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Formulation of a Hamiltonian Cauchy Problem for Solving Optimal Feedback Control Problems

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Abstract— **We propose a novel approach for solving the optimal feedback control problem. Following our previous research, we formulate the problem as a Hamiltonian system by using the necessary conditions for optimality, and treat the resultant phase flow as a canonical transformation. Then starting from the Hamilton-Jacobi equation for generating functions we derive a set of 1st order quasilinear partial differential equations with the appropriate initial or terminal conditions, which forms the well-known Cauchy problem. These equations can also be derived by applying the invariant imbedding technique to the two point boundary value problem. The solution to this Cauchy problem is utilized for solving the Hamiltonian two point boundary value problem as well as the optimal feedback control problem with hard and soft constraint boundary conditions. As suggested by the illustrative examples given, this method is promising for solving problems with control constraints, non-smooth control logic, and non-analytic cost function.**

Key Words. Optimal Feedback Control, Hamiltonian System, Generating Function, Hamilton-Jacobi Equation, Cauchy Problem

I. Introduction

Since the mid 1950s, Pontryagin's minimum principle and Bellman's dynamic programming have been two main branches of modern deterministic optimal control theory. Between these two, in general, to find the optimal feedback control for a given system requires that one should resort to Bellman's dynamic programming and solve the Hamilton-Jacobi-Bellman equation (HJBE), which is still a very active field of research. However, since it is extremely difficult to find a solution to the HJBE, there have been a myriad of other creative approaches to finding optimal feedback control laws. Many representatives can be found in the vast amount of literature on this topic; various manipulations of 1st order necessary conditions for optimality [1][2], employment of state dependent Ricatti equations [3], iterative methods based on the generalized HJBE [4][5], derivation of a new governing equation by a special transformation [6], etc. However, there has not been a noticeably superior candidate among these diverse techniques. Furthermore, many of these methods only apply to one specific type of boundary conditions or system.

Recently we have studied optimal feedback control problems in the context of Hamiltonian systems. Motivated by Guibout and Scheeres' work [7], we treated the Hamiltonian system derived from the necessary conditions for optimality as a canonical transformation and use the generating functions to solve the optimal feedback control problem. Circumventing the final time singularity by the Legendre transformation, we obtained the optimal solution in feedback form for the hard constraint problem, a difficult and general type of boundary condition that has rarely been treated in feedback sense [8]. Adapting this method, we derived an optimal control strategy for the nonlinear optimal rendezvous problem in a central gravity field [9]. Later we found that a specific kind of generating function contains the general information needed to evaluate the optimal cost function. This recognition provided us with an advantage for problems where both hard and soft constraint boundary conditions are of interest, as a single generating function is enough to treat them together and the difficult Hamilton-Jacobi equation (HJE) need not be solved repetitively [10]. However, despite these advantages, our method has a restrictive applicability to problems with control constraints due to the difficulty of treating the inherent switching structure of the control which is not known a priori, non-smoothness of the optimal control strategy and cost function, and the possible existence of singular control regimes [11].

In an attempt to overcome or mitigate these barriers, we present a new technique stemming from the generating function method. With a new set of governing equations derived from the HJE along with the appropriate boundary conditions, we show how to use them to solve the optimal feedback control problem. Unlike our previous approaches based on generating functions, this method is well suited for problems with control constraints, and thus of non-smoothness in the control scheme and cost function in general.

This document is structured as follows. In section II, we formulate the optimal control problem as a Hamiltonian system and review the properties of generating functions as a solution tool. In section III, from the HJE for the generating functions we derive a set of new governing equations with the appropriate boundary conditions, which forms a Hamiltonian Cauchy problem. We discuss how to solve the Cauchy problem to obtain the optimal feedback control. Then in section IV, we apply our method to illustrative examples including problems with control constraints. Concluding remarks are given in section V.

