Proceedings of the
44th IEEE Conference on Decision and Control, and
the European Control Conference 2005
Seville, Spain, December 12-15, 2005

Complete type Lyapunov-Krasovskii functionals with a given cross term in the time derivative

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Abstract— **A general expression for the complete type quadratic Lyapunov-Krasovskii functional with a given cross term in the time derivative is presented. Some robust stability conditions and exponential estimates are obtained.**

I. INTRODUCTION

In [3] a modification of the quadratic Lyapunov-Krasovskii functionals with a prescribed derivative has been proposed. It has been shown there that for exponentially stable systems the modified functional admits a quadratic lower bound. The modified functionals were called of *complete type* if the corresponding weight matrices in the given time derivative are definite positive. The complete type functionals prove to be useful in the robustness analysis of time delay systems, see [3], as well as in the derivation of exponential estimates for the solutions of the systems, see [4]. In this contribution we enlarge the class of functionals by adding the functionals whose time derivative includes a given cross term. Such an extension is useful in improving both the robust stability bounds and the exponential estimates obtained with the help of the functionals.

The contribution is organized as follows: some results on complete functionals with a given time derivative are recalled in Section 2. Next, in section 3, the results are extended to the case of functionals whose time derivative includes also a cross term and some concluding remarks end the contribution.

II. PREVIOUS RESULTS ON COMPLETE TYPE FUNCTIONALS

The aim of this work is to construct the Lyapunov-Krasovskii functional with a given time derivative that includes a cross term for an exponentially stable linear time delay system of the form

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-h), \quad t \geq 0 \tag{1}$$

where $A_0, A_1 \in R^{n\times n}$, $h \geq 0$. We denote by $x(t,\varphi)$ the solution of the system with a given initial function $\varphi(\theta)$, $\theta \in [-h,0]$. By $x_t(\varphi)$ we denote the segment, $\{x(t+\theta,\varphi) : \theta \in [-h,0]\}$, of the solution. When the initial function is not important or when it is evident from the context we will drop the argument φ in the above notations.

The work was partially supported by CONACyT, Project 41276-Y, Mexico

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Definition 1: System (1) is said to be exponentially stable if there exist $\lambda > 0$ and $\gamma \geq 1$ such that for every solution $x(t,\varphi)$ of system (1) with initial function $\varphi(\theta), \theta \in [-h,0]$ the following condition holds

$$\|x(t,\varphi)\| \leq \gamma \|\varphi\|_h e^{-\lambda t} \tag{2}$$

Here $\|\varphi\|_h$ denotes

$$\|\varphi\|_h = \sup_{\zeta\in[-h,0]} \|\varphi(\zeta)\|.$$

The fundamental matrix [1] $K(t)$ of the system (1) is the solution of the following matrix equation

$$\frac{d}{dt}K(t) = K(t)A_0 + K(t-h)A_1$$

with the initial condition

$$K(0) = I, \qquad K(\theta) = 0, \quad \theta \in [-h,0].$$

As the system is exponentially stable the matrix

$$U(\tau,W) = \int_0^\infty K^T(t) W K(t+\tau)dt \tag{3}$$

is well defined for all $\tau \in R$ and every constant $n \times n$ matrix W. The matrix $U(\tau,W)$ plays for the time delay system the same role as the classical Lyapunov matrices do for the case of delay free systems. We call the matrix (3) Lyapunov matrix associated with W, or simply Lyapunov matrix of system (1).

It satisfies the following properties:

$$U(-\tau,W) = U^T(\tau,W^T), \quad \tau \geq 0, \tag{4}$$

$$U'(\tau) = U(\tau)A_0 + U(\tau-h)A_1, \quad \tau \geq 0, \tag{5}$$

$$A_0^T U(0,W) + U^T(0,W)A_0 + A_1^T U(h,W) + U^T(h,W)A_1 = -W. \tag{6}$$

The following statement provides an explicit expression for quadratic functionals whose time derivative is a quadratic form of the present state of the system.

