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Global asymptotic stabilization by output feedback for some non minimum phase non linear systems

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***Abstract*— We consider systems with well defined inverse dynamics that can be made L^p-ISS via the output. We assume also the existence of an observer providing L^p-correction terms. For such systems, by propagating the L^p-ISS property by backstepping, we are able to design a globally asymptotically stabilizing output feedback.**

I. INTRODUCTION

A. The context

The question of global asymptotic stabilizability by output feedback for non minimum phase systems has received a first systematic answer only recently in [7] and [12]. The corresponding results are within a context very similar to what follows. However, thanks to a different analysis, we can deal with a different set of assumptions (see Section I-B for details). Specifically we prove :

The problem of global asymptotic stabilization by dynamic output feedback is solvable for systems which satisfy the following three assumptions :

Assumption A1 : *The dynamics of the system admit the following global form :*

$$\left\{\begin{array}{rcl} \dot{z} &=& F(z,\xi_1)\ , \\ \dot{\xi}_1 &=& \xi_2\ , \\ \dot{\xi}_2 &=& \xi_3 \\ & \vdots & \\ \dot{\xi}_n &=& f(z,\xi_1,\dots,\xi_n)\ +\ g(\xi_1)\,u\ , \\ y &=& \xi_1\ . \end{array}\right. \tag{1}$$

with input u, output y and state $(z,\xi_1,\dots,\dots,\xi_n)$ where z is in $\mathbb{R}^m$ and ξ_i is in $\mathbb{R}$, and where the function F is C^n, the functions f and g are C^1 and we have :

$$F(0,0)=0\ ,\quad f(0,0,\dots,0)=0\ ,\quad g(\xi_1)>0\ . \tag{2}$$

The form (1) is defined up to a change of coordinates in z. Also, by introducing $2(n-1)$ arbitrary sufficiently smooth functions $\{a_i\}_{1\leq i\leq n-1}$ and $\{b_i\}_{1\leq i\leq n-1}$, there exist other ones, a_n and b_n, such that the system (1) can be rewritten always as :

$$\left\{\begin{array}{rl} \dot{z} =& F(z,y)\ , \\ \dot{y} =& a_1(z,y)\,y_2\ +\ b_1(z,y)\ , \\ \dot{y}_2 =& a_2(z,y,y_2)\,y_3\ +\ b_2(z,y,y_2)\ , \\ \vdots & \\ \dot{y}_{n-1} =& a_{n-1}(z,y,y_2,\dots,y_{n-1})\,y_n \\ & \qquad +\ b_{n-1}(y,y_2,\dots,y_{n-1})\ , \\ \dot{y}_n =& a_n(y)\,u\ +\ b_n(z,y,y_2,\dots,y_n)\ . \end{array}\right. \tag{3}$$

Note however we impose that a_n depends only on y.

With collecting z and y_2 to y_n into a single vector x in $\mathbb{R}^{m+n-1}$, (3) takes the following compact form :

$$\dot{x}\ =\ A(x,y)\ +\ B(y)\,u\quad ,\qquad \dot{y}\ =\ C(x,y)\ . \tag{4}$$

Assumption A2 : *The coordinates for z and the functions $\{a_i\}_{1\leq i\leq n-1}$ and $\{b_i\}_{1\leq i\leq n-1}$ can be chosen such that :*

A2.1 there exist a C^n function $K:\mathbb{R}\to\mathbb{R}^{m+n-1}$ and a positive definite symmetric matrix P satisfying

$$P\,\tfrac{\partial A-KC}{\partial x}(x,y)\ +\ \tfrac{\partial A-KC}{\partial x}(x,y)^T P \tag{5}$$
$$\leq\ -\ \tfrac{\partial C}{\partial x}(x,y)^T\ \tfrac{\partial C}{\partial x}(x,y)\ ;$$

A2.2 the system (4) is zero-state detectable, i.e. any bounded solution $\mathcal{X}(x,t)$ of :

$$\dot{x}\ =\ A(e,0)\quad ,\qquad C(x,0)\ =\ 0 \tag{6}$$

is defined on $[0,+\infty)$ and converges to 0 as t tends to infinity ;

A2.3 the functions a_i's take strictly positive values.