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II. Optimal Control Problem formulated as a Hamiltonian System

We consider minimization of the following performance index

$$J = \phi(x(t_f), t_f) + \int_t^{t_f} L(x(\tau), u(\tau), \tau) d\tau$$

subject to the following system with terminal boundary conditions

$$\dot{x} = F(x, u, t) \quad , \quad \psi(x(t_f), t_f) = 0 \tag{1}$$

Here $x \in \mathbf{R}^n$, $u \in \mathbf{R}^m$, $t \in \mathbf{R}$, $\phi(x(t_f), t_f) : \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}$, $L(x(\tau), u(\tau), \tau) : \mathbf{R}^n \times \mathbf{R}^m \times \mathbf{R} \to \mathbf{R}$, $F(x(t), u(t), t) : \mathbf{R}^n \times \mathbf{R}^m \times \mathbf{R} \to \mathbf{R}^n$, and $\psi(x(t_f), t_f) : \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}^{p \le n}$. The control $u = [u_1 \; u_2 \; \cdots \; u_m]^T$ is bounded by the following inequality by component:

$$|u_i| \le u_{i0} = \text{constant}$$

The unconstrained problem can be dealt with by letting $u_{i0} \to \infty, \; i = 1, 2, \cdots, m$.

Given this problem statement, our objective is to find the optimal *feedback* control law for a given domain considered in $(x, t) \in \mathbf{R}^n \times \mathbf{R}$. Then from any initial point, we can evaluate the optimal trajectory satisfying the terminal constraints by simple forward integration of the system (1), updating the control as new state measurements are made.

Instead of resorting to dynamic programming and solving the Hamilton-Jacobi-Bellman equation (HJBE), we formulate the given problem as a Hamiltonian system. First define the pre-Hamiltonian $\bar{H}$ as

$$\bar{H}(x, \lambda, u, t) = L(x, u, t) + \lambda^T F(x, u, t) \tag{2}$$

where λ represents the costates. Then, Pontryagin's principle provides the necessary conditions for optimality and defines a Hamiltonian system for states and costates only [10][12]:

$$\begin{aligned} H(x, \lambda, t) &= \bar{H}(x, \lambda, u^*(x, \lambda, t), t) && (3) \\ \dot{x} &= H_\lambda(x, \lambda, t) && (4) \\ \dot{\lambda} &= -H_x(x, \lambda, t) && (5) \\ u^*(x, \lambda, t) &= arg \min_{\bar{u}} \bar{H}(x, \lambda, \bar{u}, t) && (6) \end{aligned}$$

As is noted in the problem definition, the initial states are chosen explicitly on a given domain. For the terminal condition, suppose we have an explicit condition for ψ:

$$\psi(x(t_f), t_f) = x(t_f) - x_f = 0 \tag{7}$$

where $x_f \in \mathbf{R}^n =$ constant. Then the terminal states are completely specified, which forms the hard constraint problem. Otherwise if $\psi(x(t_f), t_f) = 0$ is given by an implicit equation or does not exist, then the following transversality condition determines the n terminal boundary conditions [12, section 2]:

$$\lambda(t_f) = \frac{\partial[\phi(x(t_f), t_f) + \nu^T \psi(x(t_f), t_f)]}{\partial x(t_f)} \tag{8}$$

where ν is a Lagrange multiplier vector adjoint to ψ. In the sense that the terminal states are not directly specified but indirectly affected by ϕ and ψ, we call this type of boundary condition the soft constraint problem[1]. In either case we have $2n$ split boundary conditions equally divided between the initial and terminal time. Therefore, the optimal control problem is reduced to a two point boundary value problem (TPBVP).

