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Linear Output-Feedback Control of Stochastic Linear Systems with State- and Control-dependent Disturbances

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Abstract— **The stochastic regulation problem for linear systems with state- and control-dependent noise and a noisy linear output equation is considered. The optimal, quadratic cost, output-feedback control law in a class of linear controllers is found. The result represent an extension to the incomplete information case of the class of optimal control problems just considered in the 70's by McLane. As a result, the optimal control law result to be similar to the Linear-Quadratic-Gaussian (LQG) regulator, but a new non-negative matrix (associated to the quadratic-variation of the control-dependent noise) must to be added to the output noise covariance.**

I. INTRODUCTION

The stochastic control problem of linear systems subjected to additive disturbances whose intensity depends on the state and control of the controlled system is important in a wide class of applications, among which we point out aerospace/aeronautic or financial ones. The problem has been widely studied in the literature and various suboptimal solution has been proposed in particular cases. At this purpose we recall the survey papers by Wohnam [1] where results are found in the complete information case (the system state is accessible without error to the measurement device), and [2] which is another survey-paper where, again for the complete information case, McLane searches directly for the *linear*-state-feedback controller for the class of linear systems with control and state dependent noise. The method used by McLane consists essentially in applying a matrix version of the minimum principle [3] after having reduced the original problem to a deterministic matrix optimization one. In the present paper we are concerned with the general case of a linear output-regulator, when state- and control-dependent noises are present in the state equation, and the output equation is corrupted by Gaussian noise (incomplete information).

Control-dependent disturbances are often encountered in control and information systems. One important example is the human operator in a control task. From and intuitive point of view, the error made in these systems in tracking an input signal depends, among other things, upon the intensity of the input signal. Another example of control-dependent disturbance occurs in aerospace engineering, when a gas-jet thrusting system is used for the attitude control of a satellite. As a matter of fact the malalignement angle of the thruster should be modeled as a stochastic process (a noise), thus the control action will add a new, and control-dependent, disturbance. As a further example we point out problems arising in reinsurance-dividend management [4], [5], where the state process is the value of the liquid assets of a company, and the control is given by the dividend rate paid out to the shareholder and by the reinsurance fraction. In this model a disturbance depending of the reinsurance fraction is present in the state equation, and thus is control-dependent. As a physical example of state-dependent disturbance, we point out again one example occurring in aerospace systems. Spacecraft are often rotated around their symmetry axis in order to enhance their aerodynamic stability upon reentry, or to create an artificial gravitational field during their stay in the deep space. In these cases, the momentum exchange method for regulating the angular precession of the rotating spacecraft introduce a disturbance that depends on the precession rates and is thus state-dependent.

It should be stressed that in all the above mentioned papers, the problem is attacked by a suboptimal methodology, in the sense of searching for the mimimum of the cost functional among the linear feedbacks of the system state or output, depending on what case – complete or incomplete information, respectively – is under consideration. Such a suboptimal methodology results often to be more convenient in practice than the more general methods where the *optimal* regulator is searched for wide classes of nonlinear stochastic systems and cost functionals by reducing the incomplete-information original problem to a complete-information but infinite-dimensional one (see for instance [6], [7]). In the paper [8] this suboptimal methodology has been fully exploited – a general *polynomial* controller has been developed – in the discrete-time case for linear systems corrupted by additive Gaussian white noises. The polynomial-optimal controller is shown to be a linear function of the polynomial-optimal state estimate, thus the control problem is solved by cascading the controller with the *polynomial filter* for linear discrete-time non-Gaussian systems [9]. As far as the problem in the complete information case is concerned, we point out the papers [10], [11] where a generalized control-system with state- and control-dependent noise, stochastic coefficients (of both system and cost index), and singular control-weight coefficient, is considered. It is shown that an optimal controller (state-feedback) exists provided a solution exists for a generalized *stochastic* Riccati equation (and particular cases are discussed where a solution indeed exists for such kind of Riccati equation). When all the coefficients reduces to deterministic ones, and the control-weight is non-

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singular, the state-feedback controller equation becomes that just studied in [2], [1]. Thus [10], [11] appears to be a generalization of [2], [1], but the generalization to output-feedback controllers remains up to now unsolved for the case of a general *noisy* observation equation (the incomplete information case).