Theorem 2: [2] Let system (1) be exponentially stable. Given a symmetric matrix W_0. Then the time derivative of the functional

$$v_0(x_t,W_0) = x^T(t)U(0)x(t) \tag{7}$$

$$+2x^T(t)\int_{-h}^0 U(-h-\theta)A_1 x(t+\theta)d\theta \tag{8}$$

$$+\int_{-h}^0 \left[\int_{-h}^0 x^T(t+\theta_1)A_1^T U_0(\theta_1-\theta_2)A_1 x(t+\theta_2)d\theta_1\right]d\theta_2 \tag{9}$$

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where

$$U(\tau) = \int_0^\infty K^T(t) W_0 K(t+\tau) dt, \quad \tau \geq 0, \tag{10}$$

along the solutions of system (1) is equal to

$$\frac{dv_0(x_t, W_0)}{dt} = -x^T(t) W_0 x(t), \quad t \geq 0. \tag{11}$$

As it has been shown in [3] the following modification of the functional $v_0(x_t, W_0)$ allows to include more terms in the time derivative. Given symmetric matrices W_0, W_1, R, then the time derivative of the functional

$$\begin{aligned} v_f(x_t) &= v_0(x_t, W_0 + W_1 + hR_1) \\ &\quad + \int_{-h}^0 x^T(t+\theta)\left[W_1 + (h+\theta)R\right] x(t+\theta) d\theta \end{aligned} \tag{12}$$

along the solutions of the system (1) is equal to

$$\begin{aligned} \frac{dv_f(x_t)}{dt} &= -x(t) W_0 x(t) - x(t-h)^T W_1 x(t-h) \\ &\quad - \int_{-h}^0 x^T(t+\theta) R x(t+\theta) d\theta, t \geq 0. \end{aligned}$$

We say that the functional $v_f(x_t)$ is of complete type if the matrices W_0, W_1, R are positive definite.

III. Functionals whose time derivative includes a given cross term

Although for any exponentially stable time delay system the existence of complete type functionals with quadratic lower bound introduced in [3] is guaranteed, a broader class of functionals can be useful for achieving, for instance, better exponential estimates or robustness margins.

We look now for functionals whose time derivative

$$\frac{dv(x_t)}{dt} = -w(x_t), t \geq 0,$$

is of the form

$$\begin{aligned} w(x_t) &= \begin{pmatrix} x^T(t) & x^T(t-h) \end{pmatrix} \begin{pmatrix} W_0 & ZA_1 \\ A_1^T Z & W_1 \end{pmatrix} \\ &\quad \begin{pmatrix} x(t) \\ x(t-h) \end{pmatrix} + \int_{-h}^0 x^T(t+\theta) R x(t+\theta) d\theta. \end{aligned} \tag{13}$$

We assume that the matrix Z is symmetric and such that

$$\begin{bmatrix} W_0 & ZA_1 \\ A_1^T Z & W_1 \end{bmatrix}$$

is positive definite. Observe that for $Z = 0$, the functional $v(x_t) = v_f(x_t)$. It is clear that one only needs to complete the functional (12) with a functional whose time derivative along the solutions of system (1) is

$$-w_c(x_t) = -x^T(t) Z A_1 x(t-h) - x^T(t-h)^T A_1^T Z x(t).$$

Due to the exponential stability of the system the additional functional can be presented in the form

$$\begin{aligned} v_c(\varphi, Z) &= \int_0^\infty \left[x^T(t-h,\varphi) A_1^T Z x(t,\varphi) \right. \\ &\quad \left. + x^T(t,\varphi) Z A_1 x(t-h,\varphi\right]) dt. \end{aligned}$$

Substituting under the integral $A_1 x(t-h)$ by $\dot{x}(t) - A_0 x(t)$ gives

$$\begin{aligned} v_c(\varphi, Z) &= \int_0^\infty \left[\left(\dot{x}(t,\varphi) - A_0 x(t,\varphi) \right)^T Z x(t,\varphi) \right. \\ &\quad \left. + x^T(t,\varphi) Z \left(\dot{x}(t,\varphi) - A_0 x(t,\varphi) \right) dt \right] \\ &= \int_0^\infty \frac{d}{dt} x^T(t,\varphi) Z x(t,\varphi) dt \\ &\quad - \int_0^\infty x^T(t,\varphi) \left(A_0^T Z + Z A_0 \right) x(t,\varphi) dt. \end{aligned}$$

It follows that

$$v_c(\varphi, Z) = -\varphi(0) Z \varphi(0) - v_0(\varphi, A_0^T Z + Z A_0),$$

or

$$v_c(x_t, Z) = -x^T(t) Z x(t) - v_0(x_t, A_0^T Z + Z A_0). \tag{14}$$

Finally, we arrive at the following statement.