There exists diverse numerical techniques for solving this TPBVP, which usually yield the open loop optimal trajectory. However, this does not fit into our purpose of obtaining a feedback control scheme on a given domain. Instead, we view the Hamiltonian phase flow $(x(t), \lambda(t))$ as a transformation between terminal coordinates (x, λ, t) and initial coordinates (x_0, λ_0, t_0), which is by definition a canonical transformation[2]. Then there exist generating functions for these transformations that can have one of the four classical forms:

$$F_1(x, x_0, t, t_0), \quad F_2(x, \lambda_0, t, t_0) \quad F_3(\lambda, x_0, t, t_0), \quad F_4(\lambda, \lambda_0, t, t_0)$$

Note that these generating functions are functions of n initial coordinates and n terminal coordinates. By definition they satisfy the given boundary value problem and provide relations between initial and terminal states and costates by the following relations [13]:

$$\begin{aligned} \lambda &= \frac{\partial F_1(x, x_0, t, t_0)}{\partial x} && (9) \\ \lambda_0 &= -\frac{\partial F_1(x, x_0, t, t_0)}{\partial x_0} && (10) \\ 0 &= H(x, \lambda, t) + \frac{\partial F_1(x, x_0, t, t_0)}{\partial t} && (11) \\ \lambda &= \frac{\partial F_2(x, \lambda_0, t, t_0)}{\partial x} && (12) \\ x_0 &= \frac{\partial F_2(x, \lambda_0, t, t_0)}{\partial \lambda_0} && (13) \\ 0 &= H(x, \lambda, t) + \frac{\partial F_2(x, \lambda_0, t, t_0)}{\partial t} && (14) \\ x &= -\frac{\partial F_3(\lambda, x_0, t, t_0)}{\partial \lambda} && (15) \\ \lambda_0 &= -\frac{\partial F_3(\lambda, x_0, t, t_0)}{\partial x_0} && (16) \\ 0 &= H(x, \lambda, t) + \frac{\partial F_3(\lambda, x_0, t, t_0)}{\partial t} && (17) \\ x &= \frac{\partial F_4(\lambda, \lambda_0, t, t_0)}{\partial \lambda} && (18) \\ x_0 &= -\frac{\partial F_4(\lambda, \lambda_0, t, t_0)}{\partial \lambda_0} && (19) \\ 0 &= H(x, \lambda, t) + \frac{\partial F_4(\lambda, \lambda_0, t, t_0)}{\partial t}. && (20) \end{aligned}$$

As can be seen, the generating functions satisfy a partial differential equation found by substituting for λ in (11) and (14), and for x in (17) and (20), which are usually referred to as the Hamilton-Jacobi (HJ) equation.

A crucial property of the generating functions related to a given transformation is that they are linked to each other

[1] Among all possible soft constraint formulations, it is still very difficult to determine the optimal *feedback* control for problems with non-trivial ψ due to the additional Lagrange multiplier ν. Therefore in this paper, we consider problems where ψ vanishes, thus $\nu = 0$.

[2] Refer to Greenwood [13], Goldstein [14], and Guibout and Scheeres [7] for a review of canonical transformations and generating functions.

via Legendre transformations, which can be represented by the following identities:

$$F_2(x, \lambda_0, t, t_0) = F_1(x, x_0, t, t_0) + \lambda_0^T x_0 \quad (21)$$

$$F_3(\lambda, x_0, t, t_0) = F_1(x, x_0, t, t_0) - \lambda^T x \quad (22)$$

$$F_4(\lambda, \lambda_0, t, t_0) = F_2(x, \lambda_0, t, t_0) - \lambda^T x \quad (23)$$

Among these generating functions, F_1 is a special quantity for the optimal control problem as it provides the optimal cost function by the following theorem:

Theorem 2.1 (Optimal Cost and Control Law from F_1): Let x_f be the (fixed) terminal state at t_f and x be the (moving) initial state at t. Also let $F_1(x_f, x, t_f, t)$ be a generating function for the given phase flow. Then, F_1 satisfies the necessary conditions of the TPBVP by definition. Also, the function

$$V(x,t) = -F_1(x_f, x, t_f, t) + \phi(x_f, t_f)$$

is the optimal cost function and satisfies the HJB equation and the sufficient conditions. Furthermore, the optimal feedback control can be expressed as

$$u = arg \min_{\bar{u}} \bar{H}\left(x, \frac{\partial V(x,t)}{\partial x}, \bar{u}, t\right)$$

Proof Refer to Park and Scheeres[10].

