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Linear Independency of Interval Vectors and Its Applications to Robust Controllability Tests

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***Abstract*— In this paper, we define linear dependency and independency of the interval vectors and present an effective method for checking the linear independency of interval vectors. As a possible application of the interval vectors to the robust control problem, the robust controllability and un-controllability problems of uncertain interval system are solved. Through the numerical examples and by comparing with the existing results, the superiority of our new robust controllability test method is presented.**

***Index Terms*— Linear independency, Interval vectors, Robust controllability, Un-controllability, Uncertain systems.**

I. Introduction

In robust control, the model uncertainty problem has been effectively and popularly handled by "interval" concept. Great amount of literatures is available under the name of "interval" for example, interval algebra [1], [2], interval polynomial [3], [4], Schur stability of interval matrices [5], [6], Hurwitz stability of interval matrices [7], [8], eigenvalues of interval matrices [9], [10], and robust control with parameter uncertainty [11], [12]. However, the linear (in)-dependency problem of interval vectors has not been attacked in robust control area. Even though the basic concepts of interval vectors were defined in [1], [2], and as a specified example of the quasivector spaces, the interval vectors have been defined in [13], its algebraic properties have not been fully understood. In fact, some basic algebraic properties of the interval vectors, for example linear (in)-dependency property by combination, were studied in [14], but the linear dependency and independency condition, on its own, was not directly investigated.

In this paper, our main interest is to check the linear (in)-dependency of interval vectors for the robust control applications. After suggesting an effective sufficient conditions of the linear (in)-dependency of interval vectors, we will show that sufficient linear (in)-dependency condition of interval vectors can be effectively used in checking the robust (un)-controllability of the uncertain linear time invariant (LTI) system. Thus, the main novelty of this paper is to investigate the linear (in)-dependency of interval vectors, and then is to check the (un)-controllability of the uncertain system with much less conservatism compared with the existing results.

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The paper consists of as follows: In Section II, we provide sufficient linear dependency and independency conditions of the interval vectors. In Section III, the developed linear (in)-dependency condition of interval vectors is used to check the sufficient (un)-controllability condition of the uncertain LTI system. Conclusions are given in Section IV.

II. Linear (In)-dependency of Interval Vectors

Throughout the paper, we need the following basic definitions. Our discussions are limited to the real system.

Definition 2.1: A parameter is called *interval* if it lies between two closed extreme upper and under boundary values. So, a real interval scalar x^I can be defined as: $x^I := [\underline{x}, \overline{x}]$, where $\underline{x}$ and $\overline{x}$ are extreme under and upper values in $\mathcal{R}$.The n-dimensional real column interval vector $\mathbf{x}^I$ is defined as: $\mathbf{x}^I := (x_1^I, \ldots, x_n^I)^T$ and the $n \times m$ dimensional real interval matrix is defined from the interval vectors as: $X^I := \left(\mathbf{x}_1^I, \mathbf{x}_2^I, \ldots, \mathbf{x}_m^I\right)$. The interval vector and interval matrix can be written as: $\mathbf{x}^I = [\underline{\mathbf{x}}, \overline{\mathbf{x}}]$ and $X^I = \left[\underline{X}, \overline{X}\right]$ where $\underline{\mathbf{x}} = (\underline{x_1}, \ldots, \underline{x_n})^T$, $\overline{\mathbf{x}} = (\overline{x_1}, \ldots, \overline{x_n})^T$, $\underline{X} = \left(\underline{\mathbf{x}_1}, \ldots, \underline{\mathbf{x}_m}\right)$, and $\overline{X} = (\overline{\mathbf{x}_1}, \ldots, \overline{\mathbf{x}_m})$. Or, they can be written as: $\mathbf{x}^I = [\mathbf{x}_0 - \Delta\mathbf{x}, \mathbf{x}_0 + \Delta\mathbf{x}]$ and $X^I = [X_0 - \Delta X, X_0 + \Delta X]$, where $\mathbf{x}_0 = \frac{\overline{\mathbf{x}}+\underline{\mathbf{x}}}{2}$, $X_0 = \frac{\overline{X}+\underline{X}}{2}$, $\Delta\mathbf{x} = \frac{\overline{\mathbf{x}}-\underline{\mathbf{x}}}{2}$, and $\Delta X = \frac{\overline{X}-\underline{X}}{2}$.

Based on [1], [2], the following interval arithmetics are used in this paper.

Definition 2.2: The intersection of two real interval scalars x^I and y^I is defined as: $x^I \cap y^I := \left\{z \mid z \in x^I \text{ and } z \in y^I\right\}$. The union of two real interval scalars x^I and y^I is defined as: $x^I \cup y^I := \left\{z \mid z \in x^I \text{ or } z \in y^I\right\}$.