Theorem 3: Let system (1) be exponentially stable. Given positive definite matrices W_0, W_1, R and a symmetric matrix Z such that

$$\begin{pmatrix} W_0 & ZA_1 \\ A_1^T Z & W_1 \end{pmatrix}$$

is positive definite, then the time derivative of the functional

$$\begin{aligned} v(x_t) &= -x^T(t) Z x(t) + v_0(x_t, W_0 + W_1 - A_0^T Z - Z A_0 + hR) \\ &\quad + \int_{-h}^0 x^T(t+\theta)\left[W_1 + (h+\theta)R\right] x(t+\theta) d\theta \end{aligned} \tag{15}$$

along the solution of system (1) is equal to $-w(x_t)$,

$$\frac{dv(x_t)}{dt} = -w(x_t), t \geq 0.$$

Next, it is shown that the functional admits a quadratic lower bound.

Lemma 4: Let system (1) be exponentially stable and let symmetric matrices W_0, W_1, R and Z such that

$$\begin{pmatrix} W_0 & ZA_1 \\ A_1^T Z & W_1 \end{pmatrix}$$

is positive definite be given. Then there exists a constant $\alpha_0 > 0$ such that along the solutions of the system the functional (7) admits a quadratic lower bound of the form

$$\alpha_0 \|x(t)\|^2 \leq v(x_t),\, t \geq 0. \tag{16}$$

Proof: Let us define the functional

$$\hat{v}(x_t) = v(x_t) - \alpha \|x(t)\|^2. \tag{17}$$

Then

$$\frac{d\hat{v}(x_t)}{dt} = -\hat{w}(x_t) \tag{18}$$

where

$$\hat{w}(x_t) = w(x_t) + 2\alpha x^T(t)\left[A_0 x(t) + A_1 x(t-h)\right]$$

or

$$\begin{aligned}\hat{w}(x_t) &= \begin{pmatrix} x^T(t) & x^T(t-h) \end{pmatrix} \times \\ &\quad \begin{pmatrix} W_0 & ZA_1 \\ A_1^T Z & W_1 \end{pmatrix} \begin{pmatrix} x(t) \\ x(t-h) \end{pmatrix} \\ &\quad + \int_{-h}^{0} x^T(t+\theta) R x(t+\theta) d\theta \\ &\quad + 2\alpha x(t)\left[A_0 x(t) + A_1 x(t-h)\right],\end{aligned}$$

equivalently,

$$\begin{aligned}\hat{w}(x_t) &= \begin{pmatrix} x^T(t) & x^T(t-h) \end{pmatrix} L(\alpha) \begin{pmatrix} x(t) \\ x(t-h) \end{pmatrix} \\ &\quad + \int_{-h}^{0} x^T(t+\theta) R x(t+\theta) d\theta.\end{aligned}$$

where

$$L(\alpha) = \begin{pmatrix} W_0 & ZA_1 \\ A_1^T Z & W_1 \end{pmatrix} + \alpha \begin{pmatrix} A_0 + A_0^T & A_1 \\ A_1^T & 0 \end{pmatrix}.$$

As the matrix $L(0) = \begin{pmatrix} W_0 & ZA_1 \\ A_1^T Z & W_1 \end{pmatrix}$ is positive definite, then there exists $\alpha_0 > 0$ such that the matrix $L(\alpha_0)$ is also positive definite. For $\alpha = \alpha_0$, observe that $\hat{w}(x_t) \geq 0$. Integrating the equality (18) along the solutions of the system (1) one arrives at the conclusion that

$$\hat{v}(\varphi) = \int_0^{\infty} \hat{w}(x_t(\varphi)) dt \geq 0.$$

The last inequality ends the proof of the lemma. ∎

Remark 5: Let α^* be the first positive eigenvalue of the matrix pencil

$$L(\beta) = \begin{pmatrix} W_0 & ZA_1 \\ A_1^T Z & W_1 \end{pmatrix} + \alpha \begin{pmatrix} A_0 + A_0^T & A_1 \\ A_1^T & 0 \end{pmatrix},$$

then the inequality (16) holds for all $\alpha_0 \in (0, \alpha^*]$.