In our previous works [8][9][10], we have used generating functions and their Legendre transformations to develop a systematic methodology to solve a class of optimal control problems where the performance index and the system are analytic, and thus expandable as Taylor series. However, as is observed in [11], though we obtain a consistent result for problems with control constraints and singular optimal control problems, the applicability of our solution techniques is restrained by the unknown switching structure a priori, non-smoothness of cost function and control scheme, and possible singularities in the cost function. In an effort to overcome these difficulties, we derive a new set of equations from the HJE and employ their solution to obtain the optimal feedback control, which we detail in the next section.

III. Hamiltonian Cauchy Problem for Solving Optimal Feedback Control Problems

Derivation of Governing Equations

We start from the HJEs for generating functions. First consider the HJE for F_1 in (9) and (11). Regarding x_0 and t_0 as constants (which is consistent with the definition of F_1) and taking partial differentiation of (11) with respect to x, we have

$$\frac{\partial}{\partial x}\left(\frac{\partial F_1}{\partial t} + H\right) = 0$$

Here note that the Hamiltonian $H(x, \lambda, t) = H(x, \lambda(x, x_0, t, t_0), t)$ from (9). Using the chain rule for the Hamiltonian and the exactness property of F_1 yields

$$\frac{\partial^2 F_1}{\partial t \partial x} + H_x + H_\lambda \frac{\partial \lambda}{\partial x} = 0$$

Substituting $\lambda = \partial F_1/\partial x$ into the first term, we obtain a system of PDEs for the costate λ:

$$\frac{\partial \lambda}{\partial t} + \frac{\partial \lambda}{\partial x} H_\lambda = -H_x, \quad (24)$$

which is our new governing equation for the Hamiltonian system. Also starting from the HJE for F_2 in (12) and (14), taking λ_0 and t_0 as constants, and following the similar procedure, yields the same result.

Now we derive a similar equation for the state x from the HJEs for F_3 and F_4. From the HJE for F_3 in (15) and (17), if we regard x_0 and t_0 as constants and take partial derivatives of (17) with respect to λ, we have

$$\frac{\partial}{\partial \lambda}\left(\frac{\partial F_3}{\partial t} + H\right) = 0$$

Observing that the $H(x, \lambda, t) = H(x(\lambda, x_0, t, t_0), \lambda, t)$ from (15) and using the chain rule for the Hamiltonian and the exactness property of F_3 yields

$$\frac{\partial^2 F_3}{\partial t \partial \lambda} + H_\lambda + H_x \frac{\partial x}{\partial \lambda} = 0$$

Substituting $x = -\partial F_3/\partial \lambda$ into the first term, we obtain a system of PDEs for the state x:

$$\frac{\partial x}{\partial t} - \frac{\partial x}{\partial \lambda} H_x = H_\lambda, \quad (25)$$

which is another set of governing equations for the Hamiltonian system. Finally starting from the HJE for F_4 in (18) and (20) and following a similar procedure yields the same result.

Note that (24) and (25) are n simultaneous first order quasilinear PDEs. In order to solve these equations, we need to derive at least n initial or terminal conditions to form an initial value problem (Cauchy problem). We see that the states x and costates λ in (24) and (25) are simply the same quantities as those in the Hamiltonian formulation in (3)-(5). Hence the boundary conditions for (24) and (25) should be compatible with those of the Hamiltonian system, that is, (7) and (8). We note that it is the type of boundary conditions (7) and (8) that determines which equations to use between (24) and (25), and how to use them to derive the optimal feedback control.

First we consider the hard constraint boundary condition (7). As this should be satisfied by the state x in (25) at the terminal time, we obtain

$$x(t = t_f, \lambda) = x_f \quad (26)$$

Similarly for the soft constraint boundary condition (8), as this should be satisfied by (24) at the terminal time, we have

$$\lambda(t = t_f, x) = \frac{\partial[\phi(x, t_f) + \nu^T \psi(x), t)]}{\partial x} \quad (27)$$

With these terminal conditions (26) and (27), the governing PDEs (25) and (24), respectively, constitute the initial (or terminal) value problems, or so called Cauchy problems[3].