Definition 2.3: For nonempty closed intervals, the addition of two real interval scalars x^I and y^I is defined and calculated as: $x^I \oplus y^I = \left[\underline{x} + \underline{y}, \overline{x} + \overline{y}\right]$, the substraction is $x^I \ominus y^I = \left[\underline{x} - \overline{y}, \overline{x} - \underline{y}\right]$, and the multiplication is

$$x^I \otimes y^I = \left[\min\left\{\underline{x}\underline{y}, \underline{x}\overline{y}, \overline{x}\underline{y}, \overline{x}\overline{y}\right\}, \max\left\{\underline{x}\underline{y}, \underline{x}\overline{y}, \overline{x}\underline{y}, \overline{x}\overline{y}\right\}\right]$$

The division should be carefully defined, based on [2], as:

$$\begin{aligned}\frac{1}{x^I} &= \emptyset \text{ iff } x^I = [0,0] \\ &= \left[\frac{1}{\overline{x}}, \frac{1}{\underline{x}}\right] \text{ iff } 0 \notin x^I \\ &= \left[\frac{1}{\overline{x}}, \infty\right) \text{ iff } \underline{x} = 0 \text{ and } \overline{x} > 0\end{aligned}$$

$$= \left(-\infty, \frac{1}{\overline{x}}\right] \text{ iff } \underline{x} < 0 \text{ and } \overline{x} = 0$$
$$= (-\infty, \infty) \text{ iff } \underline{x} < 0 \text{ and } \overline{x} > 0 \quad (1)$$

Then, the division of two interval scalars is simply defined and calculated as: $x^I \oslash y^I = x^I \otimes \frac{1}{y^I}$.

Definition 2.4: The *ratio* $\mathbf{r}_{xy}$ between two interval vectors is defined and calculated as:

$$\mathbf{r}_{xy} := \mathbf{x}^I \setminus \mathbf{y}^I = \left(x_1^I \oslash y_1^I, \ldots, x_n^I \oslash y_n^I\right)$$

The addition, substraction, dot-product, and cross-product of two interval vectors and interval matrices can be defined based on above scalar arithmetics.

Remark 2.1: The interval arithmetics defined above can be defined in the set concepts. For example, the summation of two interval scalars x^I and y^I can be defined as:

$$z^I = \{z \mid z = x + y, \ \forall x \in x^I, \ \forall y \in y^I\}$$

However, in this paper, we use, simply, the interval arithmetic such as: $z^I = x^I \oplus y^I$. This is clear without notational confusion and effective to represent our idea. Note $x^I \oplus y^I = \{x + y, \ \forall x \in x^I, \ \forall y \in y^I\}$.

The interval arithmetics of a real interval scalar by itself should be differentiated from the arithmetics of two different scalar intervals. For the LTI system [1], we use the following definitions:

Definition 2.5: If x^I is not time dependent (i.e., time invariant), the addition of a real interval scalar x^I by itself is defined and calculated as: $x^I \oplus x^I = [\underline{x} + \underline{x}, \overline{x} + \overline{x}]$, the substraction is $x^I \ominus x^I = [0, 0]$, and the multiplication is $x^I \otimes x^I = \left[\alpha^2, \beta^2\right]$, where $\alpha = \min\{|\underline{x}|, |\overline{x}|\}$; $\beta = \max\{|\underline{x}|, |\overline{x}|\}$. The division is defined as: $x^I \oslash x^I = [1, 1]$ if $x^I \neq [0, 0]$.

Now, with the basic definitions given above, we define the linear (in)-dependency of interval vectors.

Definition 2.6: Let us suppose we have n different interval column vectors given as: $\mathbf{x}_1^I, \ldots, \mathbf{x}_n^I$. They are called *linearly independent* iff there exist only a trivial solution ($a_1 = a_2 = \cdots = a_n = 0$) such that $a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + \cdots + a_n\mathbf{x}_n = 0_m$ for all $\mathbf{x}_i \in \mathbf{x}_i^I$. Otherwise, we say that the interval vectors are *linearly dependent*.

Remark 2.2: In Definition 2.6, the linear independency of interval vectors was defined using the following notation: $a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + \cdots + a_n\mathbf{x}_n = 0_m$, for all $\mathbf{x}_i \in \mathbf{x}_i^I$. However, in this paper, since we use interval arithmetic, using notation $a_1\mathbf{x}_1^I \oplus a_2\mathbf{x}_2^I \oplus \cdots \oplus a_n\mathbf{x}_n^I = \mathbf{o}^I$ makes us deliver our ideas more easily. In other words, when we say there exist only a trivial solution for $a_1\mathbf{x}_1^I \oplus a_2\mathbf{x}_2^I \oplus \cdots \oplus a_n\mathbf{x}_n^I = \mathbf{o}^I$, this is equivalent to the linear independency condition of Definition 2.6.

Before considering the general case, let us first consider the linear (in)-dependency of two interval vectors. Supposing that two interval vectors are given as: $\mathbf{x}_1^I$ and $\mathbf{x}_2^I$, and based on Definition 2.6, two interval vectors are linearly independent iff there exist only trivial solutions $a_1 = a_2 = 0$ such that

$$a_1\mathbf{x}_1^I \oplus a_2\mathbf{x}_2^I = \mathbf{o}^I. \quad (2)$$

Here, notice that it is not easy to get solutions for (2) directly. However, if we use "ratio" concept, we can check the linear (in)-dependency property easily, which is expressed in the following theorem:

[1]For linear time varying case, we have to use Definition 2.3.

Theorem 2.1: Two n dimensional LTI interval vectors $\mathbf{x}^I, \mathbf{y}^I$ with $0 \notin x_1^I \cap x_2^I \cap \ldots \cap x_n^I$, $0 \notin y_1^I \cap y_2^I \cap \ldots \cap y_n^I$, are linearly independent iff, from the ratio $\mathbf{r}_{xy}$ of $\mathbf{x}^I, \mathbf{y}^I$, the following equality holds:

$$(\mathbf{r}_{xy})_1 \cap (\mathbf{r}_{xy})_2 \cap \cdots \cap (\mathbf{r}_{xy})_n = \emptyset, \quad (3)$$

where $(\mathbf{r}_{xy})_i$ can be defined as $x_i^I \oslash y_i^I$.

