_2|^{1/2} sign(\sigma_2) + \\ &+ 2\beta_2 |x_2| \int sign(\sigma_2) d\tau = 2x_2 \varphi(x_1, t) \end{aligned} \quad (22)$$

To obtain a finite time convergence $\sigma_2 \to 0$ and $\dot{\sigma}_2 \to 0$ the parameters of the second order (super-twist like) sliding mode controller (21) must satisfy the following inequalities

$$\alpha_2 \geq \frac{1}{2}\sqrt{\frac{L_2 + L_3|x_1|}{2|x_2|}}, \; \beta_2 \geq 4\left(L_2 + L_3|x_1|\right) \quad (23)$$

Apparently, the super twist-like controller (21) is able to robustly enforce $\sigma_2 = 0$ and $\dot{\sigma}_2 = 0$ almost everywhere except for the small vicinities of $x_2 = 0$ (singular points), where the second order sliding mode is destroyed.

The major advantage of super-twist like control (19), (21) over traditional SMC (8), (14) is that the super-twist is continuous. Also, the stabilization accuracy of $\sigma_1 = 0$ and $\sigma_2 = 0$ is proportional to T^2 that is much higher than the stabilization accuracy achieved via traditional SMC (8), (14) that is proportional to T, where T is a sampling time of a computer-implemented control [7], [8]. These features are very important for the controlled harmonic oscillators because high frequency switching control could destroy the sinusoidal output waveform or it could make phase noise, which the lower it is, the better the oscillator works. In addition, the robustness to the external disturbances along with the significant improvement in the accuracy of the sliding stabilization yields much more accurate sinusoidal waveform at Harmonic Oscillator output.

V. IMPLEMENTATION ISSUE

Based on the discussion given in Remark 3, the initial conditions are prohibited to be at the zero state $x_1(0) = x_2(0) = 0$. In reality all the time circuits are effected by very small (usually unwanted) signals like thermal noise, which make the unwanted zero initial conditions to be barely met. In this particular case those unwanted signals become useful because they jump-start the oscillator. Consequently the condition $x_1(0) = x_2(0) = 0$ discussed in Remark 3 will never occur in reality.

VI. EXAMPLE

The harmonic oscillator (3) with $\omega_0 = 6.28\ rad/\sec$, $\tilde{\omega} = 1.0\ rad/\sec$ $f(t) = \sin t$, $\psi(t) = \sin 2t$, $x_1(0) = 0.01\ rad$ and $x_2(0) = 0.01\ rad/\sec$ has been simulated. The amplitude of the oscillations in (5) and (6) that must be robustly stabilized by sliding mode control is chosen $a_0 = R = 2$. The parameter boundaries of the perturbed plant (3) are computed $L_1 = 2.0, L_2 = 1.0, L_3 = 20.0$ and $\rho = \varepsilon = 2.0$. The parameters of the super-twist sliding mode controllers are chosen $\alpha_1 = 1.0$, $\beta_1 = 5.0$ for u_1 and $\alpha_2 = 20\sqrt{1 + 20|x_1|}$, $\beta_2 = 5(1 + 20|x_1|)$ for u_2. The control magnitude is limited by 10.0 for the both controllers. The simulation sampling time is set to be 10^{-5} sec. The results of the simulations for both traditional and higher order SMC are shown in figures 1-12. Both controllers provide robust accurate control of Harmonic Oscillator (Figures 1-4).