[3]Whereas our derivation originates from the Hamiltonian system theory, one can also derive the same results from the so-called invariant imbedding method based on characteristic theory of 1st order PDEs. See Meyer [15] for details.

Finally note that the solutions to these Cauchy problems (24,27) and (25,26) are subordinate to generating functions by the relations (9), (12), (15) and (18); we can obtain the solutions to the above Cauchy problems simply by partial differentiations of generating functions, once we find them. However, as the motivation of this study suggests, it is very difficult to solve the HJE for a generating function for problems with control bounds, and thus non-smooth optimal control logic and cost function. In that case, we can resort to our new tools (24) and (25).

Generation of Optimal Feedback Control from the Cauchy Problems

So far we have derived a new set of governing equations and their associated terminal conditions to form a Cauchy problem. It remains how to evaluate the optimal feedback control from these new formulations. We discuss the hard and soft constraint problem separately.

Suppose we have solved the Cauchy problem (25,26) for the hard constraint problem. Then we have a solution of the form $x = x(t, \lambda)$. Again note that the state x is the same variable as that of the Hamiltonian formulation (3)-(5); the initial conditions (x_0, λ_0) at the arbitrary moment $t_0 \leq t_f$ should be satisfied by the solution to the Cauchy problem. Then given the (arbitrary) initial state x_0, the following equation should be satisfied:

$$x(t_0, \lambda_0) = x_0, \tag{28}$$

which is an n-tuple of implicit algebraic equation for the n-tuple of unknowns $\lambda_0 = [\lambda_{10} \ \ \lambda_{20} \ \ \cdots \lambda_{n0}]^T$. If we find a solution λ_0 to this equation, we can evaluate the optimal trajectory by simple forward integration of (4)-(5). Furthermore solving (28) implicitly for a given domain of initial state to construct $\lambda_0 = \lambda_0(t_0, x_0)$, we obtain the optimal feedback scheme by the optimality condition (6):

$$u^*(x, \lambda(t,x), t) = arg \min_{\bar{u}} \bar{H}(x, \lambda(t,x), \bar{u}, t) \tag{29}$$

Note that we do not solve the Cauchy problem (25,26) repetitively. Once we find a solution field for the domain of interest, the optimal feedback scheme can be obtained *algebraically*, which provides a substantial advantage over repetitive solving the TPBVP numerically for each boundary conditions.

For the soft constraint problem, the situation is more favorable, as is seen below. Similarly we first solve the Cauchy problem (24) and (27), which yields the solution of the form $\lambda = \lambda(t, x)$. Then the same arguments conclude that the following equation should be satisfied:

$$\lambda(t_0, x_0) = \lambda_0, \tag{30}$$

Here note the difference from the hard constraint problem; given the initial state x_0, λ_0 is an explicit function of t_0 and x_0, which can be more easily computed in general. Then in the same way, starting from (x_0, λ_0), we can evaluate the optimal trajectory as well as the optimal feedback control.

Finally we conclude this section by claiming that our method is truly applicable to free final (or initial) time problems. In this case, the transversality condition for the free time index [12, section 2]

$$\begin{aligned} H(t_0) - \frac{\partial \phi(x(t_0), t_0)}{\partial t_0} &= 0 \\ H(t_f) + \frac{\partial \phi(x(t_f), t_f)}{\partial t_f} &= 0 \end{aligned}$$

provides the additional algebraic equation for the varying time index.

Numerical Computation

So far we have shown that our new method is composed of two steps; first we solve the Cauchy problem (25,26) or (24,27), and then solve the associated implicit or explicit algebraic equations (28) or (30) for the hard or soft constraint problem to derive the optimal feedback control law.