Next, it is shown that the functional admits an upper bound.

Lemma 6: For some constant $\varepsilon > 0$, the functional (15) admits the following quadratic upper bound

$$v(x_t) \leq \varepsilon \left\{ \|x(t)\|^2 + \int_{-h}^{0} \|x(t+\theta)\|^2 d\theta \right\} \tag{19}$$

Proof: Let

$$\nu = \max_{\tau \in [0,h]} \left\| U(\tau, W_0 + W_1 - A_0^T Z - Z A_0 + hR) \right\|, \tag{20}$$

Using appropriate majorizations for each term of the functional described by (15) and (7) leads to

$$\begin{aligned} v(x_t) &\leq \|U(0) - Z\| \|x(t)\|^2 + \nu \|A_1\| \int_{-h}^{0} 2x(t)x(t+\theta) d\theta \\ &\quad + \nu h \|A_1\|^2 \int_{-h}^{0} \|x(t+\theta)\|^2 d\theta \\ &\quad + (\|W_1\| + h\|R\|) \int_{-h}^{0} \|x(t+\theta)\|^2 d\theta \\ &\leq \eta_1 \|x(t)\|^2 + \eta_2 \int_{-h}^{0} \|x(t+\theta)\|^2 d\theta \end{aligned} \tag{21}$$

where

$$\begin{aligned} \eta_1 &= \|U(0) - Z\| + h\nu \|A_1\| \\ \eta_2 &= \nu \|A_1\| + \|W_1\| + h\|R\| \end{aligned} \tag{22}$$

and the result holds for any ε such that $\varepsilon \geq \max\{\eta_1, \eta_2\}$. ∎

Remark 7: Using (19), and the fact that $\|x(t)\| \leq \|x_t\|_h$, it follows straightforwardly that the functional (15) also satisfies

$$v(x_t) \leq \alpha_1 \|x_t\|_h^2. \tag{23}$$

IV. Exponential Bounds

In this section, for a given initial function, we obtain an exponential estimate of the form (2) of the solution of system (1) based on the functional of complete type (15). First we prove a preliminary result.

Lemma 8: Let system (1) be exponentially stable. Given symmetric matrices W_0, W_1, R and Z, such that

$$\begin{pmatrix} W_0 & ZA_1 \\ A_1^T Z & W_1 \end{pmatrix}$$

is positive definite, then there exists a constant $\beta > 0$ such that the functional $w(x_t)$ defined in (13) satisfies

$$2\beta v(x_t) \leq w(x_t). \tag{24}$$

Proof: Consider (21) and a lower bound for $w(x_t)$ obtained as follows. Let

$$\mu = \min\left\{ \lambda_{\min} \begin{pmatrix} W_0 & ZA_1 \\ A_1^T Z & W_1 \end{pmatrix}, \lambda_{\min}(R) \right\}$$

then

$$w(x_t) \geq \mu \left(\|x(t)\|^2 + \int_{-h}^{0} \|x(t+\theta)\|^2 d\theta \right).$$

Finally observe that if we choose β as

$$2\beta = \min\left\{ \frac{\mu}{\eta_1}, \frac{\mu}{\eta_2} \right\} \tag{25}$$

where η_1 and η_2 are defined in (22), then (24) is satisfied. ∎

Theorem 9: Let system (1) be exponentially stable. Given symmetric matrices W_0, W_1, R and Z such that

$$\begin{pmatrix} W_0 & ZA_1 \\ A_1^T Z & W_1 \end{pmatrix}$$

is positive definite, then for any initial condition $\varphi(\theta), \theta \in [-h, 0]$, the solution of system (1) satisfies the inequality

$$\|x(t,\varphi)\| \leq \sqrt{\frac{\alpha_1}{\alpha_0}}\, \|\varphi\|_h e^{-\beta t}, \quad t \geq 0$$

where α_0, α_1 are the constants defined by (16) and (23), respectively, and β is defined in (25).

