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Output Regulation of Descriptor Systems with Periodic Coefficients

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***Abstract*— In this paper the output regulation problem for linear descriptor systems with periodic and almost periodic coefficients is considered. Necessary and sufficient conditions for the problem to be solvable are given in terms of the regulator equations. Simple examples are given to illustrate the theory.**

I. Introduction

The output regulation problem for a system with exogenous inputs is to find a stabilizing feedback controller such that the controlled output of the system converges to zero as time goes to infinity [26]. Problems of this type for linear time-invariant systems were studied by many authors [7], [8], [9], [12], [13] and [25]. The theory was later extended to nonlinear systems [17]. Descriptor systems appear in many real systems such as networks, electric circuits and robots see [20] and the references therein. The theory of output regulation was extended to linear descriptor systems in [20] and to nonlinear descriptor systems in [2]. We recall the main results of [20] under the setting of [26].

Consider the system

$$\begin{array}{rcl} E\dot{x} &=& Ax + B_1 w + B_2 u,\ x(0) = x_0, \\ z &=& C_1 x + D_{11} w + D_{12} u, \\ y &=& C_2 x + D_{21} w, \end{array} \tag{1}$$

where u is a control, z is the output to be regulated, y is the information available to the controller and w is the exogenous disturbance generated by an anti-Hurwitz exosystem

$$\dot{w} = Sw,\ w(0) = w_0. \tag{2}$$

The output regulation problem is to find an output feedback controller

$$\begin{array}{rcl} E_c\dot{\hat{x}} &=& A_c\hat{x} + B_c y, \\ u &=& C_c\hat{x} + D_c y \end{array} \tag{3}$$

such that the closed-loop system is asymptotically stable and that

$$\lim_{t\to\infty} z(t) = 0 \text{ for any } x_0 \text{ and } w_0. \tag{4}$$

Under the assumptions

D-1 $(E; A, B_2)$ is stabilizable,

D-2 S is anti-Hurwitz,

D-3 $\left(\begin{bmatrix} E & 0 \\ 0 & I \end{bmatrix};\ [C_2\ D_{21}],\ \begin{bmatrix} A & B_1 \\ 0 & S \end{bmatrix}\right)$ is detectable.

the following results are obtained.

Theorem 1.1: Suppose the assumptions D-1 to D-3 hold. Then the output regulation problem is solvable if and only if there exist two matrices P and Q which satisfy the regulator equations

$$\begin{array}{rcl} AP - EPS + B_1 + B_2Q &=& 0, \\ C_1P + D_{11} + D_{12}Q &=& 0. \end{array} \tag{5}$$

Under these conditions admissible controllers are given by

$$\begin{bmatrix} E & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} \dot{\hat{x}} \\ \dot{\hat{w}} \end{bmatrix} = \begin{bmatrix} A & B_1 \\ 0 & S \end{bmatrix}\begin{bmatrix} \hat{x} \\ \hat{w} \end{bmatrix} + \begin{bmatrix} B_2 \\ 0 \end{bmatrix}u + \begin{bmatrix} K_1 \\ K_2 \end{bmatrix}\left(y - [\,C_2 \quad D_{21}\,]\begin{bmatrix} \hat{x} \\ \hat{w} \end{bmatrix}\right), \tag{6}$$

$$u = [\,F \quad Q - FP\,]\begin{bmatrix} \hat{x} \\ \hat{w} \end{bmatrix}, \tag{7}$$

where F, K_1 and K_2 are any matrices such that $(E; A + B_2F)$ and

$$(\tilde{E}; \tilde{A}) = \left(\begin{bmatrix} E & 0 \\ 0 & I \end{bmatrix}; \begin{bmatrix} A - K_1C_2 & B_1 - K_1D_{21} \\ -K_2C_2 & S - K_2D_{21} \end{bmatrix}\right)$$

are exponentially stable.

In this paper we consider the output regulation problem for descriptor systems with almost periodic coefficients. Almost periodic functions [10] contain periodic functions and form an algebra and hence they appear in many practical situations. The regulator equations now involve a differential equation with almost periodic coefficients, and we give a necessary and sufficient condition for the problem to be solvable in terms of their almost periodic solution. We then consider the special case of periodic systems and refine the main results. Recently we have considered the output regulation problem for periodic (state space) systems [15] and this paper is an extension to descriptor systems.