Assumption A3 : *There exists a C^n function ϕ_z which makes the following system L^2-ISS :*

$$\dot{z}\ =\ F(z,\phi_z(z))\ +\ K_z(\phi_z(z))\,d\ , \tag{7}$$

where K_z is the component corresponding to the z-coordinates of the function K given by Assumption A2. Specifically, there exist a positive definite, radially unbounded and C^{n+1} function V_z and a positive definite continuous function α_z such that we have[1] :

$$\frac{\partial V_z}{\partial z}(z)\,[F(z,\phi_z(z))+K_z(\phi_z(z))\,d]\ \leq\ -\alpha_z(z)\ +\ |d|^2\ . \tag{8}$$

As we shall see, designing a globally stabilizing output feedback under this set of assumptions is an easy task by invoking by now standard arguments. This will be done

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[1]Remark that (8) is a coordinate dependent assumption.

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in Section III. We prefer to devote now some attention to discuss our 3 assumptions and to compare our context with published results. This discussion will be illustrated by examples in Section IV.

B. Discussion

We first observe that, by invoking the same arguments as those in [12], we can check that our new result encompasses the one in this reference but with a different feedback.

1) On Assumption A1: Systems satisfying Assumption A1 have been fully characterized, upon input scaling, by a coordinate free condition by Byrnes and Isidori in [3, Corollary 5.7]. As we observed the dynamics of such systems can always be represented in the form (3), one of the most (nominal) general form for which we know how to design a globally stabilizing output feedback (see [8], [10], [14] for instance). But we insist on the fact that, in this form, the coordinates for z and the functions $\{a_i\}_{1\leq i\leq n-1}$ and $\{b_i\}_{1\leq i\leq n-1}$ are all not given a priori.

2) On Assumption A2: There is no true novelty in Assumption A2.1. Its, maybe unusual, formulation allows a contraction analysis whose interest in particular for observers has been popularized by [11]. Also it follows from the same kind of consideration as in [17, Assumption 2.1].

As we shall see below, standard assumptions do not lead to (5) but instead to :

$$P\frac{\partial A-KC}{\partial x}(x,y) \;+\; \frac{\partial A-KC}{\partial x}(x,y)^T P \;\leq\; -I \;. \quad (9)$$

In this case it can be checked that Assumption A2.2 is automatically satisfied. However, to obtain (5) from (9), we need an extra property on the function C. Typically it is that $\left|\frac{\partial C}{\partial x}(x,y)\right|$ is bounded or more specifically that a_1 does not depend on z and $\left|\frac{\partial b_1}{\partial z}(z,y)\right|$ is bounded. We consider this as a severe restriction. In order to relax it, we shall propose in Section II another slightly different set of assumptions.

Guaranteeing the existence of a reduced order observer from Assumption A2.1 is evidently a tautology. We have given ourselves this derogation in writing since sufficient conditions for this assumption to hold are known. Specifically :

Monotonic non linearities : Following Arcak and Kokotović [2], consider the case where we can find the function K such that we have the following decomposition :

$$A(x,y) \;+\; K(y)C(x,y) \;=\; Fx \;+\; Q(y) \;+\; \sum_{i=1}^{n+m-1} G_i\gamma_i(L_i^T x,y) \quad (10)$$

where the γ_i's are C^1 function satisfying, for all (s,y) in $\mathbb{R}^2$,

$$-\infty \;<\; a_i \;\leq\; \frac{\partial\gamma_i}{\partial s}(s,y) \;\leq\; b_i \;\leq\; +\infty \;. \quad (11)$$

Proposition 1 ([2]): If there exists a positive definite matrix symmetric P and real numbers $\lambda_i \neq 0$ satisfying :

$$\sum_{i=1}^{n+m-1}\frac{b_i}{4}\left(e^T\left(\lambda_i L_i \;+\; \frac{P}{\lambda_i}G_i\right)\right)^2 - \sum_{i=1}^{n+m-1}\frac{a_i}{4}\left(e^T\left(\lambda_i L_i \;-\; \frac{P}{\lambda_i}G_i\right)\right)^2 + e^T PFe \;\leq\; -|e|^2$$

for all e in $\mathbb{R}^{m+n-1}$, then (9) holds.