Proof: Sufficiency: From $a_1\mathbf{x}_1^I \oplus a_2\mathbf{x}_2^I = \mathbf{o}^I$, we have

$$a_1 \left[x_1^I, x_2^I, \ldots, x_n^I\right]^T = \ominus a_2 \left[y_1^I, y_2^I, \ldots, y_n^I\right]^T \quad (4)$$

From Definition 2.4 and Definition 2.5, and using the commutative and associative property of interval scalars, the ratio of each elements are

$$x_i^I \oslash y_i^I = (\mathbf{r}_{xy})_i$$
$$\Leftrightarrow x_i^I \otimes \frac{1}{y_i^I} = (\mathbf{r}_{xy})_i$$
$$\Leftrightarrow x_i^I \otimes \frac{1}{y_i^I} \otimes y_i^I = (\mathbf{r}_{xy})_i \otimes y_i^I$$
$$\Leftrightarrow x_i^I = (\mathbf{r}_{xy})_i \otimes y_i^I \quad (5)$$

By inserting (5) to the left-hand side of (4), the followings are true:

$$a_1 \left[(\mathbf{r}_{xy})_1 \otimes y_1^I, (\mathbf{r}_{xy})_2 \otimes y_2^I, \ldots, (\mathbf{r}_{xy})_n \otimes y_n^I\right]^T$$
$$= \ominus a_2 \left[y_1^I, y_2^I, \ldots, y_n^I\right]^T$$
$$\Leftrightarrow a_1 \left[(\mathbf{r}_{xy})_1, (\mathbf{r}_{xy})_2, \ldots, (\mathbf{r}_{xy})_n\right]^T = \ominus a_2 \ \mathbf{i}^I$$
$$\Leftrightarrow \left[(\mathbf{r}_{xy})_1, (\mathbf{r}_{xy})_2, \ldots, (\mathbf{r}_{xy})_n\right]^T = \ominus \frac{a_2}{a_1} \ \mathbf{i}^I \quad (6)$$

Here, from (6), we have

$$(\mathbf{r}_{xy})_1 \cap (\mathbf{r}_{xy})_2 \cap \cdots \cap (\mathbf{r}_{xy})_n = \ominus \frac{a_2}{a_1}, \quad (7)$$

so, since $(\mathbf{r}_{xy})_1 \cap (\mathbf{r}_{xy})_2 \cap \cdots \cap (\mathbf{r}_{xy})_n = \emptyset$, we have $\frac{a_2}{a_1} = \emptyset$. Thus, by Definition 2.3, only $a_1 = 0$ is the solution, henceforth, since $0 \notin y_1^I \cap y_2^I \cap \ldots \cap y_n^I$, from (4), we have $a_2 = 0$.

Necessity: Let us suppose that

$$(\mathbf{r}_{xy})_1 \cap (\mathbf{r}_{xy})_2 \cap \cdots \cap (\mathbf{r}_{xy})_n \neq \emptyset,$$

then we can have $a_2 = 0$ and $a_1 \neq 0$, or $a_2 \neq 0$ and $a_1 \neq 0$. Thus, by definition, this is not linearly independent. ■

Let us further think the case $0 \in x_1^I \cap x_2^I \cap \ldots \cap x_n^I$ or $0 \in y_1^I \cap y_2^I \cap \ldots \cap y_n^I$.

Theorem 2.2: If $0 \in x_1^I \cap x_2^I \cap \ldots \cap x_n^I$ or $0 \in y_1^I \cap y_2^I \cap \ldots \cap y_n^I$, two interval vectors are then linearly dependent.

Proof: With any a_1 and $a_2 = 0$, or with $a_1 = 0$ and any a_2, the following equality can be true:

$$a_1\mathbf{x}_1^I \oplus a_2\mathbf{x}_2^I = \mathbf{o}^I.$$

So, By Definition 2.6, the proof is completed. ■

Although above theorems are effective for checking the linear (in)-dependency of two interval vectors, it is difficult to extend above theorems to more than 3 interval vectors. Let us suppose that we have three different interval vectors, which are given as: $\mathbf{x}^I, \mathbf{y}^I, \mathbf{z}^I$ and we want to check the linear (in)-dependency of them. The first task is to check the linear dependency between two interval vectors. This task can be performed from preceding results, but we also have to check the linear combination case.

That is, we have to check if there exist trivial solutions $a_1 = a_2 = a_3 = 0$ such that

$$a_1\mathbf{x}^I \oplus a_2\mathbf{y}^I \oplus a_3\mathbf{z}^I = \mathbf{o}^I.$$

However, it looks quite tough to solve this simple equation, furthermore our ultimate goal is to find the general case such as:

$$a_1\mathbf{x}_1^I \oplus a_2\mathbf{x}_2^I \oplus \cdots \oplus a_n\mathbf{x}_n^I = \mathbf{o}^I.$$

So, apparently, it is almost impossible to check the linear (in)-dependency of the interval vectors [2]. In the sequel, we suggest one simple but very effective sufficient condition for checking the linear (in)-dependency of interval vectors $\mathbf{x}_1^I, \mathbf{x}_2^I, \cdots, \mathbf{x}_n^I$ where an $\mathbf{x}_i^I$ is an interval vector in $\mathcal{R}^m$. For the accurate description of our idea, we separately consider three different cases.

