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Stabilization of Single-Input Nonlinear Systems Using Higher Order Compensating Sliding Mode Control

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Abstract— **In this paper we investigate a new approach for the stabilization of nonlinear systems using *sliding mode control*. The idea is based on defining a local nonlinear coordinate transformation for a system with a given *robust relative degree* (c.f. [2]) and then finding a control law, in the new coordinates, which steers the state of the system to the origin. Our results on sliding mode controller are compared to a sliding mode control design based on *approximate feedback linearization*.**

I. Introduction

Stabilization of single-input systems around equilibrium points is a basic problem in control theory. It is a relatively straightforward task in the case of linear controllable systems [5], [7]. For nonlinear systems, it is natural to attempt to locally transform these systems into linear ones and apply the above ideas for controlling. The linearization can be accomplished by finding a Taylor expansion of the system vector field at the equilibrium point (c.f. [7]). Another way of getting a linear approximation is *approximate feedback linearization* (AFBL) due to Hauser and Sastry [2], a generalization of *feedback linearization* [1]. They showed that a trajectory tracking controller based on AFBL performs better than one based on Taylor expansion linearization. There is a large class of systems which are approximate feedback linearizable including such important examples as the inverted pendulum and the ball and beam apparatus.

In the last few decades *sliding mode control* (c.f. [3], [6], [9]) has become a very popular control strategy for trajectory tracking and stabilization of dynamical systems. It is used both for linear and nonlinear systems. The reason for the popularity of this type of controller is robustness and system order reduction. For linear systems there is a well developed theory for the design of sliding mode controllers. For systems which are approximate feedback linearizable we can also design a sliding mode controller (SMC) for the linear system approximation [9].

In the present work, we modify a sliding mode controller based on the standard AFBL, i.e. a stabilizing SMC designed for a system linear approximation, so that the higher order terms are not neglected while finding a feedback controller. Here we present a sliding mode control strategy based on *robust coordinate transformations*. We describe our new control algorithm and compare it to a standard sliding mode controller based on the approximate feedback linearization.

We use the ball and beam example for simulation comparison of the two approaches. Using sliding mode control after robust coordinate transformation results in a greatly increased capture region than is possible via approximate feedback linearization.

The paper is organized as follows. In Section II we present some preliminary information and basic assumptions, and review of approximate feedback linearization. In Section III we present the design procedure for a higher order sliding mode controller (or sliding mode control based on robust coordinate transformation). We shall show the stability of the system under this type of controller and describe the design procedure. In Section IV we illustrate the control strategy on the example of the ball and beam apparatus, we also compare the performances of the system under sliding mode controllers based on robust coordinate transformation and approximate feedback linearization.

II. Preliminaries

A. Notation and Basic Assumptions

Through out this paper we deal with a control-affine single-input system, whose states evolve on $\mathbb{R}^n$. We model our system dynamics as follows:

$$\begin{aligned} \dot{x}(t) &= f(x(t)) + g(x(t))u(t), \\ x(0) &= x_0, \end{aligned} \tag{1}$$

where $x(t)$ is a vector representation of the states for the dynamical system at any fixed time t, $x(t) \in \mathbb{R}^n$, $u : \mathbb{R} \to \mathbb{R}$ is the control. A solution of (1) is called a *state trajectory*. The drift vector field $f(x)$ and the control vector field $g(x)$ are assumed to be smooth. Let 0 be an equilibrium point of the system, i.e. $f(0) = 0$. Define $y(t) = h(x(t))$ to be *a smooth output for system* (1) where h is a real valued infinitely differentiable function on $\mathbb{R}^n$ and $h(0) = 0$. Denote by $\dot{y}$ the time derivative of h along the system trajectory, that is $\dot{y}(t) = dh_{x(t)} \cdot (f + gu(t))\,(x(t))$.