The aim of this paper is to provide the solution of the linear regulator problem for the class of stochastic systems with state- and control-dependent noise in the incomplete-information case with a linear noisy output equation. We consider a system described by Ito equations with deterministic coefficients, and a quadratic-cost index with deterministic weights. We show that the resulting control-scheme is very similar to the one described in [1], [2] (and [10], [11]) for the complete information case but the *linear optimal state-estimate* replaces the current state in the expression of the feedback-control. We stress that this result can be interpreted as a *separation theorem* in a "suboptimal" sense. Moreover, when the incomplete-information case and a control-dependent noise are considered, the controller results in an additional term, depending on the quadratic-variation of the control-dependent noise itself. The method used here is different from that used by McLane in [2], and allows one to find the results in the complete-information case in a more simple way (see for instance [12] where only a state-dependent noise has been considered). Our method to derive the linear optimal regulator is based upon an extension of the suboptimal filter described in the paper [13]. In particular, the linear optimal filter is here derived for a bilinear system with an additional control-dependent noise and an additional observable term (the control) in the state equation. The linear-optimal controller defined in the present paper represents a first, and necessary, step towards the fully exploitment of the suboptimal methodology above outlined, that is the definition of a general *polynomial controller*.

The paper is organized as follows. In §2 the precise setting of the considered control problem is given. In §3 the solution of the suboptimal linear-feedback control problem is given. Finally, in §4 the solution of the filtering problem associated to the control problem is presented.

II. SETTING OF THE PROBLEM

Let us consider the following stochastic system:

$$\begin{aligned} dX_t &= A(t)X_t dt + H(t)u_t dt + \sum_{k=1}^{q} B^k(t)X_t dW_t^k \\ &+ \sum_{k=1}^{d} D^k(t)u_t dN_t^k, \qquad X_{t_0} = \overline{X}, \qquad (1) \\ dY_t &= C(t)X_t dt + dW'_t, \qquad Y_{t_0} = 0, \qquad (2) \end{aligned}$$

where W, W' and N_t are, respectively, the standard q, m, and d-dimensional Brownian motions. Moreover, $t \in I$, $I = [t_0, t_f] \subset \mathbb{R}$, $A(t) \in \mathbb{R}^{n\times n}$, $H(t) \in \mathbb{R}^{n\times p}$, $C(t) \in \mathbb{R}^{m\times n}$, $B^k(t) \in \mathbb{R}^{n\times n}$, $k = 1, ..., q$, are continuous matrix functions. The control function $u : \Omega \times I \to \mathbb{R}^p$ is assumed to be adapted to the non-decreasing family $\{\mathcal{F}_t^Y\}_{t\in I}$. We will denote with $\mathcal{L}_t^i(Y)$ the set of $\mathbb{R}^i$-valued linear transformations of $\{Y_s;\ s \in I, s \le t\}$. One has that $\mathcal{L}_t^i(Y)$ is a closed linear subspace of $L^2(\mathbb{R}^i)$ and hence it is well defined $\widehat{X}_t = \mathbf{\Pi}\left\{X_t/\mathcal{L}_t^n(Y)\right\}$, where $\mathbf{\Pi}$ denotes the orthogonal projection onto a subspace of random variables. that will be referred as the *linear-optimal estimate* of the state X. We shall represent with $\mathcal{L}^i(Y)$ the set of functions $\xi : I \times \Omega \to \mathbb{R}^i$ such that $\xi_t \in \mathcal{L}_t^i(Y)$, for all $t \in I$. The statement of the suboptimal linear-feedback control problem that we will consider in the present paper is the following:

$$\begin{aligned} &\min_{u\in\mathcal{L}^p(Y)} J(u), \\ J(u) &= \frac{1}{2}\mathbf{E}\Big\{ \int_{t_0}^{t_f} \Big(X_t^T Q(t) X_t + u_t^T R(t) u_t\Big) dt \\ &+ X_{t_f}^T F X_{t_f} \Big\}, \qquad (3) \end{aligned}$$

where $\forall t$, $Q(t) = Q(t)^T \ge 0$, $R(t) = R(t)^T > 0$, and $F = F^T \ge 0$, under the differential constraints represented by system (1), (2).