Proof: Observe that (24) implies that

$$\frac{dv(x_t)}{dt} \leq -w(x_t) \leq -2\beta v(x_t).$$

Solving for $v(x_t)$ yields $v(x_t) \leq e^{-2\beta t} v(\varphi)$. Then, (16) and (23) imply that

$$\alpha_0 \|x(t,\varphi)\|^2 \leq v(x_t) \leq v(\varphi) e^{-2\beta t} \leq \alpha_1 e^{-2\beta t} \|\varphi\|_h^2$$

and the result follows from the first and last terms of these inequalities. ■

V. Robust Stability

In this section we show that the functional (15) can be used to derive robust stability conditions. Consider the perturbed system

$$\dot{y}(t) = (A_0 + \Delta_0)\, y(t) + (A_1 + \Delta_1)\, y(t-h), \tag{26}$$

where

$$\|\Delta_0\| \leq \rho_0, \text{ and } \|\Delta_1\| \leq \rho_1 \tag{27}$$

and let the nominal system (1) be stable. Using the functional (15) we want to derive conditions under which the perturbed system (26) remains stable. Consider the derivative of $v(y_t)$ along the trajectories of (26):

$$\begin{aligned} \frac{dv(y_t)}{dt} = & -\dot{y}^T(t) Z y(t) - y^T(t) Z \dot{y}(t) \\ & + \frac{dv_0(y_t, W_0 + W_1 - A_0^T Z - Z A_0 + hR)}{dt} \\ & + \frac{d}{dt}\left[\int_{-h}^{0} x^T(t+\theta)\, [W_1 + R(h+\theta)] \times \right. \\ & \left. x^T(t+\theta) d\theta \right]. \end{aligned}$$

Making the change of variable $s = t + \theta$ in the integral term, calculating de derivative of v_0 along to the trajectories of (26) and substituting the system (26) yields

$$\begin{aligned} & \frac{dv(y_t)}{dt} = -w(y_t) + 2\,[\Delta_0 y(t) + \Delta_1 y(t-h)]^T \times \\ & [-Zy(t) + U(0)y(t) \\ & \left. + \int_{-h}^{0} U(-h-\theta) A_1 y(t+\theta) d\theta \right]. \end{aligned}$$

This expression can be rewritten as

$$\begin{aligned} \frac{dv(y_t)}{dt} = & -\begin{pmatrix} y^T(t) & y^T(t-h) \end{pmatrix} \begin{pmatrix} W_0 & ZA_1 \\ A_1^T Z & W_1 \end{pmatrix} \times \\ & \begin{pmatrix} y(t) \\ y(t-h) \end{pmatrix} - \int_{-h}^{0} y^T(t+\theta) R y(t+\theta) d\theta \\ & + 2\,[\Delta_0 y(t) + \Delta_1 y(t-h)]^T\, [-Zy(t) \\ & + U(0) y(t) + \int_{-h}^{0} U(-h-\theta) A_1 y(t+\theta) d\theta]. \end{aligned}$$

Let $a_0 = \|A_0\|$, $a_1 = \|A_1\|$, $z = \|Z\|$ and consider (20), (27), then

$$\begin{aligned} \frac{dv(y_t)}{dt} \leq & -\begin{pmatrix} y^T(t) & y^T(t-h) \end{pmatrix} \begin{pmatrix} W_0 & ZA_1 \\ A_1^T Z & W_1 \end{pmatrix} \times \\ & \begin{pmatrix} y(t) \\ y(t-h) \end{pmatrix} - \int_{-h}^{0} y^T(t+\theta) R y(t+\theta) d\theta \\ & + \rho_1 \nu a_1 \left(\left\| y^T(t-h) \right\|^2 + \int_{-h}^{0} \|y(t+\theta)\|^2 d\theta \right) \\ & + \rho_0 \nu a_1 \left(\left\| y^T(t) \right\|^2 + \int_{-h}^{0} \|y(t+\theta)\|^2 d\theta \right) \\ & + 2 y^T(t)\, (z\rho_0 + \rho_0 \nu)\, y(t) \\ & + 2 y^T(t)\, (z\rho_1 + \rho_1 \nu)\, y(t-h), \end{aligned}$$