$$\text{Case}-1: m > n. \quad \text{Case}-2: m = n. \quad \text{Case}-3: m < n$$

We only investigate Case-1. In fact, Case-2 and Case-3 can be investigated using the analysis method of Case-1. For convenience, the following concepts are necessary. In the $m \times n$ matrix ($M = [m_{ij}],\ \ i = 1, \cdots, m$ and $j = 1, \cdots, n$ with $m > n$), let us select whole possible $n \times n$ sub-matrices. It is easy to notice that the total number of possible sub-matrices S^i is calculated by: $k = \begin{pmatrix} m \\ n \end{pmatrix} = \frac{m(m-1)(m-2)\cdots(m-n+1)}{n!}$. Sub-matrices S^i are composed of n different row vectors of M. The index of n different row vectors of S^i is represented by a set such as: $s^i = \{\text{index of row vectors of } M \text{ for } \mathrm{S^i}\}$, $i = 1, \ldots, k$. For the accurate translation of our idea, we make a definition as follows:

Definition 2.7: In this paper, we call sub-matrices $S_M = \left\{S^i,\ i = 1, \ldots, k\right\}$ as *square set* and S^i as *sub-square matrices*, and $s_M = \{s^i,\ i = 1, \ldots, k\}$ is called *index set* and s^i is called *index*.

Then, further definition can be made without proof for the linear (in)-dependency test of the interval vectors as follows:

Definition 2.8: The rank of M is maximum rank of S_i, that is, $\mathrm{rank}(M) = \max\{\mathrm{rank}(S_i), i = 1, \ldots, k\}$.

Now, we are ready to present our main result. Considering the interval vectors $\mathbf{x}_1^I, \mathbf{x}_2^I, \cdots, \mathbf{x}_n^I$, let us write these interval vectors in an interval matrix form such as:

$$X^I := \left(\mathbf{x}_1^I, \mathbf{x}_2^I, \cdots, \mathbf{x}_n^I\right) \tag{8}$$

Then, X^I is an $m \times n$ interval matrix, so based on Definition 2.7, the corresponding square set of X^I can be found as: $S_X = \left\{S^i,\ i = 1, \ldots, k\right\}$ where $k = \begin{pmatrix} m \\ n \end{pmatrix}$, and the corresponding index set of X^I can be found as $s_X = \{s^i,\ i = 1, \ldots, k\}$. Here, we introduce the center square matrices S_{X_c} such as:

$$S_{X_c} := \left\{ S_0^i = \frac{\underline{S^i} + \overline{S^i}}{2},\ i = 1, \ldots, k \right\},$$

and introduce the radius square matrices ΔS_X such as:

$$\Delta S_X := \left\{ \Delta S^i = \frac{\overline{S^i} - \underline{S^i}}{2},\ i = 1, \ldots, k \right\}$$

[2]As far as authors are concerned, nobody has suggested this kind of questions and there is no existing solution.

For our main result, notating the element-wise absolute value of a matrix A by $|A| = (|a_{ij}|)$, the following lemma can be adopted from [15].

Lemma 2.1: For interval square matrix X^I, let its center matrix X_0 be nonsingular [3] and the spectral radius $\rho\left(\left|(X_0)^{-1}\right| \Delta X\right) < 1$, then X^I is nonsingular [4].

Now, for the linear independency test of the interval vector set, we suggest the following theorem:

Theorem 2.3: For $S^I \in S_X$, if there exists at least one corresponding $S_0 \in S_{X_c}$ and $\Delta S \in \Delta S_X$ such that S_0 is nonsingular and $\rho\left(\left|(S_0)^{-1}\right| \Delta S\right) < 1$, then the interval vectors $\mathbf{x}_1^I, \mathbf{x}_2^I, \cdots, \mathbf{x}_n^I$ are linearly independent.

Proof: Let us consider $X^I = \left(\mathbf{x}_1^I, \mathbf{x}_2^I, \cdots, \mathbf{x}_n^I\right)$, which is an $m \times n,\ m > n$, interval matrix composed of the interval vectors. It is a fact that the column vectors are linearly independent if (and only if in the point of "rank") the rank of X^I is n. Also from the fact that the row rank is equal to the column rank, so if S^I has rank n, then the column rank of X^I is also n. Therefore, if any one of $S^I \in S_X$ has row rank n, then X^I has n column rank by Definition 2.8. So, by Lemma 2.1, for S_0 and ΔS corresponding to S^I, if S_0 is nonsingular and $\rho\left(\left|(S_0)^{-1}\right| \Delta S\right) < 1$, then X^I has full column rank, because the nonsingular condition is equivalent to the full rank condition. Thus, since the full column rank indicates the linear independency, the proof is completed. ■

Remark 2.3: Theorem 2.3 checks the linear independency of the interval vector set using finite interval matrix set. The key idea of Theorem 2.3 is to investigate the linear independency of the interval vectors on the form of interval matrix. Using the fact that the row rank is equal to column rank and the full rank condition is equivalent to the linear independency condition, Theorem 2.3 easily checks the linear independency of the interval vectors.

However, although Theorem 2.3 is represented in a simple form, the result could be conservative in checking the condition $\rho\left(\left|(S_0)^{-1}\right| \Delta S\right) < 1$, because $\left|(S_0)^{-1}\right|$ is used. To reduce the conservatism, the following result can be obtained based on Theorem 2.3.

Corollary 2.1: For at least one $S^I \in S_X$ and for its corresponding $S_0 \in S_{X_c}$ and $\Delta S \in \Delta S_X$, if there exists a matrix R such that

$$\rho\left(|I - RS_0| + |R|\, \Delta S\right) < 1,$$

then the interval vectors $\mathbf{x}_1^I, \mathbf{x}_2^I, \cdots, \mathbf{x}_n^I$ are linearly independent.