III. SOLUTION OF THE LINEAR OUTPUT-FEEDBACK CONTROL PROBLEM

Lemma 3.1. *Let $V(t) \in \mathbb{R}^{n\times n}$, $\forall t$, M_t the martingale defined by:*

$$M_t = \sum_{k=1}^{q} \int_{t_0}^{t} B^k(\tau) X_\tau dW_\tau^k + \sum_{k=1}^{d} \int_{t_0}^{t} D^k(\tau) u_\tau dN_\tau^k,$$

where W, N are independent Wiener processes in $\mathbb{R}^q$, $\mathbb{R}^d$ respectively, and X is a process such that X_t is independent of $\{W_s, N_s; s \le t\}$. Then the following equality holds:

$$\begin{aligned} &\tfrac{1}{2}\textstyle\sum_{i,j=1}^{n} \left(\frac{\partial^2 (x^T V(t) x)}{\partial x_i \partial x_j}\right)_{x=X_t} d\langle M^i, M^j\rangle_t \\ &= X_t^T \Lambda(t) X_t dt + u_t^T \Delta(t) u_t dt, \qquad (4) \end{aligned}$$

with the matrices $\Lambda(t)$, $\Delta(t)$ given by:

$$\Lambda(t) = \sum_{k=1}^{q} B^k(t)^T V(t) B^k(t), \qquad (5)$$

$$\Delta(t) = \sum_{k=1}^{d} D^k(t)^T V(t) D^k(t). \qquad (6)$$

Proof. For the sake of simplicity, we omit the time dependence of the matrix functions. We denote by $G_{(i)}^k$ the i-th row of the matrix G^k. Let us compute the mutual quadratic variation process of the i-th and j-th entries of the process

M:

$$d\langle M^i, M^j\rangle_t = \sum_{k=1}^{q}\sum_{\sigma=1}^{n}\sum_{\sigma'=1}^{n} B^k_{i,\sigma} B^k_{j,\sigma'} X^\sigma_t X^{\sigma'}_t dt$$
$$+ \sum_{k=1}^{d}\sum_{\sigma=1}^{p}\sum_{\sigma'=1}^{p} D^k_{i,\sigma} D^k_{j,\sigma'} u^\sigma_t u^{\sigma'}_t dt$$
$$= \sum_{k=1}^{q} X_t^T {B^k_{(i)}}^T B^k_{(j)} X_t dt + \sum_{k=1}^{d} u_t^T {D^k_{(i)}}^T D^k_{(j)} u_t dt.$$

Since $(1/2)\partial^2(x^T V x)/\partial x_i \partial x_j = V_{ij}$, the first member of (4) results in:

$$\frac{1}{2}\sum_{i,j=1}^{n}\left(\frac{\partial^2(x^T V x)}{\partial x_i \partial x_j}\right)_{x=X_t} d\langle M^i, M^j\rangle_t$$
$$= \sum_{k=1}^{q}\sum_{i,j=1}^{n} V_{ij} X_t^T {B^k_{(i)}}^T B^k_{(j)} X_t dt$$
$$+ \sum_{k=1}^{d}\sum_{i,j=1}^{p} V_{ij} u_t^T {D^k_{(i)}}^T D^k_{(j)} u_t dt.$$

Eq. (4) follows by $\sum_{i,j=1}^{n} V_{ij} {B^k_{(i)}}^T B^k_{(j)} = {B^k}^T V B^k$, and $\sum_{i,j=1}^{p} V_{ij} {D^k_{(i)}}^T D^k_{(j)} = {D^k}^T V D^k$. •

Theorem 3.2. *The solution of the suboptimal control problem* (3)*, is given by:*

$$u^o_t = L^o(t)\widehat{X}_t,$$

where $\widehat{X}_t$ is the optimal linear estimate of X_t and

$$L^o(t) = -(R(t) + \Delta(t))^{-1} H(t)^T V(t),$$

where $\Delta(t)$ is defined in (6)*, and $V(t)$ is the (symmetric) matrix solution of the following backward Riccati equation:*

$$\dot{V}(t) = -A(t)^T V(t) - V(t)A(t) - Q(t) - \sum_{k=1}^{q} B^k(t)^T V(t) B^k(t) + V(t)H(t)(R(t)+\Delta(t))^{-1} H^T(t) V(t), \quad (7)$$
$$V(t_f) = F. \quad (8)$$