To formulate the output regulation problem we need stabilizability, detetability, stabilizing feedback controllers and observers. Basic materials concerning these notions for descriptor systems can be found in [1], [3], [4], [11], [18], [21], [23], [24] and [27]. The H_∞ theory for descriptor systems is developed in [22]. Generalized algebraic Riccati equations for descriptor systems covering H_2 and H_∞ problems are studied in [19] and necessary and sufficient conditions for the existence of stabilizing solutions are given.

Stabilizing periodic feedback controllers can be obtained from quadratic problems and H_∞ problems. Quadratic control problems for general linear periodic systems are considered in [5], [6] and the Riccati theory is established. Stability by Liapunov equations, H_2 and H_∞ problems for periodic systems are discussed in [14]. To obtain periodic

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feedback controllers for descriptor systems extensions of these problems are necessary and this will be done elsewhere.

Section 1 is an introduction. In Section 2 we introduce almost periodic systems and the basic assumptions and give the main results. In Section 3 we consider the special case of periodic systems. In Section 4 we give simple examples to illustrate our theory. Section 5 is the conclusion. In Appendix we collect some preliminary results.

II. Almost periodic systems

In this section we consider the output regulation problem for descriptor systems with almost periodic coefficients. Recall that a continuous function defined on the real line R^1 is almost periodic [10] if for any sequence $a = (a_n)$ there exists a subsequence $a' = (a_{n'})$ such that $f(t + a_{n'})$ converges uniformly on R^1. For example $\sin t + \sin\sqrt{2}t$ is almost periodic. We denote by $f_{a'}(t)$ the limit function. Almost periodic functions are bounded and form an algebra. For ease of notation we write $f_a(t)$ for $f_{a'}(t)$ when no confusion arises. Then $f_a(t)$ is almost periodic. We say that a matrix function is almost periodic if its components are almost periodic.

Consider

$$\begin{array}{rcl} E\dot{x} &=& Ax + B_1 w + B_2 u,\ x(t_0) = x_0, \\ z &=& C_1 x + D_{11} w + D_{12} u, \\ y &=& C_2 x + D_{21} w, \end{array} \tag{8}$$

where $x \in \mathbf{R}^n$, $w \in \mathbf{R}^{m_1}$, $u \in \mathbf{R}^{m_2}$, $z \in \mathbf{R}^{p_1}$, $y \in \mathbf{R}^{p_2}$ and $E,\ A \in \mathbf{R}^{n\times n}$ are almost periodic with rank$E = r < n$ and all other matrices are almost periodic and appropriately defined. The controller (3) is the same

$$\begin{array}{rcl} E_c\dot{\hat{x}} &=& A_c\hat{x} + B_c y, \\ u &=& C_c\hat{x} + D_c y \end{array} \tag{9}$$

but all matrices are almost periodic. Concerning $(E; A)$ we assume:

AP-0 There exists constant nonsingular matrices M and N such that $MEN = \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix}$, where Σ is an $r \times r$ almost periodic nonsingular matrix with $r < n$.

Introducing $\bar{x} = N^{-1}x = [\,x_1 \quad x_2\,]'$ in the homogeneous equation $E\dot{x} = Ax$, we obtain

$$\begin{array}{rcl} \Sigma\dot{x}_1 &=& A_{11}x_1 + A_{12}x_2, \\ 0 &=& A_{21}x_1 + A_{22}x_2. \end{array} \tag{10}$$

As in the time-invariant case we introduce the following.

Definition 2.1: The system (8) is exponentially stable if the system (10) satisfies the following conditions:

(i) A_{22} is nonsingular.

(ii) $\Sigma^{-1}(A_{11} - A_{12}A_{22}^{-1}A_{21})$ is exponentially stable, i.e., the transition matrix associated with it is exponentially stable. In this case $(E; A)$ is said to be exponentially stable.

Definition 2.2: $(E; A, B_2)$ is ap-stabilizable if there exists an almost periodic matrix F such that $(E; A + B_2F)$ is exponentially stable.

Definition 2.3: $(E; C_2, A)$ is ap-detectable if there exists an almost periodic matrix H such that $(E; A - HC_2)$ is exponentially stable.

Let $W(t, t_0)$ be the transiton matrix associated with $S(t)$. Instead of D-1 to D-3 we assume the following:

AP-1 $(E; A, B_2)$ is ap-stabilizable.

AP-2 $\| W(t_0, t) \| \le c_1$ for any $t \ge t_0$.