Proof: The proof can be completed by the proof of Theorem 2.3 and theorem 3.1 of [15]. ■

Using the proof of Theorem 2.3 and using the results of [15], we also can find the sufficient condition for linear dependency of the interval vectors $\mathbf{x}_1^I, \mathbf{x}_2^I, \cdots, \mathbf{x}_n^I$. Let us use the following lemma for this purpose.

Lemma 2.2: For interval matrix X^I, there exist a matrix R and a natural number p such that, element-wisely,

$$\left(I + |I - X_0 R|\right)_p \le \left(\Delta X\, |R|\right)_p$$

where $p \in \{1, \ldots, n\}$ and $(\cdot)_p$ represents p^{th} column, then interval matrix X^I is singular [5].

Proof: See theorem 3.3 of [15]. ■

[3]Nonsingular means that it is invertible.
[4]"Nonsingular" is equivalent to "full rank".
[5]Singular means it is not full rank.

Theorem 2.4: For all $S^I \in S_X$ and for all its corresponding $S_0 \in S_{X_c}$ and $\Delta S \in \Delta S_X$, if there exist a matrix R and a natural number p such that, element-wisely,

$$(I + |I - S_0 R|)_p \leq (\Delta S \, |R|)_p \, ,$$

then the interval vectors $\mathbf{x}_1^I, \mathbf{x}_2^I, \cdots, \mathbf{x}_n^I$ are linearly dependent.

Proof: Theorem 2.3 shows that the interval vectors are linearly independent if there exists at least one S^I such that the conditions of Theorem 2.3 hold. So, to eliminate the case of Theorem 2.3, we have to check all $S^I \in S_X$ for the linearly dependent test. That is, if all S_i are singular, then $\max(\text{rank}(S_i)) < n$, so from Definition 2.8, since $\text{rank}(M) = \max(\text{rank}(S_i))$, we have $\text{rank}(M) < n$. Thus, if $(I + |I - S_0 R|)_p \leq (\Delta S \, |R|)_p$, then by Lemma 2.2 and by Definition 2.6, interval vectors are linearly dependent. ■

Above results use the inverse of S_0, but, as commented in [15], this approach may be ineffective in the calculation of S_0^{-1}. Without using the inverse, we can derive the sufficient conditions for checking the linear dependency or independency. Based on theorem 4.1 of [15], the following result can be derived.

Corollary 2.2: For any $S^I \in S_X$, if there exist at least one corresponding $S_0 \in S_{X_c}$ and $\Delta S \in \Delta S_X$ such that

$$\lambda_{max} \left(\Delta S^T \Delta S \right) < \lambda_{min} \left(S_0^T S_0 \right),$$

then the interval vectors $\mathbf{x}_1^I, \mathbf{x}_2^I, \cdots, \mathbf{x}_n^I$ are linearly independent.

Proof: By theorem 3.3 of [15] and due to the same reason as Theorem 2.3, the proof is straightforward. ■

The sufficient condition for the linear dependency can also be obtained using eigenvalues as:

Corollary 2.3: For all $S^I \in S_X$, if there exist corresponding $S_0 \in S_{X_c}$ and $\Delta S \in \Delta S_X$ such that

$$\lambda_{max} \left(S_0^T S_0 \right) \leq \lambda_{min} \left(\Delta S^T \Delta S \right),$$

then the interval vectors $\mathbf{x}_1^I, \mathbf{x}_2^I, \cdots, \mathbf{x}_n^I$ are linearly dependent.

Proof: By theorem 3.3 of [15] and due to the same reasons as Theorem 2.3 and Theorem 2.4, the proof is straightforward. ■

In this section, we defined "linear dependency" and "linear independency" of "interval vectors" and suggested sufficient checking methods. Even though checking linear (in)-dependency of interval vectors looks NP hard problem, we solved these problems by forming interval matrices from interval vectors. Our key idea is straightforward, hence the suggested sufficient conditions are very simple. Notice that in interval vector, in addition to the linear dependency and independency problems discussed in this paper, there exist many interesting issues such as "interval vector norm", "null space of interval matrices", "interval multi-input control problem", and etc. Authors observe that the linear (in)-dependency problem of interval vectors can be attacked in other mathematical frameworks. These works will be further studied in our future efforts. In next section, we will show that the linear (in)-dependency property of interval vectors can be effectively used in checking the robust controllability and un-controllability of the uncertain interval LTI system.

III. Robust Controllability Test of Interval System

The robust controllability problem of uncertain linear system has been steadily studied in [16], [17], [18], [19] and therein references. Most notably, the methods suggested in [17], [19] provide algebraically elegant derivations. However, unfortunately, their methods, in instinct, cannot avoid the conservatism; hence regardless the algebraic simplification, their contribution may be limited. In this section, we provide an alternative method developed based on interval vectors, which is very simple but much less conservative. The following LTI uncertain system is considered:

$$\dot{x} = Ax + Bu \tag{9}$$

where $x \in \mathcal{R}^n$, $u \in \mathcal{R}^r$, $A \in \mathcal{R}^{n \times n}$, $B \in \mathcal{R}^{n \times r}$, $\text{rank}(B) = r$, and $A \in A^I = [\underline{A}, \overline{A}]$ and $B \in B^I = [\underline{B}, \overline{B}]$. We call the interval uncertain system (9) is controllable if $\text{rank}(C) = n$ for all $C \in C^I$

$$C^I = [B^I, A^I \otimes B^I, A^I \otimes A^I \otimes B^I, \ldots, \underbrace{A^I \otimes \cdots \otimes A^I}_{n-r} \otimes B^I],$$

which is $n \times (n - r + 1) \cdot r$ interval matrix. For convenience, $m \equiv (n - r + 1) \cdot r$. In fact, the main source of conservatism of [17], [19] is due to the fact that they used C^I without any modification for the controllability test. We explain this in more detail in the sequel.