therefore

$$\begin{aligned} \frac{dv(y_t)}{dt} \leq & -\begin{pmatrix} y^T(t) & y^T(t-h) \end{pmatrix} \times \\ & \begin{pmatrix} W_0 - \Gamma_1 I_n & ZA_1 - \Gamma_2 I_n \\ A_1^T Z - \Gamma_2 I_n & W_1 - \Gamma_3 I_n \end{pmatrix} \begin{pmatrix} y(t) \\ y(t-h) \end{pmatrix} \\ & - \int_{-h}^{0} y^T(t+\theta)\, (R - \Gamma_4 I_n)\, y(t+\theta) d\theta \end{aligned}$$

where

$$\Gamma_1 = 2\rho_0 (z + \nu) + \rho_0 \nu a_1 \tag{28}$$

$$\Gamma_2 = \rho_1 (z + \nu) \tag{29}$$

$$\Gamma_3 = \rho_1 \nu a_1 \tag{30}$$

$$\Gamma_4 = \nu a_1 (\rho_1 + \rho_0) \tag{31}$$

and we can write the following result.

Theorem 10: Let system 1 be exponentially stable. Then the system (26) remains stable for all perturbations which satisfy the conditions (27) if there exist positive definite matrices

$$\begin{pmatrix} W_0 & ZA_1 \\ A_1^T Z & W_1 \end{pmatrix} \text{ and } R$$

such that

$$\begin{pmatrix} W_0 - \Gamma_1 I_n & ZA_1 - \Gamma_2 I_n \\ A_1^T Z - \Gamma_2 I_n & W_1 - \Gamma_3 I_n \end{pmatrix} > 0 \text{ and } R - \Gamma_4 I_n > 0$$

where Γ_1, Γ_2, Γ_3 and Γ_4 are defined in (28), (29), (30) and (31), respectively

Example 11: Consider the stable system (1) with

$$A_0 = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix},\ A_1 = \begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix} \qquad (32)$$

introduced in [3]. For

$$W_0 = \begin{pmatrix} 1.3989 & 0 \\ 0 & 1.3989 \end{pmatrix},\ W_1 = \begin{pmatrix} 1.3989 & 0 \\ 0 & 1.3989 \end{pmatrix}$$
$$Z = \begin{pmatrix} 1.3989 & 0 \\ 0 & 0.27979 \end{pmatrix} \text{ and } R = I_2,$$

the matrix

$$\begin{pmatrix} W_0 & ZA_1 \\ A_1^T Z & W_1 \end{pmatrix}$$

is positive definite. So, in order to construct the functional given by (15), we calculate the matrix

$$W = W_0 + W_1 - A_0^T Z - ZA_0 + hR,$$

namely,

$$W = \begin{pmatrix} 3.7979 & -1.1191 \\ -1.1191 & 4.9171 \end{pmatrix}.$$

Using the semianalytic procedure given in [5] based on the solution of the system given by (4), (5) and (6) we determine the matrix function $U(\tau)$, $\tau \in [0,1]$. The components of the matrix U are shown in the next figure.

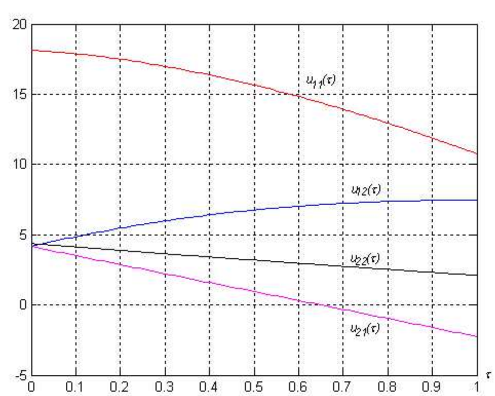

Figure 1: Components of the matrix U.

Then using the results of Theorem (9) it follows by direct calculations that the solutions of the system (1) with A_0 and A_1 defined in (32) satisfy

$$\|x(t,\varphi)\| \leq 32.394 \|\varphi\|_h e^{-0.011t}, \quad t \geq 0.$$

Next, using the results of Theorem (10) we obtain that the robust stability margins for the system (26) with A_0 and A_1 defined in (32) are $\rho_0 = 0.016$ and $\rho_1 = 0.02$. These bounds for the perturbations are less conservative than those given in [3].

VI. Concluding remarks

The form of complete type functionals whose derivative includes a cross term associated to any exponentially stable linear time delay system is determined. Exponential estimates and robustness conditions are given.

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