Definition 2.1: Let $y(t) = h(x(t))$ be a smooth output of system (1). The **robust relative degree** of y with respect to a nonlinear system (1) about $0 \in \mathbb{R}^n$ is an integer γ such that

- $\mathrm{L}_g h(0) = \mathrm{L}_g \mathrm{L}_f h(0) = \ldots = \mathrm{L}_g \mathrm{L}_f^{\gamma-2} h(0) = 0$,
- $\mathrm{L}_g \mathrm{L}_f^{\gamma-1} h(0) \neq 0$.

Here the function $\mathrm{L}_g h$ is the Lie derivative of $h(x)$ along the vector field $g(x)$ ($\mathrm{L}_g h = dh_x \cdot g(x)$), with $\mathrm{L}_g^i h = \mathrm{L}_g \mathrm{L}_g^{i-1} h$.

In this paper we restrict ourselves to systems which are *approximate feedback linearizable* at the origin of $\mathbb{R}^n$. Examples of such systems include inverted pendulum, the ball and beam apparatus. This motivates the following definition.

Definition 2.2: A single input nonlinear system (1) is called **approximate feedback linearizable** at $0 \in \mathbb{R}^n$ if there exists a smooth output $y = h(x)$ whose robust relative degree with respect to system (1) about 0 is equal to the dimension of the state space ($\gamma = n$). The output y is called an **approximately linearizing output**.

It was shown that systems which fall under the above definition in fact satisfy the sufficient condition of being *small time locally controllable* [2], [10]. We use the term *approximate feedback linearizable* for technical reasons, which will be obvious later.

B. Approximate Feedback Linearization and Robust Coordinate Transformation

In this subsection we make an overview of the approximate feedback linearization (c.f. [2]). Let system (1) be approximate feedback linearizable. Let y be an approximately linearizing output. Define a map $T : x \mapsto z$ by:

$$z_i = \mathrm{L}_f^{i-1} h(x), \quad i \in \{1, 2, \dots n\}, \tag{2}$$

with $z = (z_1 \ z_2 \ \dots \ z_n)^T$.

Remark 2.1: Note that the map T is a local diffeomorphism (see [1], [2]) with $T(0) = 0$.

Definition 2.3: Consider system (1), approximate feedback linearizable at the origin, and an approximately linearizing output $y = h(x)$. The transformation T as defined in (2) is called a **robust coordinate transformation** at $0 \in \mathbb{R}^n$.

In approximate feedback linearization one simplifies the model by assuming that the functions

$$\mathrm{L}_g h(x), \ \mathrm{L}_g \mathrm{L}_f h(x), \ \dots \ \mathrm{L}_g \mathrm{L}_f^{n-2} h(x),$$

which vanish at $x = 0$ are in fact identically equal to zero [2]. Then the dynamical system in the new coordinates can be approximated by the following system model:

$$\begin{aligned} \dot{z}_1 &= z_2, \\ \dot{z}_2 &= z_3, \\ &\vdots \\ \dot{z}_{n-1} &= z_n, \\ \dot{z}_n &= \mathrm{L}_f^n h(T^{-1}(z)) + \mathrm{L}_g \mathrm{L}_f^{n-1} h(T^{-1}(z)) u. \end{aligned} \tag{3}$$

Observation 2.1: If we choose the control as follows

$$u = -\mathrm{L}_f^n h(T^{-1}(z)) / \mathrm{L}_g \mathrm{L}_f^{n-1} h(T^{-1}(z)) + v / \mathrm{L}_g \mathrm{L}_f^{n-1} h(T^{-1}(z)),$$

(T^{-1} denotes the inverse of T which is well defined on sufficiently small neighborhoods of 0), then the dynamics of the system (3) become

$$\begin{aligned} \dot{z}_1 &= z_2, \\ \dot{z}_2 &= z_3, \\ &\vdots \\ \dot{z}_{n-1} &= z_n, \\ \dot{z}_n &= v. \end{aligned} \tag{4}$$

One can see that equations (4) describe a linear controllable system (c.f. [5], [7]), with the input v.

The technique described above is called *approximate feedback linearization* (c.f.[2]).

