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Solution to High-order Generalized Sylvester Matrix Equations

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***Abstract*— This paper provides a complete general parametric solution (V, W) to the high-order generalized Sylvester matrix equation $\sum_{i=0}^{m} A_i V F^i = BW$, with F being an arbitrary square matrix. The primary feature of this solution is that the matrix F does not need to be in any canonical form, or may be even unknown *a priori*. The results provide great convenience to the computation and analysis of the solutions to this class of equations, and can perform important functions in many analysis and design problems involving high-order dynamical systems.**

I. INTRODUCTION

A general second-order linear system can be represented by the following model

$$M\ddot{x} + D\dot{x} + Kx = Bu, \tag{1}$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^r$ are the state vector and the control vector, respectively; M, D, $K \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times r}$ are the system coefficient matrices, and are usually called the mass matrix, the structural damping matrix and the stiffness matrix, respectively. Second-order linear systems have found applications in many fields, such as robotics, mechanical engineering, aerospace engineering, etc., and have attracted much attention ([1]-[7]).

When dealing with some more complicated applications, high-order dynamical linear systems described by the following model is often encountered ([8]):

$$A_m x^{(m)} + \cdots + A_1\dot{x} + A_0 x = Bu, \tag{2}$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^r$ are the state vector and the control vector, respectively; $A_i \in \mathbb{R}^{n\times n},\ i = 0, 1, 2, \cdots, m$, and $B \in \mathbb{R}^{n\times r}$ are the system coefficient matrices. Such a system reduces to a second-order descriptor linear system and a first-order descriptor linear system when m takes the value of 2 and 1, respectively. Furthermore, let s denote the differential operator, then the system can also be represented in the following operator form:

$$A(s)x(s) = Bu(s),$$

where $x(s)$ and $u(s)$ are the Laplace transforms of $x(t)$ and $u(t)$, respectively, and

$$A(s) = s^m A_m + \cdots + sA_1 + A_0. \tag{3}$$

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It can be shown that certain control problems, such as pole/eigenstructure assignment and observer design, of the high-order linear system (2), are closely related with a type of matrix equations in the following form:

$$A_m V F^m + \cdots + A_1 V F + A_0 V = BW, \tag{4}$$

where $F \in \mathbb{C}^{p\times p}$ is an arbitrary square matrix, and $V \in \mathbb{C}^{n\times p}$ and $W \in \mathbb{C}^{r\times p}$ are the matrices to be determined. Such a type of equations are called high-order generalized Sylvester matrix equations ([8]), and are direct extensions of the second-order generalized Sylvester matrix equations in the form of

$$MVF^2 + DVF + KV = BW, \tag{5}$$

and the first-order generalized Sylvester matrix equations in the form of

$$AV - EVF = BW. \tag{6}$$

For solution to the first-order generalized Sylvester matrix equation (6), there have been many results ([9]-[12]). Regarding solution to the second-order generalized Sylvester matrix equation (5) and the high-order Sylvester matrix equation (4), there are only a few results in the literature. Under the condition of $\det M \neq 0$, reference [6] proposes a complete parametric solution to the second-order generalized Sylvester matrix equation (5), and based on this parametric equation gives a general parametric approach for eigenstructure assignment in second-order dynamical systems. Reference [7] also considers the solution to the second-order generalized Sylvester matrix equation (5) and eigenstructure assignment in second-order dynamical systems, and generalizes the results in [6] into the singular case by removing the condition $\det M \neq 0$. Very recently, reference [8] proposes, under the condition of $\det A_m \neq 0$, a complete parametric solution to the high-order Sylvester matrix equation (4), and based on this parametric equation gives a general parametric approach for eigenstructure assignment in the high-order dynamical system (2). However, all the solutions proposed in [6],[7] and [8] require the matrix F to be in Jordan form. In some applications, the matrix F is not a Jordan matrix, but an arbitrary square one. Although an arbitrary matrix can always be transformed into a Jordan matrix, the transformation process not only gives additional computation load, but is also known to be generally not numerically reliable ([13]).