Output dependent incremental rate : We restrict our attention to systems in the form (3) where the a_i's depend only on y and each component of F can be decomposed as

$$F_\ell(z,y) \;=\; f_\ell(y,z_1,\ldots,z_\ell) \;+\; c_\ell(y)\,z_{\ell+1} \;. \quad (12)$$

To simplify our forthcoming condition, we define the functions $\phi_{i,j}$ and ψ_i :

$$\phi_{i,j} \;=\; \begin{cases} \frac{\partial b_i}{\partial y_j} & 2\leq j\leq i\leq n \\ 0 & n+1\leq i\leq m\;,\; 2\leq j\leq n \\ \frac{\partial f_{i-n}}{\partial z_{j-n}} & n+1\leq j\leq i\leq n+m \end{cases} \quad (13)$$

$$\psi_i \;=\; \begin{cases} a_i & 2\leq i\leq n-1 \\ c_{i-n} & n\leq i\leq n+m-1 \end{cases} \quad (14)$$

Proposition 2 ([9, Lemma 1]): If there exists a positive real number ρ, such that, for all (x,y) in $\mathbb{R}^{n+m}$, we have :

$$\rho \;\leq\; \psi_i\;,\; 2\leq i\leq n+m-1\;, \quad (15)$$

$$\rho\,\psi_i \;\leq\; \psi_{i-1}\;,\; 3\leq i\leq n+m-1\;, \quad (16)$$

$$\rho\,|\phi_{i,j}| \;\leq\; \psi_i\;,\; 2\leq i\leq n+m-1\;, \quad (17)$$

$$\rho\,|\phi_{n+m,j}| \;\leq\; \psi_{n+m-1}\;,\; 2\leq i\leq n+m-1\;, \quad (18)$$

then there exists a vector $K(y)$ and a matrix P such that (9) is satisfied.

Moreover, following [8], we could expect (17) and (18) could be removed if we were to use the technique of dynamic scaling introduced in [16]. However, its application in the present context is made difficult by the presence of the z-dynamics for which we have assumed no specific structure.

3) On Assumption A3: In the classical framework of global asymptotic stabilization by output feedback which originated from [6] and [13] (see for instance [8], [10], [14], [18]), i.e. all the contributions before [7], Assumption A3 was that the inverse dynamics are ISS, i.e. the function ϕ_z was imposed to be zero.

The existence of the function ϕ_z as required by our new Assumption A3 is almost necessary. Indeed, assume the existence of a globally stabilizing dynamic output feedback for the system (1). By collecting the state of this feedback as well as the ξ_i's in a vector ζ, the closed loop system can be written as :

$$\dot z = F(z,\xi_1)\;,\quad \dot\zeta = \Phi(z,\zeta)\;,\quad \xi_1 = H(\zeta)\;. \quad (19)$$

Global asymptotic stability implies the existence of a positive definite, radially unbounded and C^1 function U whose derivative along the solutions of (19) is negative definite. So, for each z, the function $\zeta \mapsto U(z,\zeta)$ admits a global minimum reached at some point in the z dependent set $\mathtt{Argmin}_\zeta(U(z,\zeta))$. It can then be shown that, if there exists a Hölder selection $M : z \mapsto \mathtt{Argmin}_\zeta(U(z,\zeta))$ of order strictly greater than $\frac{1}{2}$, then the function V_z defined by :

$$V_z(z) \;=\; U(z, M(z)) \tag{20}$$

is positive definite, radially unbounded and C^1 and the function ϕ_z defined by :

$$\phi_z(z) \;=\; H(M(z)) \tag{21}$$

is continuous and satisfies :

$$\frac{\partial V_z}{\partial z}(z)\, F(z, \phi_z(z)) \;<\; 0 \qquad \forall z \neq 0\ . \tag{22}$$

So existence of ϕ_z is necessary up to the existence of this Hölder selection.

It follows that the main restriction imposed by Assumption A3 is the fact that the function ϕ_z not only stabilizes asymptotically the origin of the z-subsystem but also provides the L^2-ISS property. This restriction is, conceptually, very similar to [17, Assumption (21)] although the latter was about an L^1-ISS property (See Assumption A3' below).

In our mind it is about this specific point that our contribution differs significantly from the one in [7]. Specifically, importing the ideas of [7], would lead us to assume that the following system is input-to-state stable (ISS) :

$$\dot{z} \;=\; F(z, \phi_z(z + d_1) + d_2)\ . \tag{23}$$

As we shall see later this difference follows from the fact that we use an estimate of z and not z itself as argument of V_z (see (40)).

4) Last comment: Many other results are available in the literature. In particular it is worth mentioning those which rely on the fact that we know how to design output feedback for a simple chain of linear/homogeneous integrators but able to cope with significant perturbation. The most advanced results of this kind are for instance [8], [19].