III. Sliding Mode Control Based on Robust Coordinate Transformations

In this section we present a method of stabilization of a nonlinear system using sliding mode control, which is based on *robust coordinate transformation*. Unlike the above treatment we do not suppose that the functions

$$\mathrm{L}_g h(x), \ \mathrm{L}_g \mathrm{L}_f h(x), \ \dots \ \mathrm{L}_g \mathrm{L}_f^{n-2} h(x),$$

vanish identically but use their values in our controller design. Instead of simplified model (3) we use the exact model (see below system (6)).

Let $y = h(x)$ be a smooth output for system (1) such that $\gamma = n$. Let T be the robust coordinate transformation (see Definition 2.3), i.e.

$$z_i = \mathrm{L}_f^{i-1} h(x), \ i \in \{1, 2, 3 \dots, n\} \tag{5}$$

Suppose that the system is defined on an open neighborhood U of 0 such that T maps U diffeomorphically onto its image. Then the system dynamics in the new coordinates becomes:

$$\begin{aligned} \dot{z}_1 &= z_2 + \mathrm{L}_g h(T^{-1}(z)) u \\ \dot{z}_2 &= z_3 + \mathrm{L}_g \mathrm{L}_f h(T^{-1}(z)) u \\ \dot{z}_3 &= z_4 + \mathrm{L}_g \mathrm{L}_f^2 h(T^{-1}(z)) u \\ &\vdots \\ \dot{z}_{n-1} &= z_n + \mathrm{L}_g \mathrm{L}_f^{n-2} h(T^{-1}(z)) u \\ \dot{z}_n &= \mathrm{L}_f^n h(T^{-1}(z)) + \mathrm{L}_g \mathrm{L}_f^{n-1} h(T^{-1}(z)) u, \\ z_0 &= T(x_0). \end{aligned} \tag{6}$$

The dynamical system thus obtained is not an approximation as in the previous section since we do not neglect the sequence of functions $\mathrm{L}_g \mathrm{L}_f^i h(x), \ i = 0 \dots n-2$. However system (6) is nonlinear, so that we may not be able to use the existing linear technique to locally stabilize the system.

From the fact that T is a local diffeomorphism at 0 with $T(0) = 0$ it follows that there is an open neighborhood V of $0 \in \mathbb{R}^n$ such that asymptotic stability of (6) implies asymptotic stability of (1). Thus we need to define the control algorithm which locally stabilizes the dynamical system (6). Next we describe a sliding mode controller design to stabilize the above dynamical system at $z = 0$.

A sliding mode controller u causes the state trajectory to reach a surface $S \subset \mathbb{R}^n$ in finite time and stay on S thereafter. The set S, often referred to as a *sliding surface*, is chosen so that any system trajectory restricted to it is asymptotically stable. Here we use the notion of asymptotic stability in the small. Thus a system trajectory $x(t)$ is asymptotically stable at $0 \in \mathbb{R}^n$ if, for any $\varepsilon > 0$, there is $\delta > 0$ such that $\|x(0)\| \leq \delta$ implies that $\|x(t)\| \leq \varepsilon$ for any $t \in [0, \infty)$. Moreover, for any $\epsilon > 0$, there is $\tau > 0$ such that $\|x(t)\| \leq \epsilon$ for $t \geq \tau$. Here $\|\cdot\|$ denotes the usual norm on $\mathbb{R}^n$, i.e. for any $x \in \mathbb{R}^n$, $\|x\| = \sqrt{x^T I_n x}$, here I_n denotes an n-by-n identity matrix, x^T is the transpose of x. The following lemma specifies a control algorithm which forces a single input system to "reach" a sliding surface in finite time (c.f. [3]).