First let us consider the case without interval such as:

$$\dot{x} = A_0 x + B_0 u \tag{10}$$

and corresponding controllability matrix like

$$C_0 = [B_0, A_0 B_0, (A_0)^2 B_0, \ldots, (A_0)^{n-r} B_0].$$

If the system is controllable, then always $\text{rank}(C_0) = n$. To distinguish the interval case from the without interval case, let us suppose that the rank of following sub-matrix of C_0

$$C_0' = [B_0, A_0 B_0, (A_0)^2 B_0, \ldots, (A_0)^{n-r-q} B_0]$$

where $q \geq 1$, is n (i.e., $\text{rank}(C_0') = n$). Then, without interval, it is always true that $\text{rank}(C_0') = \text{rank}(C_0) = n$. Now, let us include interval. In this case, we have to check the rank of C^I, but since C^I is $n \times m$ interval matrices, it is not easy to find the rank of C^I. Thus, in [17], [19], inevitably, they tried to find some inequality conditions in matrix norm to guarantee the sufficient conditions of LTI interval system (see Eq. (3.9) in [19] and Eq. (10) in [17]). Using these inequalities, they found the upper boundaries for sufficient condition, but in this upper boundary calculation, the formula is so conservative (see the derivation of Theorem 1 of [19] and Eq. (3.6) of [17]). So, even there is ignorable interval uncertainty in $(C^I)'$, which is defined as:

$$(C^I)' = [B^I, A^I \otimes B^I, \ldots, \underbrace{A^I \otimes \cdots \otimes A^I}_{n-r-q} \otimes B^I]$$

the overall upper bounds are calculated based on the maximum interval uncertainty of C^I. So, the controllability checking methods of [17], [19] instinctively are very conservative. However, if we can check the rank of C^I using its sub-matrices $(C^I)'$, the result could be much less conservative. In fact, this can be done by checking the linear independency property of the interval vectors based on the results of Section II. For this, we make a formula as follows:

Corollary 3.1: If the controllability matrix C^I satisfies the linear independency conditions of Theorem 2.3, then the uncertain interval system is controllable.

Proof: From the fact that the interval system is controllable if its controllability matrix has rank n and the full rank condition is equivalent to the linear independency condition, the proof is immediate. ■

Corollary 3.2: If the controllability matrix C^I satisfies the linear independency conditions of Corollary 2.1, then the uncertain interval system is controllable.

Next, let us check the superiority of the suggested method. For the comparison with the existing results, the three examples given in [19] are used. Note, in the following examples, $a \pm \delta a$ denotes that a is an interval such as $a \in [a-\delta a, a+\delta a]$.

Example 1:

$$A \in A^I = \begin{pmatrix} 1\pm 0.05 & 0 & 0 \\ 0 & 1\pm 0.04 & 1\pm 0.03 \\ 0 & -2\pm 0.08 & 4\pm 0.4 \end{pmatrix}; B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$$

The controllability matrix C is calculated from the interval arithmetics as:

$$C \in C^I = \begin{pmatrix} 1 & 0 & 1\pm 0.05 & 0 \\ 0 & 0 & 0 & 1\pm 0.03 \\ 0 & 1 & 0 & 4\pm 0.4 \end{pmatrix}$$

So, we have four sub-square matrices:

$$S^1 \in \begin{pmatrix} 1 & 0 & 1\pm 0.05 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}; S^2 \in \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1\pm 0.03 \\ 0 & 1 & 4\pm 0.4 \end{pmatrix};$$

$$S^3 \in \begin{pmatrix} 1 & 1\pm 0.05 & 0 \\ 0 & 0 & 1\pm 0.03 \\ 0 & 0 & 4\pm 0.4 \end{pmatrix}; S^4 \in \begin{pmatrix} 0 & 1\pm 0.05 & 0 \\ 0 & 0 & 1\pm 0.03 \\ 1 & 0 & 4\pm 0.4 \end{pmatrix}$$

Then, from S^2, we have

$$S_0^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 4 \end{pmatrix}; \quad \Delta S^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0.03 \\ 0 & 0 & 0.4 \end{pmatrix}$$

Therefore, since S_0^2 is nonsingular and $\rho\left(\left|(S_0^2)^{-1}\right|\Delta S^2\right) = 0.03 < 1$, easily we confirm that the interval system is controllable. However, in [19], they conclude that their method cannot check the controllability directly, which is due to the conservatism of their method as already explained. Clearly, our method is much less conservative. In [19], the following sign variant problem was also given:

Example 2:

$$A \in A^I = \begin{pmatrix} 0\pm 0.05 & 0 & 0 \\ 0 & 1\pm 0.04 & 1\pm 0.03 \\ 0 & 0\pm 0.08 & 0\pm 0.4 \end{pmatrix}; B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Using the same method as Example 1, we found that from S^2, S_0^2 is nonsingular and $\rho\left(\left|(S_0^2)^{-1}\right|\Delta S^2\right) = 0.03 < 1$. So, regardless the sign variation, easily we find that the interval system is controllable. However, in [19], they used controller K to guarantee the controllability, but as resulted from our method, the system is already controllable. So, their approach requires the extra work, which is not necessary in our method. The following example includes the interval in B:

Example 3:

$$A \in A^I = \begin{pmatrix} 1\pm 0.02 & 0 & 0 \\ 0 & 1\pm 0.02 & 1\pm 0.02 \\ 0 & -2\pm 0.05 & 4\pm 0.09 \end{pmatrix}$$

$$B \in B^I = \begin{pmatrix} 1\pm 0.025 & 0 \\ 0 & 0 \\ 0 & 1\pm 0.02 \end{pmatrix}$$

Using the same method, we found that from S^2, S_0^2 is nonsingular and $\rho\left(\left|(S_0^2)^{-1}\right|\Delta S^2\right) = 0.04 < 1$, so the system is controllable. From these examples, it is clear that our method is very simple and much less conservative than the existing method in checking the robust controllability of the uncertain LTI system.