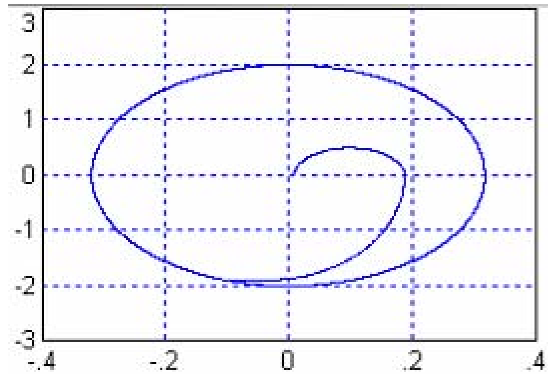

Figure 1 Phase portrait: Traditional SMC

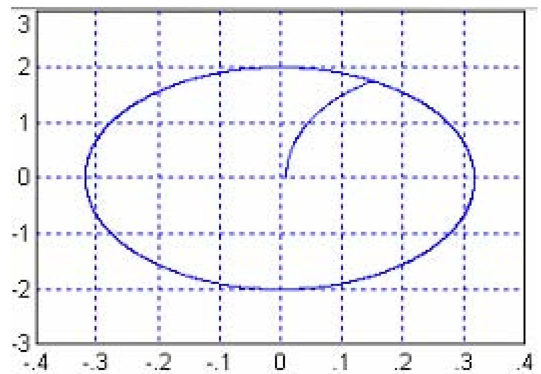

Figure 2 Phase portrait : Super-twist control

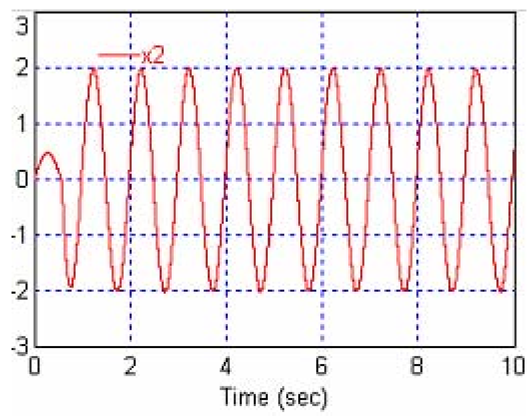

Figure 3 x_2 vs time: Traditional SMC

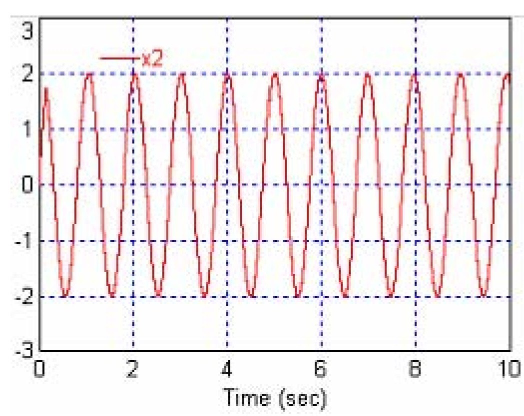

Figure 4 x_2 vs time: Super-twist control

As demonstrated in Figures 5-8, the super-twist (second order sliding mode) control provides much better stabilization accuracy rather than traditional sliding mode control.