Lemma 3.1: Let $\tilde{s}(t) = s(x(t))$ be a smooth output of a single input system (1), with $\dot{\tilde{s}}(t) = a_s(x(t)) + u(t)b_s(x(t))$. Suppose that the set S defined by

$$S = \{x | s(x) = 0\}$$

is embedded co-dimension 1 submanifold of $\mathbb{R}^n$ (e.g. 0 is a regular value of s). Consider D, a compact convex subset of $\mathbb{R}^n$ with nonempty interior, such that $D \cap S \neq \emptyset$. Set

$$u_*(x) = -K\text{sign}(s(x))\text{sign}(b_s(x)), \tag{7}$$

where $K > 0$ and $K|b_s(x)| \geq |a_s(x)|$ for all $x \in D$. Then there exists an open set $D_0 \subset D$ with $D_0 \cap S \neq \emptyset$ such that a system trajectory starting at $x_0 \in D_0$ with $u = u_*$ satisfies the following: there is a $\tau > 0$ for which $x(t) \in S$, $\forall t \geq \tau$.

This motivates the following theorem.

Theorem 3.1: Consider the system (6) with sliding surface:

$$S = \{z \in \mathbb{R}^n | \; s(z) = z_n + s_{n-1}z_{n-1} + s_{n-2}z_{n-2} \ldots + s_1 z_1 = 0\} \tag{8}$$

such that the polynomial equation

$$P_s(\sigma) = \sigma^{n-1} + s_{n-1}\sigma^{n-2} + \ldots + s_2\sigma + s_1 = 0$$

has roots with negative real parts. Let D be a convex compact neighborhood of $z = 0$. Set

$$u_{sm} = -K\text{sign}\left(\mathrm{L}_g\mathrm{L}_f^{n-1}h(T^{-1}(z)) + \textstyle\sum_{i=1}^{n-1} s_i\mathrm{L}_g\mathrm{L}_f^{i-1}h(T^{-1}(z))\right)\text{sign}(s(z)), \tag{9}$$

with $K > 0$ such that

$$K \; |\textstyle\sum_{i=1}^{n-1} s_i\mathrm{L}_g\mathrm{L}_f^{i-1}h(T^{-1}(z)) + \mathrm{L}_g\mathrm{L}_f^{n-1}h(T^{-1}(z))| \geq |\textstyle\sum_{i=1}^{n-1} s_i z_{i+1} + \mathrm{L}_f^n h(T^{-1}(z))|,$$

for all z in D. Then there exists an open neighborhood $D_0 \subset D$ of $z = 0$, such that the system (6) with $u = u_{sm}$ and $z_0 = z(0) \in D_0$ is asymptotically stable.

Proof Here we provide a sketch of the proof.

Set $\tilde{s}(t) = s(z(t))$. Then

$$\dot{\tilde{s}} = \mathrm{L}_{f_z}s(z) + \mathrm{L}_{g_z}s(z),$$

where

$$f_z(z) = \begin{pmatrix} z_2 \\ z_3 \\ \vdots \\ z_n \\ \mathrm{L}_f^n h(T^{-1}(z)) \end{pmatrix},$$

$$g_z(z) = \begin{pmatrix} \mathrm{L}_g h(T^{-1}(z)) \\ \mathrm{L}_g\mathrm{L}_f h(T^{-1}(z)) \\ \vdots \\ \mathrm{L}_g\mathrm{L}_f^{n-1}h(T^{-1}(z)) \end{pmatrix}.$$

In particular

$$\dot{\tilde{s}}(t) = a_s(z) + b_s(z)u,$$

where

$$a_s = \textstyle\sum_{i=1}^{n-1} s_i z_{i+1} + \mathrm{L}_f^n h(T^{-1}(z)),$$

$$b_s(z) = \mathrm{L}_g\mathrm{L}_f^{n-1}h(T^{-1}(z)) + \textstyle\sum_{i=1}^{n-1} s_i\mathrm{L}_g\mathrm{L}_f^{i-1}h(T^{-1}(z)).$$

Thus from Lemma 3.1, there is an open neighborhood D_0 of $z = 0$ such that (for $z(0) = z_0 \in D_0$) there exists a $\tau > 0$ for which $z(t) \in D_0 \cap S$, $t \geq \tau$.