AP-3 $\left(\begin{bmatrix} E & 0 \\ 0 & I \end{bmatrix};\ [C_2\ D_{21}],\ \begin{bmatrix} A & B_1 \\ 0 & S \end{bmatrix} \right)$ is ap-detectable.

The regulator equations (5) are relaced by

$$E\dot{P} = AP - EPS + B_1 + B_2Q, \tag{11}$$

$$\lim_{t\to\infty}[C_1P + D_{11} + D_{12}Q](t)W(t, t_0) = 0. \tag{12}$$

Our main result is the following.

Theorem 2.1: Suppose the assumptions AP-1 to AP-3 hold. Then the output regulation problem is solvable if and only if there exist almost periodic matrices P and Q which satisfy the regulator equations (11) and (12). Under these conditions admissible controllers are given by

$$\begin{bmatrix} E & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} \dot{\hat{x}} \\ \dot{\hat{w}} \end{bmatrix} = \begin{bmatrix} A & B_1 \\ 0 & S \end{bmatrix}\begin{bmatrix} \hat{x} \\ \hat{w} \end{bmatrix} + \begin{bmatrix} B_2 \\ 0 \end{bmatrix} u + \begin{bmatrix} K_1 \\ K_2 \end{bmatrix}\left(y - [\,C_2 \quad D_{21}\,]\begin{bmatrix} \hat{x} \\ \hat{w} \end{bmatrix} \right), \tag{13}$$

$$u = [\,F \quad Q - FP\,]\begin{bmatrix} \hat{x} \\ \hat{w} \end{bmatrix}, \tag{14}$$

where F, K_1 and K_2 are any matrices such that $(E; A + B_2F)$ and

$$(\tilde{E}; \tilde{A}) = \left(\begin{bmatrix} E & 0 \\ 0 & I \end{bmatrix}; \begin{bmatrix} A - K_1C_2 & B_1 - K_1D_{21} \\ -K_2C_2 & S - K_2D_{21} \end{bmatrix} \right)$$

are uniformly exponentially stable. Moreover, (12) implies

$$\lim_{t\to\infty}[C_1P + D_{11} + D_{12}Q](t) = 0. \tag{15}$$

The condition (12) can be replaced by (15) if the following condition holds:

AP-4 $\| W(t, t_0) \| \le c_2$ for any $t \ge t_0$.

Corollary 2.1: Suppose the assumptions AP-1 to AP-4 hold. Then the output regulation problem is solvable by an almost periodic controller if and only if there exist almost periodic matrices P and Q which satisfy the regulator equations (11) and (15).

Proof. It is enough to note that AP-4 and (15) imply (12).

We prepare two lemmas for the proof of Theorem 2.1.

Lemma 2.1: Suppose $(E; A)$ is exponentially stable. Then for each almost periodic matrix R there is a unique almost periodic solution of the matrix equation

$$E\dot{P} = AP - EPS + R. \tag{16}$$

Proof. Without loss of generality we consider (16) in the coordinate system of $\bar{x}$ given by

$$\begin{array}{rcl} \Sigma\dot{P}_1 &=& A_{11}P_1 + A_{12}P_2 - \Sigma P_1S + R_1, \\ 0 &=& A_{21}P_1 + A_{22}P_2 + R_2, \end{array} \tag{17}$$

where $N^{-1}P = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}$ and $R = \begin{bmatrix} R_1 \\ R_2 \end{bmatrix}$. Since $(E;A)$ is exponentially stable, A_{22} is nonsingular and we obtain

$$\Sigma \dot{P}_1 = (A_{11} - A_{12}A_{22}^{-1}A_{21})P_1 - \Sigma P_1 S + R_1 - A_{12}A_{22}^{-1}R_2. \tag{18}$$

This equation has a unique almost periodic solution given by

$$P_1(t) = \int_{-\infty}^{t} U_1(t,r)(R_1 - A_{12}A_{22}^{-1}R_2)(r)W(r,t)dr, \tag{19}$$

where $U_1(t,s)$ is the transition matrix generated by $\Sigma^{-1}(A_{11} - A_{12}A_{22}^{-1}A_{21})$. Note that $\| U_1(t,s) \| \leq M_0 \exp[-\alpha(t-s)],\ t \geq s$ for some $M_0 \geq 1$ and $\alpha > 0$. Note also that $W(r,t),\ r < t$ is bounded by AP-2. Thus the integral is well-defined. By Lemma 1.2 $P_1(t)$ is almost periodic. Since $P_2 = -A_{22}^{-1}(A_{21}P_1 + R_2)$ is almost periodic, the equation (17) has a unique almost periodic solution.