Though the well-posed Cauchy problem is guaranteed to have a unique solution [15, pages 9-17], it is by no means easy to solve a system of 1st order quasilinear PDEs numerically for most non-trivial problems. Suppose we consider one of the traditional finite difference methods, for example. Then, we are first faced with the obstacle of dimensionality. If we assign M grids for one spatial dimension and N grids for the time span of interest, then we need NM^n storage points for a $2n$-dimensional Hamiltonian system representing the necessary conditions for optimality (3)-(5).

For the hard constraint problem, this curse of dimensionality becomes even more significant, as we need to solve the algebraic equations (30) *implicitly*. In general, we do not know *a priori* where in the λ-domain the solutions exist for the corresponding x-domain of interest. Thus, we need to solve the Cauchy problem (25,26) for a large enough domain, in the hope that the solution falls into the estimated domain. For the soft constraint problem, the problem of estimating the solution domain can be alleviated, as the expression (30) becomes an explicit function for x. The initial costate λ_0 can be evaluated by n-dimensional interpolation, which is much simpler than the case of hard constraint problem.

Despite these difficulties and limitations, note that once the Cauchy problems are solved, the solutions work as implicit (or explicit) numerical *feedback* charts for the hard (soft) constraint problem. We believe that with such charts the interpolation process can be done rapidly for relatively high dimensional problems. In other words, the solution to the Cauchy problem, the feedback chart, can be real time implementable for many practical problems.

IV. Illustrative Examples

Linear Quadratic Soft Constraint Problem

We first consider a 2nd order linear quadratic soft constraint problem: minimize

$$J = \frac{1}{2} x^T(t_f) Q_f x(t_f) + \frac{1}{2} \int_{t_0}^{t_f} (x^T Q x + u^T R u) dt$$

subject to the linear system

$$\dot{x} = Ax + Bu$$

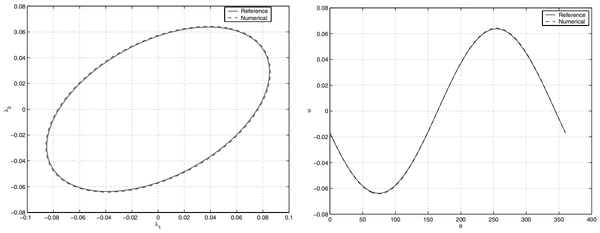

Fig. 1. Loci of Initial Costates and Optimal Control Scheme, $x_0 = [0.1\cos\theta\ 0.1\sin\theta]^T$, $\theta = 0 \sim 360$, $Q_f = 1$

where the numerical parameters are chosen such that

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \ , \ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$Q = 0_{2\times 2} \ , \ R = I_{1\times 1} \ , \ t_0 = 0 \ , \ t_f = 1$$

Defining the pre-Hamiltonian as

$$\bar{H} = \frac{1}{2}(x^T Q x + u^T R u) + \lambda^T (Ax + Bu)$$

and using Pontryagin's principle yields the following Hamiltonian system:

$$H(x, \lambda, t) = \frac{1}{2} x^T Q x + \lambda^T A x - \frac{1}{2}\lambda^T B R^{-1} B^T \lambda$$

$$\begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix}$$

$$u(x, t) = -R^{-1} B^T \lambda(t)$$

As this is a soft constraint problem, the terminal boundary condition is determined from the transversality condition (8) as

$$\lambda(t_f) = Q_f x(t_f)$$

Now we can set up the Cauchy problem (24,27) as follows:

$$\begin{aligned} \frac{\partial \lambda}{\partial t}(t, x) + \frac{\partial \lambda}{\partial x}(t, x)[Ax - BR^{-1}B^T\lambda] &= -(Qx + A^T\lambda) \\ \lambda(t_f, x) &= Q_f x \end{aligned}$$

For numerical computation of this Cauchy problem, the 2nd order Lax-Wendroff finite difference scheme is chosen for this problem[4]. After solving this Cauchy problem we can obtain the initial costate from the explicit function (30), and the optimal feedback control law from (29). For the illustration of the *feedback* nature of our method, we choose a set of initial conditions around the origin with the radius of $r = 0.1$, which is parameterized by $x_0 = [r\cos\theta\ \ r\sin\theta]^T$, $\theta = 0 \sim 360$degrees. Figure 1 shows the loci of initial costates and the corresponding optimal control scheme for the given set of initial conditions with $Q_f = 1$. It is seen that the numerical solutions consistently approximate the reference solution, which is provided by the sweep method in Bryson and Ho [12, pages 148-157] or the generating function method in [10].