From the expression $\dot{\tilde{s}}(t) = 0$ we find an *equivalent control* u_{eq} (c.f. [3], [7]) acting on the system while on S. To find the dynamics of the system restricted to S, we substitute u_{eq} into (6) and get:

$$\begin{aligned} \dot{z}_1 &= z_2 + q_1(z) \\ \dot{z}_2 &= z_3 + q_2(z) \\ \dot{z}_3 &= z_4 + q_3(z) \\ &\vdots \\ \dot{z}_{n-1} &= -s_1z_1 - s_2z_2 \ldots - s_{n-1}z_{n-1} + q_{n-1}(z) \\ \\ z_n &= s_1z_1 + \cdots + s_{n-1}z_{n-1}, \\ z(0) &= z_0, \end{aligned} \tag{10}$$

It can be shown that the smooth real valued functions q_i are $O^2(z)$ [1]. This proves the local stability of (10) (see [2], [7]), thus completes the proof of the theorem. □

We call the control u_{sm} defined in (9) a **higher order compensating sliding mode controller (HOCSMC)**.

Note that when we use the sliding mode control based on approximate feedback linearization (c.f. [9]), we do not account for the terms: $\mathrm{L}_g\mathrm{L}_f^i h(T^{-1}(z))$, $i \in \{0, 1, 2 \ldots, n-2\}$. That is, we design a control algorithm for the approximation (3). Therefore under approximate feedback linearization the sliding mode controller (corresponding to the sliding surface S defined in (8)) becomes:

$$u_{sm}^{approx} = -K\text{sign}(\mathrm{L}_g\mathrm{L}_f^{n-1}h(T^{-1}(z))\text{sign}(s(z)),$$

with $K > 0$ such that

$$K \; |\mathrm{L}_g\mathrm{L}_f^{n-1}h(T^{-1}(z))| \geq |\textstyle\sum_{i=1}^{n-1} s_i z_{i+1} + \mathrm{L}_f^n h(T^{-1}(z))|.$$

One can verify that the term $\sum_{i=1}^{n-1} s_i\mathrm{L}_g\mathrm{L}_f^i h(T^{-1}(z))$ does not appear explicitly in the above control algorithm in contrast to the control law (9). In the next section we illustrate the advantage of our higher order sliding mode controller u_{sm} over approximate feedback linearization based sliding mode controller u_{sm}^{approx} for the ball and beam apparatus.

[1]We recall that the class $O^n(x)$ is defined from: $O^n(x) = \{\phi \in C^\infty | \lim_{\|x\|\to 0} \frac{|\phi(x)|}{\|x\|^n} \neq 0, < \infty\}$.

IV. Stabilization of the Ball and Beam System Using Sliding Mode Control

Consider the following dynamical model of the ball and beam system, proposed in [2]:

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= B(x_1x_4^2 - G\sin(x_3))\\ \dot{x}_3 &= x_4\\ \dot{x}_4 &= u,\end{aligned} \tag{11}$$

with $B = 0.7$, $G = 9.8$. The variables x_1, x_2 denote position of the ball on the beam and the velocity of the ball respectively. The variables x_3, x_4 denote the angular position and velocity of the beam (i.e. $\dot{x}_3 = x_4$). For this system

$$f(x) = \begin{pmatrix} x_2 \\ B(x_1x_4^2 - G\sin(x_3)) \\ x_4 \\ 0 \end{pmatrix}, \qquad g(x) = \begin{pmatrix} 0\\0\\0\\1\end{pmatrix}.$$

Consider a smooth output $y = h(x) = x_1$. By straightforward computation one can verify that:

$$\begin{aligned}&\mathrm{L}_g h \equiv \mathrm{L}_g\mathrm{L}_f h \equiv 0,\\ &\mathrm{L}_g\mathrm{L}_f^2 h(x) = 2Bx_1x_4,\\ &\mathrm{L}_g\mathrm{L}_f^3 h(x) = 2Bx_2x_4 - BG\cos(x_3).\end{aligned}$$

Thus $\mathrm{L}_g\mathrm{L}_f^i h(0) = 0$ for $i = \{0,1,2\}$, but $\mathrm{L}_g\mathrm{L}_f^3 h(0) \neq 0$. Therefore according to Definition 2.1 the robust relative degree of y is 4, i.e. equal to the dimension of the system state space. Define the transformation $T : x \mapsto z$ by:

$$T(x) = \begin{pmatrix} h(x)\\ \mathrm{L}_f h(x)\\ \mathrm{L}_f^2 h(x)\\ \mathrm{L}_f^3 h(x)\end{pmatrix} = \begin{pmatrix} z_1\\z_2\\z_3\\z_4\end{pmatrix} = z. \tag{12}$$

Note that T is a robust coordinate transformation (see Definition 2.3).