II. The L^1 case

We have observed that, if, to check (5), we have to go through (9), then we end up with a restrictive assumption on C. In order to weaken this assumption, we give another possible set of assumptions :

Assumption A2' : *The coordinates for z and the functions $\{a_i\}_{1\leq i\leq n-1}$ and $\{b_i\}_{1\leq i\leq n-1}$ can be chosen such that :*

A2.1' there exists a positive definite, radially unbounded and locally Lipschitz function W, and a C^n function K satisfying :

$$D^+W(e)\,(\Delta A(x,e,y) - K(y)\Delta C(x,e,y)) \tag{24}$$
$$\leq\; -|\Delta C(x,e,y)|\ .$$

with the notations

$$\Delta A(x,e,y) = A(x+e,y) \;-\; A(x,y) \tag{25}$$
$$\Delta C(x,e,y) = C(x+e,y) \;-\; C(x,y) \tag{26}$$
$$D^+W(e)f(e,x) = \limsup_{t\searrow 0} \frac{W(e+tf(e,x))-W(e)}{t} \tag{27}$$

Moreover, K_z, the component corresponding to the z-coordinates of the function K, is bounded.

A2.2' Same as Assumption A2.2

A2.3' Same as Assumption A2.3

Assumption A3' : *There exist a C^n function ϕ_z, a positive definite, radially unbounded, C^{n+1} function V_z and a positive definite continuous function α_z satisfying :*

$$\frac{\partial V_z}{\partial z}(z)\left[F(z,\phi_z(z)) + K_z(\phi_z(z))\,d\right] \leq -\alpha_z(z) + |d| \tag{28}$$

III. Global asymptotic stabilization by output feedback

Theorem 1: If Assumptions A1, A2 and A3 or A1, A2' and A3' hold, then, there exists a globally stabilizing dynamic output feedback of dimension $m+n-1$.

The proof is mainly the same for the two set of assumptions. It relies on the following separation recipe :

If we have an observer providing L^p-correction terms, and if we have a state feedback making the system L^p-ISS, then we can cook up a globally asymptotically stabilizing output feedback.

This recipe may not have been formalized in this way previously (see however [1], [17]) but we certainly do not claim any originality. Most of the published results on output feedback, starting from [6], [13], can be reinterpreted along its lines.

We introduce the notation :

$$\begin{array}{rcll} |s|^p &=& |s| & \text{in the } L^1 \text{ case,} \\ |s|^p &=& |s|^2 & \text{in the } L^2 \text{ case.} \end{array} \tag{29}$$

To formulate Assumption A2.1 in a way similar to A2.1' we let the function W be defined (in the L^2 case) as :

$$W(s) \quad = \quad s^T P s\ . \tag{30}$$

Proof : The dynamic output feedback we propose has the structure of an observer-controller. The observer is a reduced order observer which induces L^p correction terms. By using the observer backstepping technique, the controller is built to ensure an L^p-ISS property. We end the proof by analyzing the behavior of the closed loop system.

Observer : We introduce the following observer :

$$\left\{\begin{array}{rcl} \hat{x} &=& w + M(y) \\ \dot{w} &=& A\big(\hat{x},y\big) \;+\; B(y)\,u \;-\; K(y)C\big(\hat{x},y\big)\ , \end{array}\right. \tag{31}$$

where :

$$\hat{x} \;=\; (\hat{z},\, \hat{y}_2,\, \ldots,\, \hat{y}_n) \tag{32}$$

and the function M is :

$$M(y) \;=\; \int_0^y K(s)\, ds\,. \tag{33}$$

It follows that the estimate of the state $\hat{x}$ satisfies :

$$\dot{\hat{x}} = A(\hat{x},y) + B(y)\,u + K(y)\left(C(x,y) - C(\hat{x},y)\right)\ . \tag{34}$$

By letting :

$$e = \mathcal{x} - \hat{\mathcal{x}} , \quad (35)$$

and with (5) and (30) (resp. with (24)) we get[2] :

$$\overbrace{W(e)}^{\cdot} \leq -|C(\hat{x}+e,y) - C(\hat{x},y)|^p . \quad (36)$$