Now we consider the output regulation problem with full information.

Lemma 2.2: Assume AP-1 and AP-2 and consider the case where $y = [x' \ w']'$. Then the output regulation problem is solvable by an almost periodic controller if and only if there exist almost periodic matrices P and Q which satisfy the regulator equation (11) and (12). Under these conditions admissible controllers are given by

$$u = Fx + (Q - FP)w,$$

where F is any almost periodic matrix such that $(E; A + B_2F)$ is exponentially stable.

Proof. Suppose the control $u = Fx + Gw$ is admissible so that it is stabilizing and (4) holds. Then by Lemma 2.1 there exists an almost periodic solution of

$$E\dot{P} = (A + B_2F)P - EPS + B_1 + B_2G. \tag{20}$$

Define $\tilde{x} = x - Pw$. Then

$$E\dot{\tilde{x}} = (A + B_2F)\tilde{x}$$

and $\tilde{x}(t) \to 0$ as $t \to \infty$. Since

$$z = (C_1 + D_{12}F)\tilde{x} + [C_1P + D_{11} + D_{12}(FP + G)]w \to 0$$

as $t \to \infty$ for any w_0, we have

$$\lim_{t\to\infty} [C_1P + D_{11} + D_{12}(FP + G)](t)W(t,t_0) = 0.$$

Now setting $Q = FP + G$ we have an almost periodic solution to (11) and (12). Now

$$\begin{aligned} & \| [C_1P + D_{11} + D_{12}(FP + G)](t) \| \\ = \ & \| [C_1P + D_{11} + D_{12}(FP + G)](t) \\ & \qquad \times W(t,t_0)W(t_0,t) \| \\ \leq \ & c_1 \| [C_1P + D_{11} + D_{12}(FP + G)](t)W(t,t_0) \|, \end{aligned}$$

by AP-2, and the condition (12) implies (15).

To show sufficiency, let P and Q be the solution of the regulator equations (11) and (12), and consider the stabilizing controller

$$u = Fx + (Q - FP)w.$$

Then P satisfies (12) with $G = Q - FP$ and as in the necessity part we obtain

$$\begin{aligned} z &= (C_1 + D_{12}F)\tilde{x} + [C_1P + D_{11} + D_{12}(FP + G)]w \\ &= (C_1 + D_{12}F)\tilde{x} + (C_1P + D_{11} + D_{12}Q)w \\ &= (C_1 + D_{12}F)\tilde{x} + (C_1P + D_{11} + D_{12}Q)W(t,t_0)w_0, \end{aligned}$$

where $\tilde{x}$ is defined as in the necessity part. Thus the assertion readily follows.

We are now ready to prove Theorem 2.1.

Proof of Theorem 2.1. Suppose that the almost periodic controller (9) solves the output regulation problem. Then the closed-loop system is given by (17) and the output regulation problem for the augmented system above has a feedback solution

$$u = \tilde{F}\begin{bmatrix} x \\ \hat{x} \end{bmatrix} + \tilde{G}w = [\, D_cC_2 \quad C_c \,] \begin{bmatrix} x \\ \hat{x} \end{bmatrix} + D_cD_{21}w.$$

Applying the necessity part of Lemma 2.2, we obtain almost periodic matrices Π and Θ which satisfy the following equations

$$\begin{aligned} \begin{bmatrix} E & 0 \\ 0 & E_c \end{bmatrix} \begin{bmatrix} \dot{\Pi} \\ \dot{\Theta} \end{bmatrix} &= \begin{bmatrix} A + B_2D_cC_2 & B_2C_c \\ B_cC_2 & A_c \end{bmatrix} \begin{bmatrix} \Pi \\ \Theta \end{bmatrix} \\ & \quad - \begin{bmatrix} E & 0 \\ 0 & E_c \end{bmatrix} \begin{bmatrix} \Pi \\ \Theta \end{bmatrix} S + \begin{bmatrix} B_1 + B_2D_cD_{21} \\ B_cD_{21} \end{bmatrix} \\ \lim\nolimits_{t\to\infty}([\, C_1 \quad 0 \,] & \begin{bmatrix} \Pi \\ \Theta \end{bmatrix} + D_{11} \\ & + D_{12}(\tilde{F}\begin{bmatrix} \Pi \\ \Theta \end{bmatrix} + \tilde{G}))W(t,t_0) = 0. \end{aligned}$$

From this we obtain

$$\begin{aligned} & E\dot{\Pi} = A\Pi - E\Pi S + B_1 \\ & \qquad\qquad + B_2(D_cC_2\Pi + C_c\Theta + D_cD_{21}), \\ & \lim\nolimits_{t\to\infty}[C_1\Pi + D_{11} \\ & \qquad + D_{12}(D_cC_2\Pi + C_c\Theta + D_cD_{21})]W(t,t_0) = 0. \end{aligned}$$

Setting $P = \Pi$ and $Q = D_cC_2\Pi + C_c\Theta + D_cD_{21}$, we obtain a solution to the equation (11) and (12).