In this note, solution to the high-order generalized Sylvester matrix equation (4), with F being an arbitrary square matrix, is investigated, and a complete, general parametric solution is established. The proposed solution is in a very simple and neat form, provides all the degree of freedom, and allows the matrix F to be set undetermined and used as a part of degree of freedom. By setting the matrix B in (4) to the identity matrix, a type of high-order normal Sylvester matrix equations are obtained, which includes both the Lyapunov matrix equations for the continuous-time and discrete-time systems. Through simplifying the established general solution to the high-order generalized Sylvester matrix equation (4), analytical solutions to this type of high-order normal Sylvester matrix equations are also derived. The obtained results may have important applications in many analysis and design problems in control systems theory.

II. PRELIMINARIES

A. Notations And A Technical Lemma

In this paper, we use $\sigma(A)$ to denote the set of eigenvalues of the matrix A, and $\otimes$ the Kronecker product, and we use $\varnothing$ to denote the null set. Also, for an $m\times n$ matrix $R=[r_{ij}]$, the so-called stretching function vec(R) is defined as

$$[\text{vec}\,(R)]^T = \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{m1} & \cdots & r_{1n} & r_{2n} & \cdots & r_{mn} \end{bmatrix}.$$

For matrices M, X and N with appropriate dimensions, we have the following well-known result related with the stretching operation:

$$\text{vec}\,(MXN) = \left(N^T\otimes M\right)\text{vec}\,(X)\,. \tag{7}$$

Let $Z\in\mathbb{R}^{n\times p}$, and $P(s)\in\mathbb{R}^{m\times n}[s]$ be defined as

$$P(s)=\sum_{i=0}^{\omega}P_i s^i.$$

For convenience, in this paper the following operations are used:

$$P(F)=\sum_{i=0}^{\omega}P_iF^i,\quad \forall F\in\mathbb{R}^{n\times p},$$

$$P_{\circledast Z}(s)=\sum_{i=0}^{\omega}(P_iZ)\,s^i.$$

Lemma 1: Let $A(s)\in\mathbb{R}^{n\times n}[s]$ be given in (3), $B\in\mathbb{R}^{n\times r}$. Then there exist two right coprime polynomial matrices $N(s)\in\mathbb{R}^{n\times r}[s]$ and $D(s)\in\mathbb{R}^{r\times r}[s]$ satisfying

$$A(s)N(s)-BD(s)=0. \tag{8}$$

Proof: Omitted due to paper length. ■

When the polynomial det$A(s)$ is not identically zero, $A(s)$ is invertible. Under the assumption that $D(s)$ is invertible, the above equation can be rewritten in the form of the following so-called right factorization

$$A^{-1}(s)B=N(s)\,D^{-1}(s)\,. \tag{9}$$

B. Controllability And Uncontrollable Modes

Definition 1: The original m-th order system (2) is called R-controllable if its corresponding extended first-order state-space model

$$E_{ec}\dot z = A_{ec}z + B_e u, \tag{10}$$

with

$$E_{ec}=\text{Blockdiag}\,(I_n,\cdots,I_n,A_m)\,, \tag{11}$$

$$A_{ec}=\begin{bmatrix} 0 & I_n & & & \\ & 0 & I_n & & \\ & & \ddots & \ddots & \\ & & & 0 & I_n \\ -A_0 & -A_1 & \dots & -A_{m-2} & -A_{m-1}\end{bmatrix}, \tag{12}$$

$$B_e=\begin{bmatrix} 0 & \cdots & 0 & B^T\end{bmatrix}^T \tag{13}$$

is R-controllable, i.e.,

$$\text{rank}\begin{bmatrix} sE_e-A_e & B_e\end{bmatrix}=mn,\ \forall s\in\mathbb{C}.$$

In this case, we also say that $(A(s),B)$ is R-controllable.

The following lemma gives a necessary and sufficient condition of R-controllability of $(A(s),B)$.