Controller design : To design the controller, we aim at making the following system L^2(resp L^1)-ISS :

$$\begin{cases} \dot{\hat{z}} &= F(\hat{z},y) + K_z(y)\, d \ , \\ \dot{y} &= a_1(\hat{z},y)\, \hat{y}_2 + b_1(\hat{z},y) + d \ , \\ \dot{\hat{y}}_2 &= a_2(\hat{z},y,\hat{y}_2)\, \hat{y}_3 + b_2(\hat{z},y,\hat{y}_2) + K_2(y)\, d \ , \\ & \vdots \\ \dot{\hat{y}}_n &= a_n(y)\, u + b_n(\hat{z},y,\hat{y}_2,\ldots,\hat{y}_n) + K_n(y)\, d \ , \end{cases} \quad (37)$$

where d is an exogenous signal taking values in $\mathbb{R}$. Using Assumption A3 (resp. A3'), we have a C^{n+1} Lyapunov function V_z satisfying :

$$\frac{\partial V_z}{\partial z}(\hat{z})\big(F(\hat{z},\phi_z(\hat{z})) + K_z(\phi_z(\hat{z}))\, d\big) \leq -\alpha_z(\hat{z}) + |d|^p . \quad (38)$$

So, by following a standard backstepping procedure with applying recursively Lemma 1 (resp. Lemma 2) in appendix, we can propagate this property up to getting a C^1 Lyapunov function V_n and a C^0 state feedback ϕ_n such that $u = \phi_n(\hat{z},y,\hat{y}_2,\ldots,\hat{y}_n)$ gives for the system (37) :

$$\overbrace{V_n(\hat{z},y,\hat{y}_2,\ldots,\hat{y}_n)}^{\cdot} \leq -\alpha_n(\hat{z},y,\hat{y}_2,\ldots,\hat{y}_n) + (n+1)\,|d|^p , \quad (39)$$

where α_n is positive definite.

Study of the closed loop system : The dynamics of the closed loop system are completely described by[3] :

$$\begin{cases} \dot{\hat{z}} &= F(\hat{z},y) + K_z(y)\Delta C \ , \\ \dot{y} &= a_1(\hat{z},y)\, \hat{y}_2 + b_1(\hat{z},y) + \Delta C \ , \\ \dot{\hat{y}}_2 &= a_2(\hat{z},y,\hat{y}_2)\, \hat{y}_3 + b_2(\hat{z},y,\hat{y}_2) + K_2(y)\Delta C \ , \\ & \vdots \\ \dot{\hat{y}}_n &= a_n(y)\, \phi_n(\hat{z},y,\hat{y}_2,\ldots,\hat{y}_n) \\ & \qquad + b_n(\hat{z},y,\hat{y}_2,\ldots,\hat{y}_n) + K_n(y)\Delta C \ , \\ \cdots & \cdots\cdots\cdots\cdots\cdots\cdots \\ \dot{e} &= A(\hat{x}+e,y) - A(\hat{x},y) - K(y)\Delta C \ , \end{cases} \quad (40)$$

with the notation (26), (32) and :

$$K(y) = (K_z(y), K_2(y), \ldots, K_n(y)) . \quad (41)$$

The top part of this system (40) is nothing but the system (37) with :

$$d = \Delta C . \quad (42)$$

[2]Here and implicitly in (44), we denote by $\overbrace{W(e)}^{\cdot}$ the expression $D^+W(e)\,(\Delta A(\mathcal{x},e,y) - K(y)\Delta C(\mathcal{x},e,y))$.

[3]In [7], [12], some similar dynamics are written but with z instead of $\hat{z}$.

We introduce the locally Lipschitz Lyapunov function :

$$U(e,\hat{\mathcal{x}},y) = 2\,W(e) + \frac{1}{n+1}V_n(\hat{\mathcal{x}},y) . \quad (43)$$

Along the trajectories of the closed loop system (40), by using (36) and (39), we get :

$$\overbrace{U(e,\hat{\mathcal{x}},y)}^{\cdot} \leq -\alpha_n(\hat{\mathcal{x}},y) - |C(\hat{x}+e,y) - C(\hat{x},y)|^p . \quad (44)$$

This implies that the system (40) is complete in positive time and that the origin is globally stable. Moreover, since α_n is positive definite we obtain that any solution $(E(e,t), \hat{\mathcal{X}}(\hat{\mathcal{x}},t), Y(y,t))$ converges to the largest invariant set contained in the set

$$\{(e,\hat{\mathcal{x}},y) \ : \ C(e,0) = 0\,,\ \hat{\mathcal{x}} = 0\,,\ y = 0\} .$$

By following the same arguments as in [4, p.44] we can conclude with Assumption A2.2 (resp. A2.2'), that each solution converges to the origin. So we have established global asymptotic stability. □