To show sufficiency, let P and Q be the solution of the regulator equations. Then by Lemma 2.2 a full information controller is given by

$$u = Fx + (Q - FP)w,$$

where F is stabilizing. The controller (13) and (14) is a realization of this controller via an observer and is stabilizing. In fact the estimation errors $e_1 = x - \hat{x}$ and $e_2 = w - \hat{w}$ satisfy the following equation

$$\begin{bmatrix} E & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \dot{e_1} \\ \dot{e_2} \end{bmatrix} = \begin{bmatrix} A - K_1C_2 & B_1 - K_1D_{21} \\ -K_2C_2 & S - K_2D_{21} \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}.$$

Hence by assumption $e_i(t) \to 0$ as $t \to \infty$. Now

$$\begin{aligned} E\dot{x} &= Ax + B_1w + B_2\,[\, F \quad Q - FP \,] \begin{bmatrix} x \\ w \end{bmatrix} \\ & \qquad - B_2\,[\, F \quad Q - FP \,] \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}, \\ z &= C_1x + D_{11}w + D_{12}\,[\, F \quad Q - FP \,] \begin{bmatrix} x \\ w \end{bmatrix} \\ & \qquad - D_{12}\,[\, F \quad Q - FP \,] \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}. \end{aligned}$$

Since the controller

$$u = Fx + (Q - FP)w$$

fulfills output regulation, so does the controller (13) and (14).

III. Periodic systems

Consider

$$\begin{array}{rcl} E\dot{x} &=& Ax + B_1 w + B_2 u, \quad x(t_0) = x_0, \\ z &=& C_1 x + D_{11} w + D_{12} u, \\ y &=& C_2 x + D_{21} w, \end{array} \tag{21}$$

where all matrices are continuous functions of time and T-periodic. We assume that AP-0 is satisfied and Σ in (10) is T-periodic.

Without loss of generality we assume $t_0 = 0$. We also assume that S is continuous and T-periodic. Now we modify assumptions AP-1 to AP-3.

P-1 $(E; A, B_2)$ is p-stabilizable.

P-2 $\| W(0,T) \| \leq 1$.

P-3 $\left(\begin{bmatrix} E & 0 \\ 0 & I \end{bmatrix}; \ [C_2 \ D_{21}], \ \begin{bmatrix} A & B_1 \\ 0 & S \end{bmatrix} \right)$ is p-detectable.

Here p-stabilizability of $(E; A, B_2)$ is defined by continuous T-periodic feedback and p-detectability is defiend in a similar manner.

The regulator equations (11) and (12) are relaced by

$$E\dot{P} = AP - EPS + B_1 + B_2 Q, \tag{22}$$

$$C_1 P + D_{11} + D_{12} Q = 0. \tag{23}$$

Our main result is the following.

Theorem 3.1: Suppose the assumptions P-1 to P-3 hold. Then the output regulation problem is solvable by a T-periodic controller if and only if there exist continuous T-periodic matrices P and Q which satisfy the regulator equations (22) and (23). Under these conditions admissible controllers are given by (13) and (14), where F and K_i are T-periodic and continuous matrices such that $(E; A + B_2 F)$ and $(\tilde{E}; \tilde{A})$ are exponentially stable.