Proof. Let $\overline{R} > 0$ such that

$$R(t) = \overline{R}(t) - \Delta(t). \quad (9)$$

Equation (7), endowed with the final condition (8), admits a unique solution $V(t) \geq 0$, absolutely continuous in $[t_0, t_f]$ (see [2], [14]). Let us define the process $\xi_t = X_t^T V(t) X_t$. Since, by (8), $\int_{t_0}^{t_f} d\xi_\tau = X_{t_f}^T F X_{t_f} - X_{t_0}^T V(t_0) X_{t_0}$, we can rewrite index $J(u)$ of (3) as follows:

$$J(u) = \frac{1}{2}\mathbf{E}\Bigg\{\int_{t_0}^{t_f}\Big(X_t^T Q(t) X_t + u_t^T R(t) u_t\Big) dt + \int_{t_0}^{t_f} d\xi_\tau + X_{t_0}^T V(t_0) X_{t_0}\Bigg\}. \quad (10)$$

By applying the Ito formula to the process ξ, we obtain

$$d\xi_t = X_t^T \dot{V}(t) X_t dt + dX_t^T V(t) X_t + X_t^T V(t) dX_t + \frac{1}{2}\sum_{i,j}\left(\frac{\partial^2(x^T V(t) x)}{\partial x_i \partial x_j}\right)_{x=X_t} d\langle M^i, M^j\rangle_t, \quad (11)$$

where

$$dM_t = \sum_{k=1}^{q} B^k(t) X_t dW^k_t + \sum_{k=1}^{d} D^k(t) u_t dN^k_t.$$

Now, using Lemma 3.1, one has:

$$\frac{1}{2}\sum_{i,j=1}^{n}\left(\frac{\partial^2(x^T V(t) x)}{\partial x_i \partial x_j}\right)_{x=X_t} d\langle M^i, M^j\rangle_t = X_t^T \Lambda(t) X_t dt + u_t^T \Delta(t) u_t dt, \quad (12)$$

with the matrices $\Lambda(t)$, $\Delta(t)$ given by (5), (6). Substituting (12) and the right-hand side of (1) in (11) results in:

$$\begin{aligned} d\xi_t = \; & \big(X_t^T \dot{V}(t) X_t + X_t^T A(t)^T V(t) X_t \\ & + u_t^T H(t)^T V(t) X_t + X_t^T V(t) A(t) X_t \\ & + X_t^T V(t) H(t) u_t + X_t^T \Lambda(t) X_t + u_t^T \Delta(t) u_t\big) dt \\ & + \sum_{k=1}^{q} X_t^T B^k(t)^T V(t) X_t dW^k_t \\ & + \sum_{k=1}^{q} X_t^T V(t) B^k(t) X_t dW^k_t \\ & + \sum_{k=1}^{d} u_t^T D^k(t)^T V(t) X_t dN^k_t \\ & + \sum_{k=1}^{d} X_t^T V(t) D^k(t) u_t dN^k_t. \end{aligned} \quad (13)$$

By substituting (13) in (10), taking into account that the expectations of the noise-dependent terms in (13) vanish, and finally taking into account (7), it follows that:

$$\begin{aligned} J(u) = \; & \frac{1}{2}\mathbf{E}\Bigg\{\int_{t_0}^{t_f}\Big(u_t^T \overline{R}(t) u_t + u_t^T H(t)^T V(t) X_t \\ & + X_t^T V(t) H(t) u_t + X_t^T V(t)(H\overline{R}^{-1} H^T)(t) \\ & \cdot V(t) X_t\Big) dt + X_{t_0}^T V(t_0) X_{t_0}\Bigg\}. \end{aligned} \quad (14)$$