Lemma 2: The high-order dynamical system (2) is R-controllable if and only if

$$\text{rank}\begin{bmatrix} A(s) & B\end{bmatrix}=n,\ \forall s\in\mathbb{C}.$$

Proof: This result has been shown in [8] for the special case of $\det A_m\neq 0$. The general case can also be shown along the same lines using the well-known PHB criterion. ■

The above lemma indicates that $(A(s),B)$ is R-controllable if and only if $A(s)$ and B are left coprime. Furthermore, it follows from this lemma that $(A(s),B)$ is not R-controllable if and only if there exists a $\lambda\in\mathbb{C}$ such that

$$\text{rank}\begin{bmatrix} A(\lambda) & B\end{bmatrix}<n. \tag{14}$$

It is obvious that such a λ must be the zeros of $\det A(s)$. For convenience, we give a term for such complex scalars.

Definition 2: Let $A(s)\in\mathbb{R}^{n\times n}[s]$ be given in (3), $B\in\mathbb{R}^{n\times r}$. Then any $\lambda\in\mathbb{C}$ satisfying (14) is called an uncontrollable mode of $(A(s),B)$, or an uncontrollable pole of system (2).

In the following we use $\mathcal{U}_c(A(s),B)$ to denote the set of uncontrollable modes of $(A(s),B)$, that is

$$\mathcal{U}_c(A(s),B)=\left\{\lambda \mid \text{rank}\begin{bmatrix} A(\lambda) & B\end{bmatrix}<n\right\}.$$

The following further characterizes $\mathcal{U}_c(A(s),B)$.

Lemma 3: Let $A(s)\in\mathbb{R}^{n\times n}[s]$ be given in (3), $B\in\mathbb{R}^{n\times r}$, and let $\Sigma(s)\in\mathbb{R}^{n\times n}[s]$ be the Smith form of $\begin{bmatrix} A(s) & B\end{bmatrix}$. Then

$$\mathcal{U}_c(A(s),B)=\{s\mid \det\Sigma(s)=0\}\,.$$

Proof: Omitted due to paper length. ■

This lemma states that all uncontrollable modes of $(A(s),B)$ coincides with the zeros of $\det\Sigma(s)$.

C. F-left (right) Coprimeness

The following definition performs an important role through out of this paper.

Definition 3: Let $F \in \mathbb{C}^{p\times p}$ be an arbitrary matrix, then

1) a pair of polynomial matrices $N(s) \in \mathbb{R}^{n\times r}[s]$ and $D(s) \in \mathbb{R}^{r\times r}[s]$ is said to be F-right coprime if

$$\operatorname{rank}\begin{bmatrix} N(s) \\ D(s) \end{bmatrix} = r,\ \forall s \in \sigma(F); \tag{15}$$

2) a pair of polynomial matrices $H(s) \in \mathbb{R}^{m\times n}[s]$ and $L(s) \in \mathbb{R}^{m\times m}[s]$ is said to be F-left coprime if

$$\operatorname{rank}\begin{bmatrix} H(s) & L(s) \end{bmatrix} = m,\ \forall s \in \sigma(F). \tag{16}$$

According to the above definition, $A(s)$ and B are F-left coprime if

$$\operatorname{rank}\begin{bmatrix} A(s) & B \end{bmatrix} = n, \quad \forall s \in \sigma(F). \tag{17}$$

From this we can observe that the R-controllability of $(A(s), B)$ implies that $A(s)$ and B are F-left coprime for arbitrary given F matrix of appropriate dimension. Following directly from the definition of uncontrollable modes, we immediately have the following conclusion.

Proposition 4: Let $A(s) \in \mathbb{R}^{n\times n}[s]$ be given in (3), $B \in \mathbb{R}^{n\times r}$. Then $A(s)$ and B are F-left coprime if and only if the eigenvalues of matrix F are different from the uncontrollable modes of $(A(s), B)$, that is,

$$\mathcal{U}_c(A(s), B) \cap \sigma(F) = \varnothing. \tag{18}$$

To verify the F-coprimeness of two polynomial matrices $N(s)$ and $D(s)$ according to the definition, we need to get the eigenvalues of the matrix F. This may involve bad conditioning. To end this section, we now introduce a criterion which needs only the coefficient matrices $N_i, D_i, i = 0, 1, \cdots, \omega$, and F.