To prove Theorem 3.1 we modify the proof of Theorem 2.1. We need to show the T-periodicity of $P_1(t)$ of Lemma 2.1. This can be shown as follows.

$$\begin{array}{rl} & P_1(t+T) \\ = & \int_{-\infty}^{t+T} U_1(t+T, r)(R_1 - A_{12}A_{22}^{-1}R_2)(r) \\ & \qquad \times W(r, t+T)dr, \\ = & \int_{-\infty}^{t} U_1(t+T, \sigma+T)(R_1 - A_{12}A_{22}^{-1}R_2)(\sigma+T) \\ & \qquad \times W(\sigma+T, t+T)d\sigma, \\ = & \int_{-\infty}^{t} U_1(t, \sigma)(R_1 - A_{12}A_{22}^{-1}R_2)(\sigma)W(\sigma, t)d\sigma \\ = & P_1(t), \end{array}$$

where we have used the property $U_1(t+T, s+T) = U_1(t, s)$. Next we shall derive (23) from (12)

$$\lim_{t\to\infty} [C_1 P + D_{11} + D_{12}(FP + G)](t)W(t, 0) = 0.$$

Choose $t = \tau + kT$, $0 \leq \tau \leq T$. Then $W(0,t) = W(0, \tau + kT) = W(0,T)^k W(0,\tau)$. Now

$$\begin{array}{rl} & \| [C_1P + D_{11} + D_{12}(FP+G)](\tau) \| \\ = & \| [C_1P + D_{11} + D_{12}(FP+G)](t) \| \\ = & \| [C_1P + D_{11} + D_{12}(FP+G)](t) \\ & \qquad W(t,0)W(0,t) \| \\ = & \| [C_1P + D_{11} + D_{12}(FP+G)](t) \\ & \qquad \times W(t,0)W(0,T)^k W(0,\tau) \| \\ \leq & \| [C_1P + D_{11} + D_{12}(FP+G)](t)W(t,0) \| \\ & \qquad \times \| W(0,\tau) \| \to 0, \end{array}$$

where we have used the assumption P-2. Hence$[C_1P + D_{11} + D_{12}(FP+G)](t) = 0$ since it is T-periodic. Now setting $Q = FP + G$ we have a T-periodic solution to (22) and (23).

IV. Examples

In this section we give simple examples to illustrate our theory.

Example 4.1: Consider the system

$$\begin{array}{rcl} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \dot{x} &=& \begin{bmatrix} -1 & \sin \pi t \\ 1 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \\ z &=& [0 \quad 1]\, x - [1 \quad 0]\, w, \\ y &=& \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} w \end{array}$$

and the exosystem

$$\dot{w} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} w, \ w(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

This corresponds to the tracking problem which requires $x_2(t) - \sin t \to 0$ as $t \to \infty$. The regulator equation (10) and (11) has a solution

$$P(t) = \begin{bmatrix} P_1(t) & P_2(t) \\ 1 & 0 \end{bmatrix}, \ Q(t) = [\, -(1 + P_1(t)) \quad -P_2(t) \,]$$

where $P_1(t)$ and $P_2(t)$ is a solution of

$$\begin{bmatrix} \dot{P}_1 \\ \dot{P}_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} (t) + \begin{bmatrix} \sin \pi t \\ 0 \end{bmatrix}.$$

If we set $P_1(0) = -0.3058$ and $P_2(0) = 0.0620$, then we obtain a periodic solution $P_1(t)$ and $P_2(t)$ of the above differential equation (Figure 1). The feedback control $u = [\,0 \quad 1\,]\, x$ stabilizes the system and yields an eigenvalue $-(1 + 0.5 \sin \pi t)$ while the observer gain

$$K = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix}'$$

makes the three observer poles -1. The simulation result of the system with the designed controller for $x(0) = 0$ is given in Figure 2.

Example 4.2: Consider the system

$$\begin{array}{rcl} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \dot{x} &=& \begin{bmatrix} 0 & 1 \\ \sin 2\pi t & 2 + \sin t \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \\ z &=& [1 \quad 0]\, x - [1 \quad 0]\, w, \\ y &=& \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} w \end{array}$$