Remark: From our proof, we readily see that our conclusion still holds

- when the component K_i of the function K, given by Assumption A2 or A2', depends also on $\hat{y}_2$ to $\hat{y}_i$, subject to an at most linear growth in $|\hat{y}_i|$ in the L^1 case (see (68));
- or when we use a full order observer of the form :

$$\begin{cases} \dot{\hat{\mathcal{x}}} = A(\hat{\mathcal{x}},y) + K_{\mathcal{x}}(y,\hat{\mathcal{x}})(y-\hat{y}) \ , \\ \dot{\hat{y}} = C(\hat{\mathcal{x}},y) + K_y(y,\hat{\mathcal{x}})(y-\hat{y}) \ , \end{cases} \quad (45)$$

  provided K_y and the components of $K_{\mathcal{x}}$ have the appropriate triangular dependence (see the above point) and both ΔC and $y - \hat{y}$ are in L^p.
- or, following [17], when $K_z\Delta C$ can be factorized as (and similarly for $K_i\Delta C$)

$$K_z\Delta C = L_z\,\delta \quad (46)$$

  where L_z is a matrix and δ is a vector provided L_z replaces K_z in Assumption A3 or A3' and, along the solution, δ is in L^p;
- or when $\dot{z}$ depends on $(y, y_1, \ldots, y_p)$ but we know a feedback law ϕ_p such that $y_{p+1} = \phi_p(z,y,y_1,\ldots,y_p)$ makes the $(z,y,y_1,\ldots,y_p)$ subsystem L^p-ISS.

IV. EXAMPLES

A. Example 1

Consider the system :

$$\begin{cases} \dot{z} &= z^2 + y \ , \\ \dot{y} &= y_2 + z^3 \ , \\ \dot{y}_2 &= u + 2z^3 + z^2 + 2z \ . \end{cases} \quad (47)$$

Assumptions A1, A2.2, A2.3 and A3 are satisfied.

Suppose that Assumption A2.1 is satisfied. Then there exist 3 real numbers (p,q,r) and 2 functions K_z and K_2

such that, for each e_z, e_2, z and y, we get :

$$\left(e_z\ e_2\right)\begin{pmatrix} p\ q \\ q\ r \end{pmatrix}\begin{pmatrix} 2z+3K_z(y)z^2 & K_z(y) \\ 6z^2+2z+2+3K_2(y)z^2 & K_2(y) \end{pmatrix}\begin{pmatrix} e_z \\ e_2 \end{pmatrix} \leq -\left(\left(3z^2\ 1\right)\begin{pmatrix} e_z \\ e_2 \end{pmatrix}\right)^2 . \quad (48)$$

When $e_2 = 0$, this gives, for all (z, y),

$$p\,(2z+3K_z(y)z^2)+q\,(6z^2+2z+2+3K_2(y)z^2) \leq -9z^4 \quad (49)$$

which is impossible. So Assumption A2.1 cannot be satisfied for the given representation (47).

To show that Assumption A2.1' is satisfied, we take $K_z = -2$ and $K_2 = -4$ and obtain the observer :

$$\left\{\begin{array}{rcl} \dot{w}_z &=& \hat{z}^2 + y - 2(\hat{y}_2 + \hat{z}^3)\ , \\ \dot{w}_2 &=& u + 2\hat{z}^3 + \hat{z}^2 + 2\hat{z} - 4(\hat{y}_2 + \hat{z}^3)\ , \\ \hat{z} &=& w_z + 2\,y\ , \\ \hat{y}_2 &=& w_2 + 4\,y\ . \end{array}\right. \quad (50)$$

Then by picking :

$$W(\hat{z}, z, \hat{y}_2, y_2) \ = \ |z-\hat{z}| + |(y_2-\hat{y}_2)-(z-\hat{z})|\ , \quad (51)$$

we get :

$$D^+W \ \leq \ -|\hat{y}_2 - y_2 + \hat{z}^3 - z^3|\ . \quad (52)$$

Consequently Assumption A2.1' is satisfied.

We conclude that we can construct a globally asymptotically stabilizing output feedback.