[4]For a comprehensive discussion of finite difference methods, we cite [16].

Time Optimal Control of the Double Integrator System

As another example, consider minimizing

$$J = \int_{t_0}^{t_f} dt$$

subject to the double-integrator system with control constraints:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ u \end{bmatrix} \quad , \quad |u| \le 1$$

The initial and terminal boundary conditions are given by

$$\begin{bmatrix} x_1(t_0) \\ x_2(t_0) \end{bmatrix} = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix} \quad , \quad \begin{bmatrix} x_1(t_f) \\ x_2(t_f) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Here for convenience, we fix the terminal time $t_f = 0$ and vary the initial time t_0, which does not change the intrinsic property of the problem. Then, defining the pre-Hamiltonian as

$$\bar{H}(x, \lambda, u, t) = 1 + \lambda_1 x_2 + \lambda_2 u$$

and using the Pontryagin's principle yields the following necessary conditions for optimality with the transversality condition for free initial time:

$$\begin{aligned} H &= 1 + \lambda_1 x_2 - |\lambda_2| \\ \dot{x}_1 &= x_2 \qquad\quad x_1(t_0) = x_{10} \quad x_1(t_f) = 0 \\ \dot{x}_2 &= -\mathrm{sign}(\lambda_2) \quad x_2(t_0) = x_{20} \quad x_2(t_f) = 0 \\ \dot{\lambda}_1 &= 0 \\ \dot{\lambda}_2 &= -\lambda_1 \\ u &= -\mathrm{sign}(\lambda_2) \\ H(t_0) &= 1 + \lambda_1(t_0) x_2(t_0) - |\lambda_2(t_0)| = 0 \end{aligned}$$

From the transversality condition, we can show that there does not exist singular intervals and that the optimal control should be $u = \pm 1$ [11]. The Cauchy problem (25,26) for this hard constraint problem can be written as

$$\begin{aligned} \begin{bmatrix} \frac{\partial x_1}{\partial t} \\ \frac{\partial x_2}{\partial t} \end{bmatrix} + \begin{bmatrix} \frac{\partial x_1}{\partial \lambda_1} & \frac{\partial x_1}{\partial \lambda_2} \\ \frac{\partial x_2}{\partial \lambda_1} & \frac{\partial x_2}{\partial \lambda_2} \end{bmatrix} \begin{bmatrix} 0 \\ -\lambda_1 \end{bmatrix} &= \begin{bmatrix} x_2 \\ -\mathrm{sign}(\lambda_2) \end{bmatrix} \\ [x_1(t_f, \lambda) \ \ x_2(t_f, \lambda)]^T &= [\ 0 \quad 0\ ]^T, \end{aligned}$$

which can be solved numerically. In fact, this problem can be used to show that the solution to the above Cauchy problem can be derived from the associated generating functions, which has been computed in [11]. For example, starting from F_2 generating function, we can obtain the initial state x_0 as a function of terminal state x_f and initial costate λ_0[5]:

$$\begin{aligned} F_2(x_f, \lambda_0, t) &= \frac{\pm\lambda_{20}^2 - 2\lambda_{20} \mp 1}{2\lambda_{10}} \mp x_{2f} + x_{1f}\lambda_{10} \\ &\quad \mp \frac{x_{2f}^2 \lambda_{10}}{2}, \qquad (u = \mp 1 \to \pm 1) \end{aligned}$$

[5]For the effective domain for each case of the solution, see [11].