Lemma 5: Let $F \in \mathbb{C}^{p\times p}$ be an arbitrary matrix and

$$\begin{cases} N(s) = \sum\limits_{i=0}^{\omega} N_i s^i,\ N_i \in \mathbb{R}^{n\times r} \\ D(s) = \sum\limits_{i=0}^{\omega} D_i s^i,\ D_i \in \mathbb{R}^{r\times r} \end{cases}. \tag{19}$$

Then $N(s)$ and $D(s)$ are F-right coprime if and only if

$$\operatorname{rank}\left[\sum_{i=0}^{\omega}\left(F^i \otimes \begin{bmatrix} N_i \\ D_i \end{bmatrix}\right)\right] = rp. \tag{20}$$

Proof: Let the Jordan form of the matrix F be as follows:

$$\begin{cases} J = \operatorname{Blockdiag}(J_1, \cdots, J_w) \\ J_i = \begin{bmatrix} s_i & 1 & & \\ & s_i & \ddots & \\ & & \ddots & 1 \\ & & & s_i \end{bmatrix}_{p_i \times p_i} \end{cases},$$

where $s_i, i = 1, \cdots, w$, are obviously the eigenvalues of the matrix F (which are not necessarily distinct). Further, let the corresponding eigenvector matrix of F be P, then we have (27). Substituting equation (27) into the left side of (20), yields

$$\begin{aligned} &\sum_{i=0}^{\omega}\left(F^i \otimes \begin{bmatrix} N_i \\ D_i \end{bmatrix}\right) \\ =\ &\sum_{i=0}^{\omega}\left(\left(PJP^{-1}\right)^i \otimes \begin{bmatrix} N_i \\ D_i \end{bmatrix}\right) \\ =\ &(P \otimes I_{n+r})\sum_{i=0}^{\omega}\left(J^i \otimes \begin{bmatrix} N_i \\ D_i \end{bmatrix}\right)\left(P^{-1} \otimes I_r\right). \end{aligned}$$

Since $P \otimes I_{n+r}$ and $P^{-1} \otimes I_r$ are nonsingular, we have

$$\operatorname{rank}\left[\sum_{i=0}^{\omega} F^i \otimes \begin{bmatrix} N_i \\ D_i \end{bmatrix}\right] = \sum_{j=0}^{w} \operatorname{rank}\left[\sum_{i=0}^{\omega} J_j^i \otimes \begin{bmatrix} N_i \\ D_i \end{bmatrix}\right]. \tag{21}$$

Define a nilpotent matrix

$$E_j = \begin{bmatrix} 0 & I_{p_j-1} \\ 0 & 0 \end{bmatrix}_{p_j\times p_j}, \quad j = 1, \cdots, w,$$

then we have

$$E_j^l = \begin{bmatrix} 0 & I_{p_j-l} \\ 0 & 0 \end{bmatrix}. \tag{22}$$

Noting that

$$J_j = s_j I_{p_j} + E_j,\ j = 1, \cdots, w,$$

and the matrices pair $s_j I_{p_j}$ and E_j are commutative, and using the Binomial Theorem, we further have

$$J_j^i = s_j^i I_{p_j} + s_j^{i-1} C_i^1 E_j^1 + \cdots + s_j^0 C_i^i E_j^i. \tag{23}$$

Substituting the above matrices into (21), we get

$$\begin{aligned} &\sum_{i=0}^{\omega}\left(J_j^i \otimes \begin{bmatrix} N_i \\ D_i \end{bmatrix}\right) \\ =\ &\sum_{i=0}^{\omega}\left(s_j^i I_{p_j} + \cdots + s_j^0 C_i^i E_j^i\right) \otimes \begin{bmatrix} N_i \\ D_i \end{bmatrix} \\ =\ &\sum_{i=0}^{\omega}\left(s_j^i I_{p_j} C_i^0 \otimes \begin{bmatrix} N_i \\ D_i \end{bmatrix}\right) \\ &+ \sum_{i=0}^{\omega-1}\left(s_j^i E_j^1 C_{i+1}^1 \otimes \begin{bmatrix} N_{i+1} \\ D_{i+1} \end{bmatrix}\right) \\ &+ \cdots + \sum_{i=0}^{1}\left(s_j^i E_j^{\omega-1} C_{i+\omega-1}^{\omega-1} \otimes \begin{bmatrix} N_{i+\omega-1} \\ D_{i+\omega-1} \end{bmatrix}\right) \\ &+ \sum_{i=0}^{0}\left(s_j^i E_j^{\omega} C_{i+\omega}^{\omega} \otimes \begin{bmatrix} N_{i+\omega} \\ D_{i+\omega} \end{bmatrix}\right). \end{aligned}$$