B. Example 2

Consider the following system, already studied in [17],

$$\left\{\begin{array}{rcl} \dot{x}_1 &=& x_2 + u\ , \\ \dot{x}_2 &=& f(x_1) + x_3 - u\ , \\ \dot{x}_3 &=& -f(x_1)\ , \\ y &=& x_1\ , \end{array}\right. \quad (53)$$

where f is a C^2 function such that $f(0) = 0$ and $f'(0) \neq 0$. Following [12], consider the fictitious output μ :

$$\mu \ = \ t_1\,x_1 + t_2\,x_2 + t_3\,x_3\ , \quad (54)$$

with $t_1 \neq t_2$. The corresponding zero dynamics are :

$$\left\{\begin{array}{rcl} \dot{x}_1 &=& \dfrac{t_1x_1 + (t_3-t_2)(x_3+f(x_1))}{t_1-t_2}\ , \\ \dot{x}_3 &=& -f(x_1)\ . \end{array}\right. \quad (55)$$

If f vanishes at another point than 0, the origin cannot be globally asymptotically stable. So the result of [12] does not apply. Nevertheless (53) can be rewritten in :

$$\left\{\begin{array}{rcl} \dot{z}_1 &=& z_1 - y\ , \\ \dot{z}_2 &=& -f(y)\ , \\ \dot{y} &=& z_1 - z_2 - y + u\ . \end{array}\right. \quad (56)$$

So we see readily that Assumption A1 and A2' are satisfied with K constant.

To check that Assumption A3' is also satisfied, we design a feedback ϕ_z which ensures an L^1-ISS property for the (z_1, z_2) subsystem. Let σ be a C^0 non decreasing function satisfying :

$$\sigma(z_1) \ \geq \ \max\left\{\frac{f(2z_1)}{z_1}\,,\,\frac{1}{2}\max_{|s|\leq 1}|f''(2z_1+s)|\right\}\ . \quad (57)$$

We consider the Lyapunov function :

$$V_z(z_1,z_2) = \ell\left(U(z_1) + 2\sqrt{1+\frac{a}{2}\zeta_2^2}\right)\ , \quad (58)$$

where :

$$\zeta_2 \ = \ \left(z_2 - \int_0^{z_1}\frac{f(2s)}{s}ds\right) \quad (59)$$

and

$$a < \frac{1}{2\left(\max_{|z_1|<1}\sigma(z_1)\right)^2}\ ,\ U(z_1) = 2\sqrt{a}\int_0^{z_1} s\sigma(s)\,ds \quad (60)$$

and ℓ is a positive definite, radially unbounded and C^1 function whose derivative is non increasing and satisfies :

$$\ell'(U(z_1)) \ \leq \ \frac{1}{U'(z_1)}\ . \quad (61)$$

By picking the feedback :

$$\phi_z(z_1,z_2) = 2z_1 + \tfrac{2}{\pi}\arctan\left(U'(z_1) - \frac{a\zeta_2}{\sqrt{1+\frac{a}{2}\zeta_2^2}}\left(-f'(2z_1)+\frac{f(2z_1)}{z_1}\right)\right) \quad (62)$$

and using the following inequality, for all z_1 and s : $|s| \leq 1$

$$|f(2z_1)+f'(2z_1)s - f(2z_1+s)| \ \leq \ \sigma(z_1)\,s^2\ , \quad (63)$$

we can obtain, when $y = \phi_z(z_1, z_2)$,

$$\dot{\overbrace{V_z(z_1,z_2)}} < 0 \qquad \forall (z_1,z_2) \neq 0\ . \quad (64)$$

So Assumption A3' holds since K_z is constant and V_z has a bounded gradient.

V. Conclusion

We have established that global asymptotic stability by output feedback can be achieved for systems whose inverse dynamics can be made L^p-ISS via their output and for which we have an observer providing L^p-correction terms. This result is obtained by getting the output feedback from this observer and from a state feedback designed by backstepping while propagating the L^p-ISS property.

Appendix : L^p-ISS propagation

In this appendix, we show how the L^2-ISS and L^1-ISS properties can be propagated through a chain of integrators.

We consider a system in the form :

$$\left\{\begin{array}{l} \dot{x}_1 = f(x_1,x_2) + K_1(x_1,x_2)\,d_1\ , \\ \dot{x}_2 = a(x_1,x_2)\,u + b(x_1,x_2) + K_2(x_1,x_2)\,d_2\ . \end{array}\right. \quad (65)$$

where x_1 is in $\mathbb{R}^{n_1}$, x_2 is in $\mathbb{R}$, u is in $\mathbb{R}$, d_1 is in $\mathbb{R}^{n_1}$, d_2 in $\mathbb{R}$, $a(x_1,x_2)$ is strictly positive.