Define the sliding surface:

$$S = \{z \in \mathbb{R}^4 |\ s(z) = z_4 + 6z_3 + 12z_2 + 8z_1 = 0\}. \tag{13}$$

Thus the time derivative of $\tilde{s}(t) = s(z(t))$ along the system trajectory $z(t)$ is equal to:

$$\dot{\tilde{s}}(t) = \mathrm{L}_f^4 h(T^{-1}(z)) + \textstyle\sum_{i=1}^3 s_i z_{i+1} + u\left(\mathrm{L}_g\mathrm{L}_f^3 h(T^{-1}(z)) + \sum_{i=1}^3 s_i \mathrm{L}_g\mathrm{L}_f^{i-1} h(T^{-1}(z))\right),$$

with $(s_1, s_2, s_3) = (8, 12, 6)$.

Define the regulating signal u according to Theorem 3.1, i.e.

$$u_{sm} = -K\mathrm{sign}\left(\mathrm{L}_g\mathrm{L}_f^3 h(T^{-1}(z)) + \textstyle\sum_{i=1}^3 s_i \mathrm{L}_g\mathrm{L}_f^{i-1} h(T^{-1}(z))\right)\mathrm{sign}(s(z)).$$

This gives us the following control algorithm for system (11):

$$\begin{aligned}u_{sm}(x) = -K&\mathrm{sign}(-BG\cos(x_3) + 2Bx_2x_4 + 12Bx_1x_4)\\ &\times\mathrm{sign}(-BGx_4\cos(x_3) + Bx_2x_4^2\\ &\quad -6BG\sin(x_3) + 6Bx_1x_4^2 + 12x_2 + 8x_1).\end{aligned} \tag{14}$$

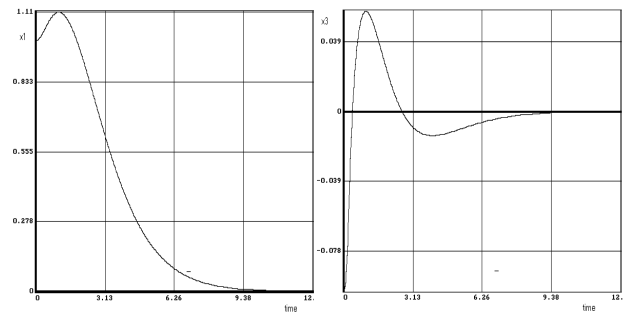

Fig. 1. Simulation results for system (11), under stabilizing sliding mode control using (14), with initial conditions $x_1(0) = 1,\ x_3(0) = -0.1$. The variables $x1$ and $x3$ are position of the ball and angular position of the beam respectively.

where $K = 2$. The simulation results for system (11) with different initial conditions are shown in Figures 1,2,3,4.

To provide a basis for comparison we define a stabilizing sliding mode control based on approximate feedback linearization of system (11). Thus we consider the following dynamical model:

$$\begin{aligned}\dot{z}_1 &= z_2\\ \dot{z}_2 &= z_3\\ \dot{z}_3 &= z_4\\ \dot{z}_4 &= \mathrm{L}_f^4 h(T^{-1}(z)) + \mathrm{L}_g\mathrm{L}_f^3 h(T^{-1}(z))u,\end{aligned} \tag{15}$$

where $T(x) = z$ was defined in (12). Define the sliding surface as in (13), then according to Theorem 3.1 the control algorithm becomes:

$$u = -K\mathrm{sign}(\mathrm{L}_g\mathrm{L}_f^3 h(T^{-1}(z)))\mathrm{sign}(s(z)) \tag{16}$$

or in x coordinates:

$$\begin{aligned}u = K&\mathrm{sign}(-BG\cos(x_3) + 2Bx_2x_4)\\ &\times\mathrm{sign}(-BGx_4\cos(x_3) + Bx_2x_4^2\\ &\quad -6BG\sin(x_3) + 6Bx_1x_4^2 + 12x_2 + 8x_1).\end{aligned}$$

The simulation results for the system under approximate feedback linearization based controller are shown in Figures 5,6,7,8.