In next examples, we check the (un)-controllability of the interval system using Theorem 2.4. For convenience, from Theorem 2.4, we make the following corollary:

Corollary 3.3: For all $S^I \in S_X$ and for all its corresponding $S_0 \in S_{X_c}$ and $\Delta S \in \Delta S_X$, if there exist a natural number p such that, element-wisely,

$$I_p \leq \left(\Delta S \left|(S_0)^{-1}\right|\right)_p,$$

then the interval system is uncontrollable.

Proof: In Theorem 2.4, by replacing R by $(S_0)^{-1}$, and based on Theorem 2.4, the proof is immediate. ■

Example 4: Let us consider the fully-populated interval A matrix and interval B matrix such as:

$$A \in A^I = \begin{pmatrix} 1\pm 1\times\alpha & 2\pm 2\times\alpha & -1\pm 1\times\alpha \\ -2\pm 2\times\alpha & 1\pm 1\times\alpha & 1\pm 1\times\alpha \\ 0.5\pm 0.5\times\alpha & -2\pm 2\times\alpha & 4\pm 4\times\alpha \end{pmatrix}$$

$$B \in B^I = \begin{pmatrix} 1\pm 1\times\alpha & 0 \\ 0 & 0 \\ 0 & 1\pm 1\times\alpha \end{pmatrix}$$

To check the conservatism, we vary α. That is, we test different percent interval uncertainties in A^I matrix and B^I matrix. The controllability and un-controllability are checked by Corollary 3.1 and Corollary 3.3, respectively. For example, with $\alpha = 0.1$, the controllability matrix is calculated as:

$$C \in C^I = \begin{pmatrix} 1\pm 0.1 & 0 & 1\pm 0.21 & -1\pm 0.21 \\ 0 & 0 & -2\pm 0.42 & 1\pm 0.21 \\ 0 & 1\pm 0.1 & 0.5\pm 0.105 & 4\pm 0.84 \end{pmatrix}$$

So, from S^1, S^2, S^3, we calculate $\rho\left(\left|(S_0^i)^{-1}\right|\Delta S^i\right)$ as 0.21, 0.21, and 0.3088, respectively; hence the system is controllable. However in Corollary 3.3, we calculate $\left(\Delta S\left|(S_0)^{-1}\right|\right)$ from S^1, S^2, S^3, and S^4 as:

$$\left(\Delta S^1\left|(S_0^1)^{-1}\right|\right) = \begin{pmatrix} 0.1000 & 0.1550 & 0 \\ 0 & 0.2100 & 0 \\ 0 & 0.0775 & 0.1000 \end{pmatrix};$$

$$\left(\Delta S^2\left|(S_0^2)^{-1}\right|\right) = \begin{pmatrix} 0.1000 & 0.3100 & 0 \\ 0 & 0.2100 & 0 \\ 0 & 1.2400 & 0.1000 \end{pmatrix}$$

$$\left(\Delta S^3\left|(S_0^3)^{-1}\right|\right) = \begin{pmatrix} 0.1000 & 0.1641 & 0.0859 \\ 0 & 0.2100 & 0.0988 \\ 0 & 0.0988 & 0.2100 \end{pmatrix};$$

$$\left(\Delta S^4\left|(S_0^4)^{-1}\right|\right) = \begin{pmatrix} 0.6300 & 0.4200 & 0 \\ 0.8400 & 0.6300 & 0 \\ 2.6350 & 1.3950 & 0.1000 \end{pmatrix}$$

Thus, since S^1, S^2, and S^3 do not satisfy $I_p \leq \left(\Delta S\left|(S_0)^{-1}\right|\right)_p$, we cannot conclude that the system is uncontrollable. Now, we increase α, and the test results are summarized in Table I. In table, $\surd$ confirms (un)-controllability, but $\cdot$ represents that (un)-controllability cannot be checked. So, in this case, there is

TABLE I

THE CONTROLLABILITY AND UNCONTROLLABILITY TESTS OF EXAMPLE-4

α	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
Controllability	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	·	·
Uncontrollability	·	·	·	·	·	·	·	$\surd$	$\surd$

TABLE II

THE CONTROLLABILITY AND UNCONTROLLABILITY TESTS OF EXAMPLE-5

α	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
Controllability	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	·	·	·	·
Uncontrollability	·	·	·	·	·	·	$\surd$	$\surd$	$\surd$

almost no conservatism in checking the controllability and un-controllability. Example-5: Let us consider the fully-populated interval B

$$A \in A^I = \begin{pmatrix} 1 \pm 1 \times \alpha & 2 \pm 2 \times \alpha & -1 \pm 1 \times \alpha \\ -2 \pm 2 \times \alpha & 1 \pm 1 \times \alpha & 1 \pm 1 \times \alpha \\ 0.5 \pm 0.5 \times \alpha & -2 \pm 2 \times \alpha & 4 \pm 4 \times \alpha \end{pmatrix}$$

$$B \in B^I = \begin{pmatrix} 1 \pm 1 \times \alpha & -0.1 \pm 0.1 \times \alpha \\ 0.1 \pm 0.1 \times \alpha & 0.1 \pm 0.1 \times \alpha \\ -0.1 \pm 0.1 \times \alpha & 1 \pm 1 \times \alpha \end{pmatrix}$$

From Corollary 3.1 and Corollary 3.3, we have the test results as shown in Table II. In this case, with 35 percent uncertainty, we cannot conclude the controllability nor un-controllability. So, with fully populated B matrix, there exists conservatism, because with 35 percent uncertainty, we cannot draw any conclusion about the system from Corollary 3.1 and Corollary 3.3.