Lemma 1 (L^2-ISS propagation): Suppose K_1 and K_2 are C^q and C^{q-1} functions respectively and there exist a positive definite, radially unbounded and C^{q+1} function $V_1 : \mathbb{R}^{n_1} \to \mathbb{R}_+$, a C^q function $\phi_1 : \mathbb{R}^{n_1} \to \mathbb{R}$, and a positive definite continuous function $\alpha_1 : \mathbb{R}^{n_1} \to \mathbb{R}_+$ such that, when $x_2 = \phi_1(x_1)$, we have :

$$\overbrace{V_1(x_1)}^{\cdot} \leq -\alpha_1(x_1) + |d_1|^2 . \tag{66}$$

Then, there exists a positive definite, radially unbounded and C^q function $V_2 : \mathbb{R}^{n_1+1} \to \mathbb{R}_+$, a C^{q-1} function $\phi_2 : \mathbb{R}^{n_1+1} \to \mathbb{R}$, and a positive definite continuous function $\alpha_2 : \mathbb{R}^{n_1+1} \to \mathbb{R}_+$ such that, by taking $u = \phi_2(x_1, x_2)$, we get for (65) :

$$\overbrace{V_2(x_1, x_2)}^{\cdot} \leq -\alpha_2(x_1, x_2) + |d_1|^2 + |d_2|^2 . \tag{67}$$

This result is well known. See [5], [18] for a proof.

Lemma 2 ($L^1 - ISS$ propagation): Suppose there exist a continuous function $M : \mathbb{R}^{n_1} \to \mathbb{R}_+$, a positive definite, radially unbounded and C^{q+1} function $V_1 : \mathbb{R}^{n_1} \to \mathbb{R}_+$, a C^q function $\phi_1 : \mathbb{R}^{n_1} \to \mathbb{R}$, and a positive definite continuous function $\alpha_1 : \mathbb{R}^{n_1} \to \mathbb{R}_+$ satisfying :

$$|K_1(x_1, x_2)| + \frac{|K_2(x_1, x_2)|}{1 + |x_2|} \leq M(x_1) \tag{68}$$

and, when $x_2 = \phi_1(x_1)$:

$$\overbrace{V_1(x_1)}^{\cdot} \leq -\alpha_1(x_1) + |d_1| . \tag{69}$$

Then, there exist a positive definite, radially unbounded and C^q function $V_2 : \mathbb{R}^{n_1+1} \to \mathbb{R}_+$, a C^{q-1} function $\phi_2 : \mathbb{R}^{n_1+1} \to \mathbb{R}$, and a positive definite continuous function $\alpha_2 : \mathbb{R}^{n_1+1} \to \mathbb{R}_+$ such that, by taking $u = \phi_2(x_1, x_2)$, we get for (65) :

$$\overbrace{V_2(x_1, x_2)}^{\cdot} \leq -\alpha_2(x_1, x_2) + |d_1| + |d_2| . \tag{70}$$

Proof : The idea we exploit here follows from a suggestion of Frederic Mazenc. He used a very similar argument in his dissertation [15, (2.412)] in order to control the time derivative of a Lyapunov function for a combined system.

Due to space limitation, we give only the control Lyapunov function and leave to the reader the fact of checking that it does give the result.

As V_1 is a radially unbounded function we can introduce $k' : \mathbb{R}_+ \to \mathbb{R}_+$ an increasing C^p function which satisfies :

$$k'(V(x_1)) \geq \max\Big\{1, \ \tfrac{1}{2}M(x_1)\,(3 + \phi_1(x_1)), \quad \left(\left|\tfrac{\partial \phi_1}{\partial x_1}(x_1)\right| + 2\left|\tfrac{\partial V_1}{\partial x_1}(x_1)\right|\right) M(x_1)\Big\} \tag{71}$$

With this data, an appropriate control Lyapunov function for establishing our result is the following positive definite, radially unbounded and C^p function :

$$W(x_1, x_2) = \ell\left(k\big(V_1(x_1)\big) + \log\left(1 + \tfrac{1}{2}(x_2 - \phi_1(x_1))^2\right)\right) \tag{72}$$

where k is the radially unbounded C^{p+1} function :

$$k(s) = \int_0^s k'(u)\, du . \tag{73}$$

and ℓ is the positive definite, radially unbounded and C^{p+1} function :

$$\ell(s) = \frac{1}{2} k^{-1}(s) , \tag{74}$$

Note that ℓ' is strictly positive. □

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