From the simulations, we note that on a small neighborhood of $0 \in \mathbb{R}^4$ (i.e. $\{x | |x_1| \leq 1.5,\ |x_3| \leq 0.2\}$) the system performance is the same for both control strategies Figures 1, 5. However as we increase the region where we want to stabilize the system, it becomes unstable under sliding mode controller based on approximate feedback linearization (Figures 6,7,8), while remaining stable when steered by the higher order sliding mode controller (14) (Figures 2,3,4). We see that for this particular example, the simulation results support our initial supposition about getting a bigger capture region while applying HOCSMC as opposed to AFBL based SMC. Using the above example as an inspiration we would like to explore the possibilities of showing semi-global results while stabilizing approximate feedback linearizable systems by means of HOCSMC.

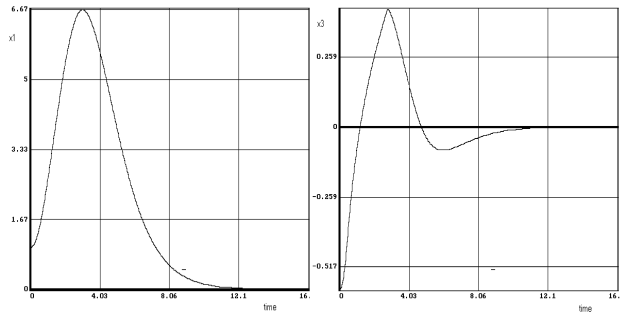

Fig. 2. Simulation results for system (11), under stabilizing sliding mode controller (14), with initial conditions $x_1(0) = 1,\ x_3(0) = -0.6$. The variables $x1$ and $x3$ are position of the ball and angular position of the beam respectively.

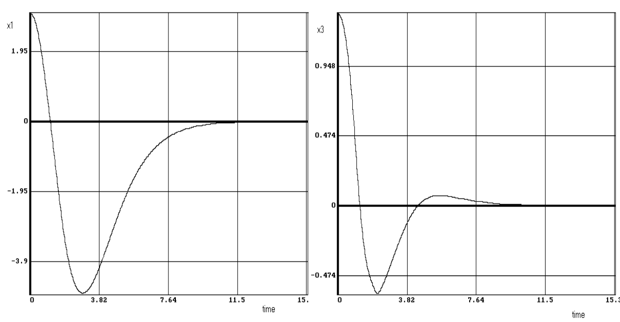

Fig. 3. Simulation results for system (11), under stabilizing sliding mode control using (14), with initial conditions $x_1(0) = 3,\ x_3(0) = 1.3$. The variables $x1$ and $x3$ are position of the ball and angular position of the beam respectively.

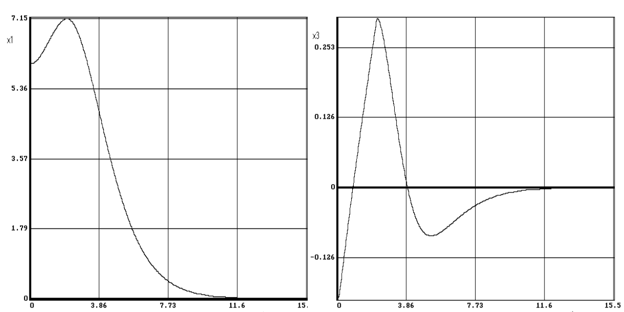

Fig. 4. Simulation results for system (11), under stabilizing sliding mode controller (14), with initial conditions $x_1(0) = 6,\ x_2(0) = -0.2$. The variables $x1$ and $x3$ are position of the ball and angular position of the beam respectively.