Let us consider the matrix

$$L^o(t) = -\overline{R}(t)^{-1} H(t)^T V(t),$$

then the following equalities hold:

$$\begin{aligned} L^o(t)^T \overline{R}(t) L^o(t) &= V(t)(H\overline{R}^{-1} H^T)(t) V(t), \\ L^o(t)^T \overline{R}(t) &= -V(t) H(t). \end{aligned}$$

Thus, substituting in (14), the index $J(u)$ is finally given by:

$$J(u) = \frac{1}{2}\mathbf{E}\Bigg\{ \int_{t_0}^{t_f} \big(u_t - L^o(t)X_t\big)^T \overline{R}(t)\big(u_t - L^o(t)X_t\big)dt + X_{t_0}^T V(t_0) X_{t_0} \Bigg\}. \quad (15)$$

Since the second and third terms in the above expression of $J(u)$ are independent of u_t, to minimize $J(u)$ is equivalent to minimizing the quantity $\mathbf{E}\left\{\|u_t - L^o(t)X_t\|^2\right\}$ for almost all $t \in [t_0, t_f]$. Hence, by definition of projection, the minimum is attained for the control law:

$$u_t^o = \mathbf{\Pi}\{L^o(t)X_t/\mathcal{L}_t^p(Y)\} = L^o(t)\widehat{X}_t = -\overline{R}(t)^{-1}H(t)^T V(t)\widehat{X}_t,$$

where $\widehat{X}_t = \mathbf{\Pi}\{X_t/\mathcal{L}_t^n(Y)\}$, and taking into account of (9), $V(t)$ is given by (7). •

IV. SOLUTION OF THE ASSOCIATED FILTERING PROBLEM

From Theorem 3.2, in order to endow the control-scheme an associated filtering problem needs to be solved: to find the linear-optimal filter for the stochastic system (1), (2), with $u_t = L^o(t)\widehat{X}_t$. This result represents an extension of the linear-optimal filter that was presented in [13] for systems with state-dependent noises and deterministic input.

Theorem 4.2. *Let us consider the stochastic system:*

$$dX_t = A(t)X_t dt + (HL^o)(t)\widehat{X}_t dt + \sum_{k=1}^{q} B^k(t)X_t dW_t^k + \sum_{k=1}^{d} D^k(t)L^o(t)\widehat{X}_t dN_t^k, \quad X_{t_0} = \overline{X}, \quad (16)$$

$$dY_t = C(t)X_t dt + dW_t', \qquad Y_{t_0} = 0, \quad (17)$$

with $L^o(t)$ as in the statement of Theorem 3.2 and $\widehat{X}_t = \mathbf{\Pi}\left\{X_t/\mathcal{L}_t^n(Y)\right\}$. Suppose that $\Psi_X(t)$ is nonsingular for any $t \in I$, and the matrix functions $B^k(t)\Psi_X(t)B^k(t)^T$, $(D^kL^o)(t)\Psi_{\widehat{X}}(t)(D^kL^o)^T(t)$ have a constant rank over the time-interval I. Then $\widehat{X}_t$ satisfies the following system of equations:

$$d\widehat{X}_t = A(t)\widehat{X}_t dt + (HL^o)(t)\widehat{X}_t dt + P(t)C(t)^T \cdot \Big(dY_t - C(t)\widehat{X}_t dt\Big), \quad \widehat{X}_{t_0} = \mathbf{E}\{\overline{X}\}, \quad (18)$$

$$P(t) = \Psi_X(t) - \Psi_{\widehat{X}}(t), \quad (19)$$

$$\begin{aligned}\dot{\Psi}_X(t) = {} & A(t)\Psi_X(t) + \Psi_X(t)A(t)^T + (HL^o)(t)\Psi_{\widehat{X}}(t) + \Psi_{\widehat{X}}(t)(L^{oT}H^T)(t) \\ & + \sum_{k=1}^{q} B^k(t)\big(\Psi_X(t) + \mu\mu^T(t)\big)B^k(t)^T \\ & + \sum_{k=1}^{d} D^k(t)L^o(t)\big(\Psi_{\widehat{X}}(t) + \mu\mu^T(t)\big) \cdot L^o(t)^T D^k(t)^T, \quad \Psi_X(t_0) = \overline{\Psi}_X, \end{aligned} \quad (20)$$

$$\dot{\Psi}_{\widehat{X}}(t) = (A(t) + (HL^o)(t))\Psi_{\widehat{X}}(t) + \Psi_{\widehat{X}}(t) \cdot (A(t) + (HL^o)(t))^T + (PC^TCP)(t), \quad \Psi_{\widehat{X}}(t_0) = 0, \quad (21)$$

$$\dot{\mu}(t) = (A(t) + (HL^o)(t))\,\mu(t), \quad \mu(t_0) = \mathbf{E}\{\overline{X}\}, \quad (22)$$

where $\mu(t) = \mu_X(t) = \mu_{\widehat{X}}(t)$.