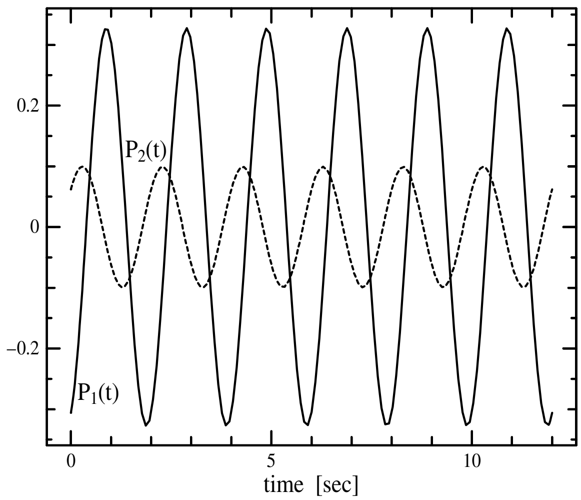

Fig. 1. $P_1(t)$ and $P_2(t)$

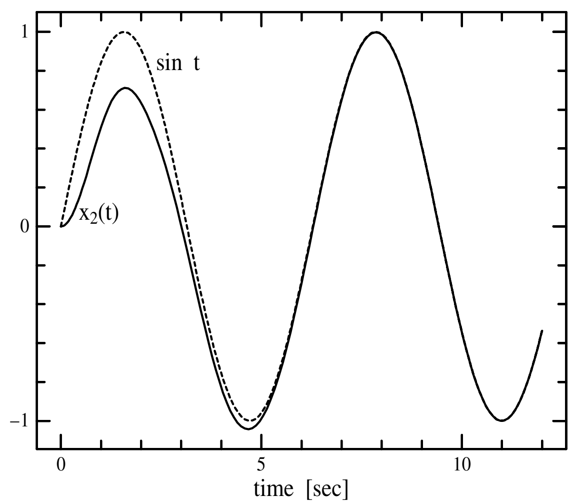

Fig. 2. Sine tracking

and consider the same tracking problem. The regulator equation (22) and (23) has a unique solution

$$P = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \; Q = [\, -\sin 2\pi t \quad 2 + \sin t \,].$$

The feedback control $u = [\, 1 - \sin 2\pi t \quad -1 - \sin t \,]\, x$ stabilizes the system and yields an eigenvalue -1 and the observer gain

$$K = \begin{bmatrix} 1 & \sin 2\pi t & 0 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix}'$$

keeps the three observer poles as -1.

V. Conclusion

In this paper the output regulation problem for linear descriptor systems with almost periodic and periodic coefficients are considered and necessary and sufficient conditions for its solvability are given in terms of the regulator equations. For illustration simple examples are given and the regulator equations are explicitly solved.

Appendix

Here we collect some basic results needed for the proof of Lemma 2.1.

Lemma 1.1: Let $A(t)$ be an almost periodic matrix and let $U(t,s)$ be the corresponding transition matrix. For any sequence $a = (a_n)$ there exists a subsequence denoted again by a such that $U(t+a_n, s+a_n)$ converges uniformly in (t,s) over each bounded set in R^2.

Proof. Let $A_a(t)$ be the uniform limit of $A(t+a_n)$ (along a subsequence) and let $U_a(t,s)$ be the corresponding fundamental matrix. Then for any $s \le t$

$$\begin{aligned} U(t,s) &= I + \textstyle\int_s^t A(r)U(r,s)dr, \\ U_a(t,s) &= I + \textstyle\int_s^t A_a(r)U_a(r,s)dr. \end{aligned}$$

Let $d_1 = \sup_t \parallel A(t) \parallel$. Then $\parallel U(t,s) \parallel, \quad \parallel U_a(t,s) \parallel \le \exp[d_1(t-s)]$. Now

$$\begin{aligned} & U(t+a_n, s+a_n) \\ =\ & I + \textstyle\int_{s+a_n}^{t+a_n} A(r)U(r, s+a_n)dr \\ =\ & I + \textstyle\int_s^t A(\sigma+a_n)U(\sigma+a_n, s+a_n)d\sigma. \end{aligned}$$

Hence

$$\begin{aligned} & U(t+a_n, s+a_n) - U_a(t,s) \\ =\ & \textstyle\int_s^t A(\sigma+a_n)[U(\sigma+a_n, s+a_n) - U_a(\sigma,s)]d\sigma \\ & \quad + \textstyle\int_s^t [A(\sigma+a_n) - A_a(\sigma)]U_a(\sigma,s)d\sigma. \end{aligned}$$

From this we obtain

$$\begin{aligned} & \parallel U(t+a_n, s+a_n) - U_a(t,s) \parallel \\ \le\ & d_1 \textstyle\int_s^t \parallel U(\sigma+a_n, s+a_n) - U_a(\sigma,s) \parallel d\sigma \\ & \quad + d_2 \sup_{s \le \sigma \le t} \parallel A(\sigma+a_n) - A_a(\sigma) \parallel, \end{aligned}$$

where $d_2 = \int_s^t \parallel U_a(\sigma,s)d\sigma \parallel \le \frac{1}{d_1} \exp d_1(t-s)$. By Gronwall's inequality

$$\begin{aligned} & \parallel U(t+a_n, s+a_n) - U_a(t,s) \parallel \\ \le\ & d_2 \sup_{s \le \sigma \le t} \parallel A(\sigma+a_n) - A_a(\sigma) \parallel \exp d_1(t-s) \\ \le\ & \sup_\sigma \parallel A(\sigma+a_n) - A_a(\sigma) \parallel \tfrac{1}{d_1} \exp 2d_1(t-s). \end{aligned}$$

Hence for each fixed $T > 0$, $U(t+a_n, s+a_n)$ converges to $U_a(t,s)$ uniformly for $0 \le t-s \le T$.