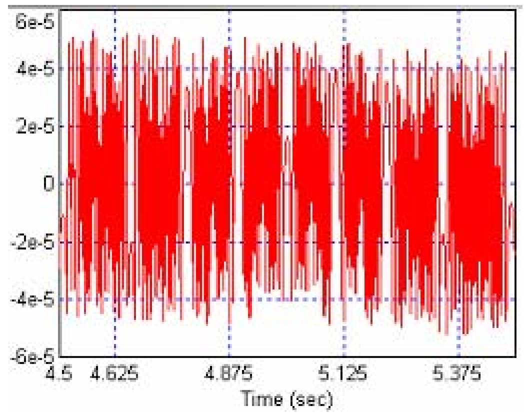

Figure 5 Sliding variable σ_1 vs time (zoomed): Traditional SMC

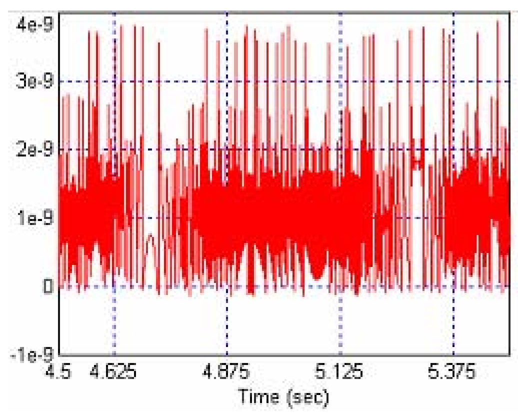

Figure 6 Sliding variable σ_1 vs time (zoomed): Super-twist control

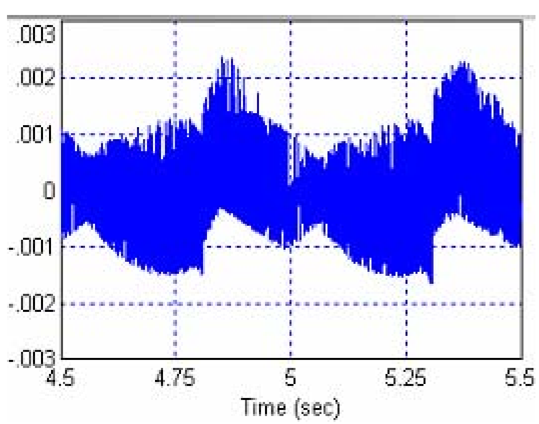

Figure 7 Sliding variable σ_2 vs time (zoomed): Traditional SMC

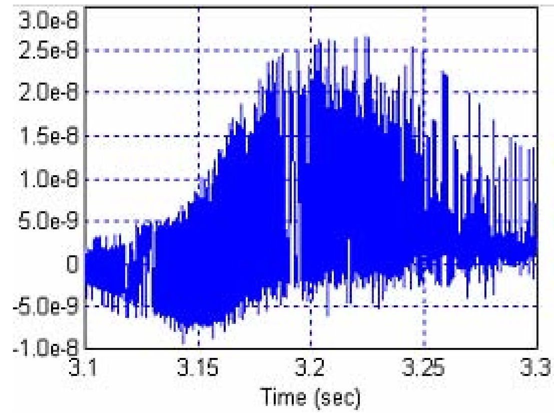

Figure 8 Sliding variable σ_2 vs time (zoomed): Super-twist control

Figures 9-11 illustrate the fact that the robust performance of the harmonic oscillator can be achieved via continuous control generated by SOSM (super-twist algorithm) and high frequency switching control generated by the traditional sliding mode controller. The spikes in the second order control law due to the aforementioned singularity can be clearly observed in Figure 12.

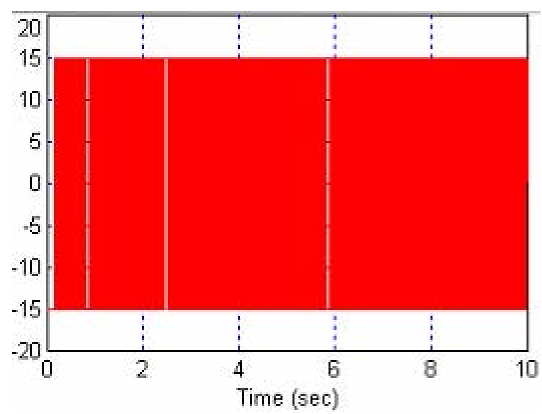

Figure 9 Integral Traditional SMC u_1

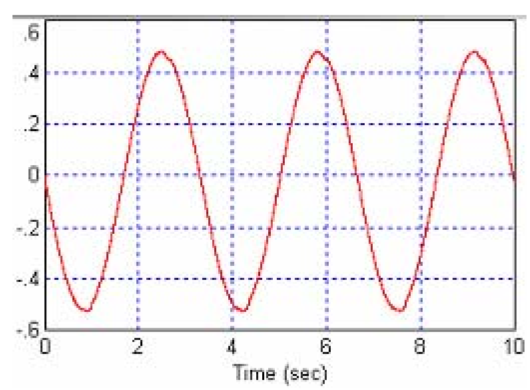

Figure 10 Integral Super-twist u_1

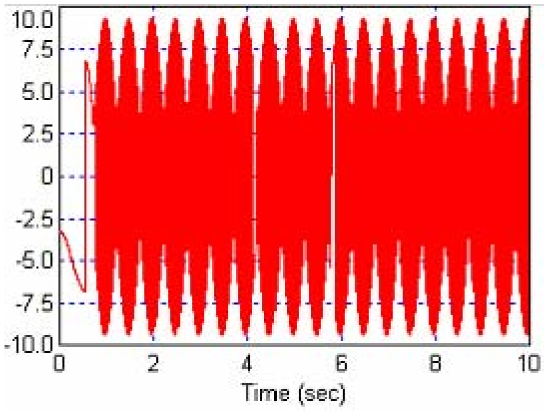

Figure 11 Control function u_2 : Traditional SMC

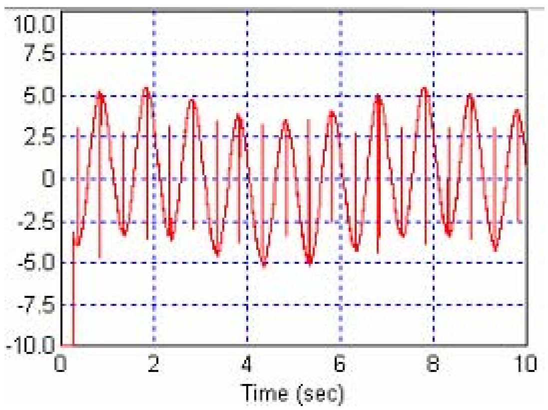

Figure 12 Control function u_2 : Super-twist algorithm

VII. CONCLUSION

Traditional and high order sling modes provides robust control of frequency and amplitude of oscillations in Harmonic Oscillators in presence of bounded disturbances and uncertainties. Using the high order (second) sliding mode control, a much more precise stabilization with a continuous control function can also be provided. It is shown that despite some singularity points in the control law, the second order sliding is restored.

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