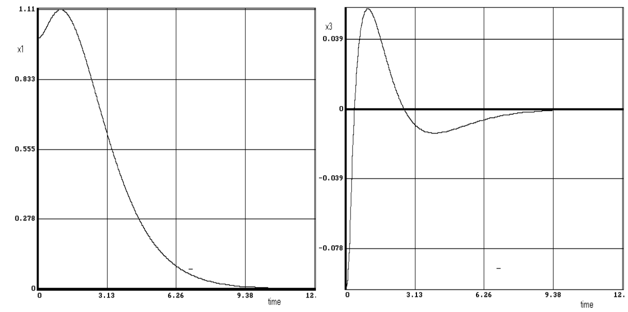

Fig. 5. Simulation results for system (11), under stabilizing sliding mode controller using approximate feedback linearization, with initial conditions $x_1(0) = 1,\ x_3(0) = -0.1$. The variables $x1$ and $x3$ are position of the ball and angular position of the beam respectively.

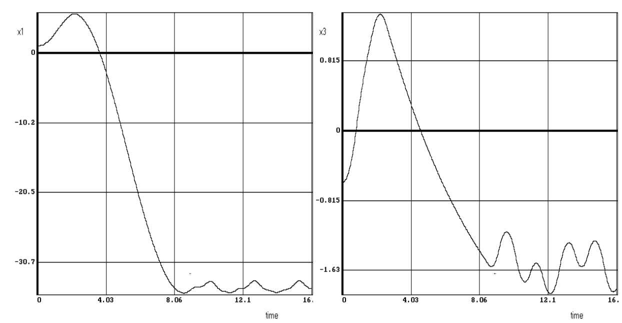

Fig. 6. Simulation results for system (11), under stabilizing sliding mode control using approximate feedback linearization, with initial conditions $x_1(0) = 1,\ x_3(0) = -0.6$. The variables $x1$ and $x3$ are position of the ball and angular position of the beam respectively.

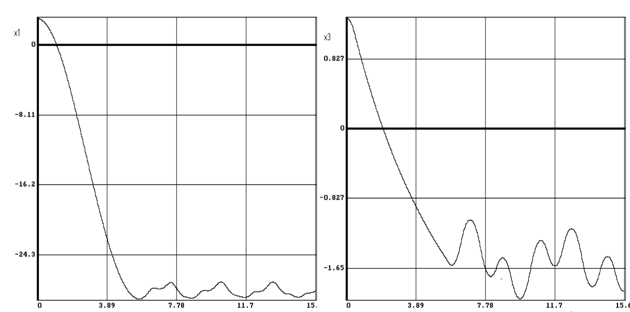

Fig. 7. Simulation results for system (11), under stabilizing sliding mode controller using approximate feedback linearization, with initial conditions $x_1(0) = 3,\ x_3(0) = 1.3$. The variables $x1$ and $x3$ are position of the ball and angular position of the beam respectively.

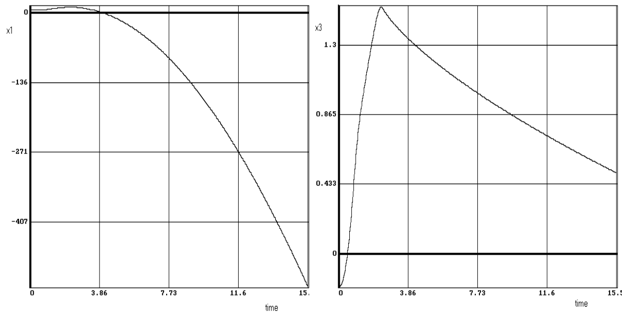

Fig. 8. Simulation results for system (11), under stabilizing sliding mode control using approximate feedback linearization, with initial conditions $x_1(0) = 6, \ x_3(0) = -0.2$. The variables $x1$ and $x3$ are position of the ball and angular position of the beam respectively.

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