Corollary 1.1: Suppose $U(t,s)$ has a bound

$$\parallel U(t,s) \parallel \le M \exp[\alpha(t-s)]$$

$M \ge 1$, α real for any $t \ge s$. Then $U_a(t,s)$ has the same bound.

Lemma 1.2: Suppose A, R and S are almost periodic, $U(t,r)$ is unformly exponentially stable i.e., $\parallel U(t,s) \parallel \le$

$M\exp[-\alpha(t-s)],\ M\geq 1,\ \alpha>0$ for any $t\geq s$ and $W(t_0,t)$ is bounded for any $t\geq t_0$. Then

$$P(t)=\int_{-\infty}^{t}U(t,r)R(r)W(r,t)dr. \qquad (24)$$

is almost periodic.

Proof. Notice that the integral is well-defined and bounded i.e., $\| P(t)\|\leq d_3$, since $R(t)$ and $W(r,t)$ for $r\leq t$ are bounded. To show that $P(t)$ is almost periodic, let $a=(a_n)$ be an arbitrary sequence. Then there is a subsequence denoted again by a such that $A(t+a_n)$, $R(t+a_n)$ and $S(t+a_n)$ converge uniformly to $A_a(t)$, $R_a(t)$ and $S_a(t)$ respectively. Now as (24) define

$$P_a(t)=\int_{-\infty}^{t}U_a(t,r)R_a(r)W_a(r,t)dr.$$

Then $\| P_a(t)\|\leq d_3$. We shall show that $P(t+a_n)\rightarrow P_a(t)$ uniformly on R^1. Consider

$$\begin{aligned} & P(t+a_n) \\ = & \int_{t-T+a_n}^{t+a_n}U(t+a_n,r)R(r)W(r,t+a_n)dr \\ & \quad +\int_{-\infty}^{t-T+a_n}U(t+a_n,r)R(r)W(r,t+a_n)dr \\ = & P_1(t+a_n)+P_2(t+a_n). \end{aligned}$$

and

$$\begin{aligned} P_a(t) = & \int_{t-T}^{t}U_a(t,r)R_a(r)W_a(r,t)dr \\ & +\int_{-\infty}^{t-T}U_a(t,r)R_a(r)W_a(r,t)dr \\ = & P_{a1}(t)+P_{a2}(t). \end{aligned}$$

$$\begin{aligned} & \| P_2(t+a_n)\| \\ = & \| \int_{-\infty}^{t-T+a_n}U(t+a_n,t-T+a_n)U(t-T+a_n,r) \\ & \qquad \times R(r)W(r,t+a_n)dr\| \\ \leq & M\exp(-\alpha T)\| \int_{-\infty}^{t-T+a_n}U(t-T+a_n,r) \\ & \qquad \times R(r)W(r,t+a_n)dr\| \\ \leq & d_3M\exp(-\alpha T). \end{aligned}$$

$P_{a2}(t)$ has the same upper bound. Now

$$\begin{aligned} P_1(t+a_n) &= \int_{t-T+a_n}^{t+a_n}U(t+a_n,r)R(r)W(r,t+a_n)dr \\ &= \int_{t-T}^{t}U(t+a_n,\sigma+a_n)R(\sigma+a_n) \\ & \qquad \times W(\sigma+a_n,t+a_n)dr. \end{aligned}$$

By Lemma 1.1 $P_1(t+a_n)$ converges to $P_{a1}(t)$ uniformly for each fixed T since each term in the integrand of $P_1(t+a_n)$ converges uniformly to the corresponding term of $P_{a1}(t)$. Taking T large and then n large we conclude that $P(t+a_n)\rightarrow P_a(t)$ uniformly on R^1. To show uniqueness set $R(t)=0$. Since $P(t)=U(t,0)P(0)W(0,t)$,

$$P(0)=U(0,t+a_n)P(t+a_n)W(t+a_n,0).$$

Letting $a_n\rightarrow-\infty$ we obtain $P(0)=0$.

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