Proof. First of all, we use a result of [13] in order to put the controlled bilinear system (16) in the form of a *linear* system with wide-sense diffusion. Indeed, by the hypotheses, as shown in Theorem 4.1 of [13], for the state-dependent noise term one has, almost surely:

$$\sum_{k=1}^{q} B^k(t)X_t dW_t^k = \sum_{k=1}^{2q} \widetilde{B}^k(t)d\widetilde{W}_t^{(k)}, \quad (23)$$

$$\sum_{k=1}^{d} D^k(t)L^o(t)\widehat{X}_t dN_t^k = \sum_{k=1}^{2d} \widetilde{D}^k(t)d\widetilde{N}_t^{(k)}, \quad (24)$$

where:

$$\widetilde{B}^k(t) \doteq \begin{cases} \big(B^k(t)\Psi_X(t)B^k(t)^T\big)^{\left(\frac{1}{2}\right)} \in \mathbb{R}^{n\times\rho_k}, \\ B^{k-q}(t)\mu(t) \in \mathbb{R}^n, \end{cases}$$

respectively for $k = 1, ..., q$, and for $k = q+1, ..., 2q$, and

$$\widetilde{D}^k(t) \doteq \begin{cases} \big((D^kL^o)(t)\Psi_{\widehat{X}}(t)(D^kL^o)^T(t)\big)^{\left(\frac{1}{2}\right)} \in \mathbb{R}^{n\times\rho_k'} \\ D^{k-d}(t)L^{o\,k-d}(t)\mu(t) \in \mathbb{R}^n, \end{cases}$$

respectively for $k = 1, ..., d$, and for $k = d+1, ..., 2d$, with $\rho_k \doteq \operatorname{rank}\{B^k(t)\Psi_X(t)B^k(t)^T\}$, $\rho_k' \doteq \operatorname{rank}\{D^k(t)L^o(t)\Psi_{\widehat{X}}(t)L^o(t)^T D^k(t)^T\}$. Moreover, the processes $\widetilde{W}^{(k)}$, $k = 1, ..., 2q$, and $\widetilde{N}^{(k)}$, $k = 1, ..., 2d$ are mutually uncorrelated wide-sense Wiener processes (and, in particular, for $k = q+1, ..., 2q$, simply $\widetilde{W}_t^{(k)} = W_t^{k-q}$, and, for $k = d+1, ..., 2d$, simply $\widetilde{N}_t^{(k)} = N_t^{k-d}$). By using the identities (23), (24) we can rewrite eq. (16) as

$$dX_t = A(t)X_t dt + (HL^o)(t)\widehat{X}_t dt + \widetilde{B}(t)d\widetilde{W}_t + \widetilde{D}(t)d\widetilde{N}_t, \quad X_{t_0} = \overline{X}, \quad (25)$$

where $\widetilde{B}(t)$, $\widetilde{D}(t)$ are the following block-matrices:

$$\widetilde{B}(t) = \left[\,\widetilde{B}^1(t) \;\vdots\; \ldots \;\vdots\; \widetilde{B}^{2q}(t)\,\right],$$

$$\widetilde{D}(t) = \left[\,\widetilde{D}^1(t) \;\vdots\; \ldots \;\vdots\; \widetilde{D}^{2d}(t)\,\right],$$

and $\widetilde{W}_t{}^T = [\widetilde{W}_t^{(1)T} \ \ldots \ \widetilde{W}_t^{(q)T} \ W_t^T]$, $\widetilde{N}_t{}^T = [\widetilde{N}_t^{(1)T} \ \ldots \ \widetilde{N}_t^{(d)T} \ N_t^T]$. Now, let us consider the *uncontrolled* system:

$$dX_t^0 = A(t)X_t^0 dt + \widetilde{B}(t)d\widetilde{W}_t + \widetilde{D}(t)d\widetilde{N}_t, \quad X_{t_0}^0 = \overline{X}, \quad (26)$$

$$dY_t^0 = C(t)X_t^0 dt + dW_t', \qquad Y_{t_0}^0 = 0. \quad (27)$$

For system (26), (27) it is possible to obtain the linear-optimal filter by using the general formulas for the linear estimation of wide-sense diffusions of Liptser-Shiryaev ([15,

§15]), or Theorem 8.2 in [13]. Thus, the following Kalman-like filter is obtained:

$$d\widehat{X}_t^0 = A(t)\widehat{X}_t^0 dt + P(t)C(t)^T d\nu_t^0, \qquad \widehat{X}_{t_0}^0 = \mathbf{E}\{\overline{X}\}, \tag{28}$$

where $d\nu_t^0 = dY_t^0 - C(t)\widehat{X}_t^0 dt$, and $P(t)$ satisfying the following equation:

$$\begin{aligned}\dot{P}(t) =& A(t)P(t) + P(t)A(t)^T - P(t)C^T C(t)P(t) \\ &+ \sum_{k=1}^{q} B^k(t)\left(\Psi_X(t) + \mu\mu^T(t)\right)B^k(t)^T \\ &+ \sum_{k=1}^{d} D^k(t)L^o(t)\big(\Psi_{\widehat{X}}(t) + \mu\mu^T(t)\big) \\ &\cdot L^o(t)^T D^k(t)^T, \quad P(t_0) = \overline{\Psi}_X. \end{aligned} \tag{29}$$

Now, let m_t be the solution of the equation:

$$dm_t = A(t)m_t dt + (HL^o)(t)m_t dt + P(t)C(t)^T d\nu_t^0, \qquad m_{t_0} = \mathbf{E}\{\overline{X}\}, \tag{30}$$

and consider the following system forced by m_t:

$$dX'_t = A(t)X'_t dt + (HL^o)(t)m_t dt + \widetilde{B}(t)d\widetilde{W}_t + \widetilde{D}(t)d\widetilde{N}_t, \qquad X'_{t_0} = \overline{X}, \tag{31}$$

$$dY'_t = C(t)X'_t dt + dW'_t, \qquad Y'_{t_0} = 0. \tag{32}$$

We will show that $m_t = \mathbf{\Pi}\left\{X'_t/\mathcal{L}_t^n(Y')\right\}$ (i.e. $m_t \equiv \widehat{X}'_t$). To this purpose, first of all let us prove that $\mathcal{L}_t^n(Y') = \mathcal{L}_t^n(Y^0)$. From (27), (32), (31) and (26), the following equation can be readily derived:

$$Y'_t - Y_t^0 = \int_{t_0}^{t} C(\tau)\int_{t_0}^{\tau}\Phi(\tau,s)(HL^o)(s)m_s ds d\tau, \tag{33}$$

where $\Phi(t,\tau)$ denotes the transition matrix of $A(t)$. Since, by (30), $m_s \in \mathcal{L}_s^n(\nu^0) = \mathcal{L}_s^n(Y^0)$, from (33) one has $Y'_t \in \mathcal{L}_t^m(Y^0)$. On the other hand, since $\nu'_t = Y'_t - \int_{t_0}^{t} C(\tau)\widehat{X}'_\tau d\tau$ is a standard wide-sense Wiener process, we can replace ν^0 in (30) with ν', thus $m_t \in \mathcal{L}_t^n(\nu') = \mathcal{L}_t^n(Y')$. Then, again from (33), it follows that $Y_t^0 \in \mathcal{L}_t^m(Y')$. We now show that $m_t = \mathbf{\Pi}\left\{X'_t/\mathcal{L}_t^n(Y^0)\right\}$. As a matter of fact, from (30) and (28), we can readily obtain $m_t = \widehat{X}_t^0 + U_t$, with $U_t = \int_{t_0}^{t}\Phi(\tau,t_0)(HL^o)(\tau)m_\tau d\tau$. Similarly, from (26) and (31), it results: $X'_t = X_t^0 + U_t$. Thus, for any linear map Λ_t, one has $\mathbf{E}\left\{\left(X'_t - m_t\right)^T\Lambda_t(Y^0)\right\} = \mathbf{E}\left\{\left(X_t^0 - \widehat{X}_t^0\right)^T\Lambda_t(Y^0)\right\} = 0$, that implies $m_t = \mathbf{\Pi}\left\{X'_t/\mathcal{L}_t^n(Y^0)\right\}$. Then, since $\mathcal{L}_t^n(Y') = \mathcal{L}_t^n(Y^0)$, it results: $m_t = \mathbf{\Pi}\left\{X'_t/\mathcal{L}_t^n(Y')\right\} = \widehat{X}'_t$. Hence (31), (32) can be rewritten as:

$$\begin{cases} dX'_t = A(t)X'_t dt + (HL^o)(t)\mathbf{\Pi}\{X'_t/\mathcal{L}_t^n(Y')\}dt \\ \qquad + \widetilde{B}(t)d\widetilde{W}_t + \widetilde{D}(t)d\widetilde{N}_t, \qquad X'_{t_0} = \overline{X}, \\ dY'_t = C(t)X'_t dt + dW'_t, \qquad Y'_{t_0} = 0, \end{cases} \tag{34}$$

whereas, by (25) and (17), the couple (X, Y) satisfies:

$$\begin{cases} dX_t = A(t)X_t dt + (HL^o)(t)\mathbf{\Pi}\{X_t/\mathcal{L}_t^n(Y)\}dt \\ \qquad + \widetilde{B}(t)d\widetilde{W}_t + \widetilde{D}(t)d\widetilde{N}_t, \qquad X_{t_0} = \overline{X}, \\ dY_t = C(t)X_t dt + dW'_t, \qquad Y_{t_0} = 0. \end{cases} \tag{35}$$

A comparison between (34), (35) immediately shows that $X'_t = X_t$, $Y'_t = Y_t$, and hence a.s.: $m_t = \widehat{X}'_t = \mathbf{\Pi}\left\{X'_t/\mathcal{L}_t^n(Y')\right\} = \mathbf{\Pi}\left\{X_t/\mathcal{L}_t^n(Y)\right\} = \widehat{X}_t$. By using the latter, and substituting ν_t^0 with $\nu_t = Y_t - \int_{t_0}^{t} C(\tau)\widehat{X}_\tau d\tau$ in eq. (30), we finally obtain the filter equation:

$$\begin{aligned} d\widehat{X}_t =& A(t)\widehat{X}_t dt + (HL^o)(t)\widehat{X}_t dt + P(t)C(t)^T \\ &\cdot\Big(dY_t - C(t)\widehat{X}_t dt\Big), \qquad \widehat{X}_{t_0} = \mathbf{E}\{\overline{X}\}, \end{aligned}$$

with $P(t)$ given by (29). Now, by the orthogonality principle, one has $\Psi_{X\widehat{X}}(t) = \Psi_{\widehat{X}}(t) = \Psi_{\widehat{X}X}(t)$, and since $P(t) = \mathbf{E}\{(X_t - \widehat{X}_t)(X_t - \widehat{X}_t)^T\}$, (19) follows. With standard calculation eq. (20) can be readily obtained, and then by (19), (21) is immediately derived. •

V. CONCLUSION

Theorem 3.2 shows that the suboptimal linear-feedback control function is a linear map of the linear-optimal state-estimate. This represents the "suboptimal" version of the well known *separation theorem*. The equations of the linear-optimal filter – that complete the control scheme – are derived in Theorem 4.2, that yields an extension of the result in [13] to systems driven by stochastic, observable, forcing terms and with an additional control-dependent noise. The suboptimal linear-feedback results are the natural extension of the control scheme for the complete-information case, that was derived by Wonham and, later, by McLane using a more direct procedure. For the incomplete-information case, the control-scheme results in the same controller of the complete-information case, applied to the linear-optimal state-estimate, in place of the current state. These are, moreover, formally very similar to the classical LQG controller, thus preserving its simplicity and meaningfulness.

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