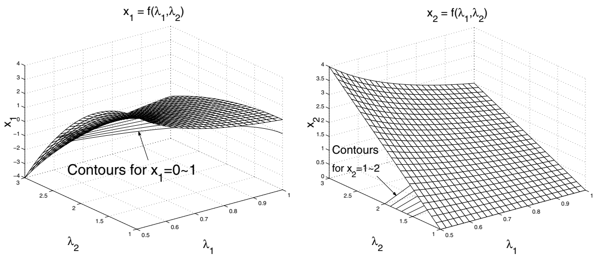

Fig. 2. The Solution to the the Cauchy Problem $(x_f = 0)$

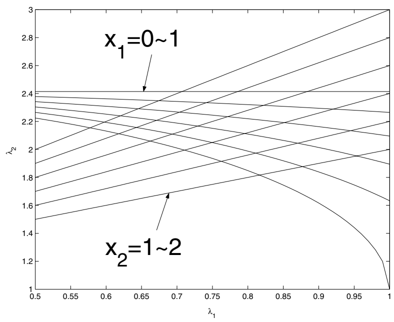

Fig. 3. Graphical Determination of Initial Costate $(x_f = 0)$
Each intersection point defines a unique pair of (λ_1, λ_2) for the optimal control at (x_1, x_2).

$$
\begin{aligned}
x_{10} &= \frac{\partial F_2}{\partial \lambda_{10}} = x_{1f} \mp \frac{1}{2} x_{2f}^2 + \frac{\mp\lambda_{20}^2 + 2\lambda_{20} \pm 1}{2\lambda_{10}^2} \\
x_{20} &= \frac{\partial F_2}{\partial \lambda_{20}} = \frac{\pm\lambda_{20} - 1}{\lambda_{10}}
\end{aligned}
$$

Imposing the given terminal boundary condition at the origin, i.e., $(x_{1f}, x_{2f}) = (0, 0)$, and removing the subscript 0 to represent the moving initial conditions, we have

$$
(x_1, x_2) = \left(\frac{\mp\lambda_2^2 + 2\lambda_2 \pm 1}{2\lambda_1^2}, \frac{\pm\lambda_2 - 1}{\lambda_1} \right)
$$

Simply by direct substitution, we can easily show that this expression satisfies the above Cauchy problem.

Now suppose that we have found this solution numerically from the Cauchy problem (25,26). Then fixing the initial states into the desired ones $(x_1, x_2) = (x_{10}, x_{20})$, we can find the loci of initial costates from each component of the solution in the $\lambda_1\lambda_2$-domain. The whole procedure can be shown graphically with ease. We first draw plots for $x_1 = x_1(\lambda_1, \lambda_2)$ and $x_2 = x_2(\lambda_1, \lambda_2)$ (Figure 2). Then for the desired initial states, we construct the contours for each plot, as in Figure 2 where the contours are drawn for $x_1 = 0 \sim 1$ and $x_2 = 1 \sim 2$, respectively. Imposing these two contours together in the $\lambda_1\lambda_2$-plane, we can find the initial costates for the given initial states by choosing the intersection of these two contours (Figure 3). Then, the optimal feedback control law can be determined from the optimality condition $u = -\text{sign}(\lambda_2)$.

V. Conclusion

We have presented a new method for solving the optimal feedback control problem. Formulating the optimal control problem as a Hamiltonian system, we have derived new governing PDEs with their associated boundary conditions and formed Cauchy problems. Then it has been shown how the solutions to the Cauchy problems can be used for solving both hard and soft constraint problems. Though the new equations have been derived from the HJEs for generating functions, they can be used independently. Furthermore, they can compensate for the limited applicability of generating functions for some problems with non-smooth control logic and cost function.

In the future we will further research how to recover the generating functions from the solutions to the Cauchy problems. Also, we are exploring the relations between the solutions to the hard and soft constraint problem and trying to find a method to recover one solution from the other, as can be done in our generating function method.

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