Remark 3.1: The robust observability is dual to the robust controllability problem and can be checked similarly as done in the robust controllability.

Remark 3.2: In [17], [19], they provided the necessary and sufficient condition for checking the robust controllability. However, this approach is not practically meaningful, and furthermore, our method can be developed for the necessary and sufficient conditions as done in [17], [19]. In above examples, we just showed that our method, on its own, is much more simple and the result is less conservative than existing results. Furthermore, from the fact that our method can check the un-controllability, it is clear that our method is advantageous over the existing methods.

IV. CONCLUSIONS

In this paper, we suggested the concept of "linear dependency" and "linear independency" of interval vectors and for the possible application, we applied our result to the robust (un)-controllability tests of the uncertain interval LTI system. From the tests with existing examples, we could verify that, from our method, the controllability of the interval system was checked with much less conservatism and in a simple manner. For the un-controllability tests, we developed two examples; even if there exists conservatism in Example-5, from the tests, we found that the suggested methods based on interval vectors can also be effectively used to check the robust un-controllability of the uncertain interval systems.

REFERENCES

[1] R. E. Moore, *Interval Analysis*, Prentice-Hall, Englewood Cliffs, NJ, 1966.

[2] Luc Jaulin, Michel Kieffer, Olivier Didrit, and Éric Walter, *Applied Interval Analysis*, Springer, 2001.

[3] P. Batra, "On necessary conditions for real robust Schur-stability," *IEEE Trans. on Automatic Control*, vol. 48, no. 2, pp. 259–261, 2003.

[4] Richard Greiner, "Necessary conditions for Schur-stability of interval polynomials," *IEEE Trans. on Automatic Control*, vol. 49, no. 5, pp. 740 – 744, 2004.

[5] Y. C. Soh and R. J. Evans, "Stability analysis of interval matrices-continuous and discrete systems," *Int. J. Control*, vol. 47, no. 1, pp. 25–32, 1988.

[6] C. I. Jiang, "Sufficient conditions for the asymptotic stability of interval matrices," *Int. J. Control*, vol. 46, pp. 1803–1810, 1987.

[7] M. Mansour, "Simplified sufficient conditions for the asymptotic stability of interval matrices," *Int. J. Control*, vol. 50, pp. 443–444, 1989.

[8] Kaining Wang, Anthony N. Michel, and Derong Liu, "Necessary and sufficient conditions for the Hurwitz and Schur stability of interval matrices," *IEEE Trans. on Automatic Control*, vol. 39, no. 6, pp. 1251–1255, 1994.

[9] David Hertz, "The extreme eigenvalues and stability of real symmetric interval matrices," *IEEE Trans. on Automatic Control*, vol. 37, no. 4, pp. 532–535, 1992.

[10] J. Rohn, "Positive definiteness and stability of interval matrices," *SIAM J. Matrix Anal. Appl.*, vol. 15, no. 1, pp. 175–184, 1994.

[11] B. Ross Barmish, *New Tools for Robustness of Linear Systems*, Macmillan Publishing Company, New York, 1994.

[12] S. P. Bhattacharyya, H. Chapellat, and L. H. Keel, *Robust Control: The Parameter Approach*, Prentice Hall, 1995.

[13] Svetoslav Markov, "Quasivector spaces and their relation to vector spaces," *Submitted to Electronic Journal on Mathematics of Computation*, 2004.

[14] Svetoslav Markov, "On the algebraic properties of intervals and some applications," *Reliable Computing*, vol. 7, no. 2, pp. 113–127, 2001.

[15] Georg Rex and Jiri Rohn, "Sufficient conditions for regularity and singularity of interval matrices," *SIAM J. Matrix. Anal. Appl*, vol. 20, no. 2, pp. 437–445, 1998.

[16] S. S. Sastry and C. A. Desoer, "The robustness of controllabiity and observability of linear time-varying systems," *IEEE Trans. on Automatic Control*, vol. 27, no. 4, pp. 933–939, 1982.

[17] Kaining Wang and Anthony N. Michel, "Necessary and sufficient conditions for the controllability and observability of a class of linear, time-invaariant systems with interval plants," *IEEE Trans. on Automatic Control*, vol. 39, no. 7, pp. 1443–1447, 1994.

[18] C. Lin, J. L. Wang, G. H. Yang, and C. B. Soh, "Robust C-controllabiity and/or C-observability for uncertain descriptor system with inerval perturbations in all matrices," *IEEE Trans. on Automatic Control*, vol. 44, no. 9, pp. 1768–1773, 1999.

[19] Bangwen Cheng and Jing Zhang, "Robust controllabiity for a class of uncertain linear time-invariant mimo systems," *IEEE Trans. on Automatic Control*, vol. 49, no. 11, pp. 2022–2027, 2004.