Denote

$$\theta_j^{\omega-k} = \sum_{i=0}^{k}\left(s_j^i C_{i+\omega-k}^{\omega-k} \begin{bmatrix} N_{i+\omega-k} \\ D_{i+\omega-k} \end{bmatrix}\right),\ k = 0, \cdots, \omega. \tag{24}$$

Comparing (19) with θ_j^0, we clearly have

$$\theta_j^0 = \begin{bmatrix} N(s_j) \\ D(s_j) \end{bmatrix}. \tag{25}$$

So (21) can be simplified as

$$
\begin{aligned}
&\sum_{i=0}^{\omega}\left(J_j^i\otimes\begin{bmatrix} N_i \\ D_i \end{bmatrix}\right) \\
=\ & I_{p_j}\otimes\theta_j^0+\cdots+E_j^{\omega}\otimes\theta_j^{\omega} \\
=\ & \begin{bmatrix}
\theta_j^0 & \theta_j^1 & \cdots & \theta_j^{\omega} & 0 & \cdots & 0 \\
 & \theta_j^0 & \theta_j^1 & \ddots & \ddots & \ddots & \vdots \\
 & & \theta_j^0 & \ddots & \ddots & \ddots & 0 \\
 & & & \ddots & \ddots & \ddots & \theta_j^{\omega} \\
 & & & & \theta_j^0 & \theta_j^1 & \vdots \\
 & & & & & \theta_j^0 & \theta_j^1 \\
 & & & & & & \theta_j^0
\end{bmatrix}.
\end{aligned}
$$

Since the above matrix has full column rank if and only if $\theta_j^0,\ j=1,2,\cdots,w$, has full column rank, i.e., the relation (15) holds, the conclusion of the lemma clearly holds true. ■

III. MAIN RESULTS

This section studies the solution to the generalized Sylvester matrix equation (4).

A. Degree of Freedom

The high-order Sylvester matrix equation (4) is clearly homogeneous, so its solution is not unique. Regarding the degree of freedom in the solution (V, W), we have the following result.

Theorem 6: Let $A(s)\in\mathbb{R}^{n\times n}[s]$ be given by (3), $F\in\mathbb{C}^{p\times p}$, $B\in\mathbb{R}^{n\times r}$, $\text{rank}B=r$. Then the degree of freedom in the solution (V, W) to the high-order Sylvester matrix equation (4) is rp if and only if $A(s)$ and B are F-left coprime, or equivalently, the eigenvalues of matrix F are different from the uncontrollable modes of $(A(s), B)$.

Proof: Putting $\text{vec}(\cdot)$ on both sides of the equation (4) and using (7), we obtain

$$\Phi\begin{bmatrix} \text{vec}\,(V) \\ \text{vec}\,(W) \end{bmatrix}=0, \tag{26}$$

where

$$\Phi=\begin{bmatrix} \sum\limits_{i=0}^{m}\left(F^T\right)^i\otimes A_i & -I_p\otimes B \end{bmatrix}.$$

Clearly, this is an equivalent form of equation (4). Let P and J be the eigenvector matrix and the Jordan form of matrix F^T, respectively, then we have

$$F^T=PJP^{-1}. \tag{27}$$

Substituting (27) into (26), yields

$$
\begin{aligned}
\Phi\ =\ & P\otimes I_n\begin{bmatrix} \sum\limits_{i=0}^{m}J^i\otimes A_i & I_p\otimes B \end{bmatrix} \\
& \times\begin{bmatrix} P^{-1}\otimes I_n & 0 \\ 0 & -P^{-1}\otimes I_r \end{bmatrix}.
\end{aligned}
$$

Noting that $P\otimes I_n$, $P^{-1}\otimes I_n$ and $P^{-1}\otimes I_r$ are all nonsingular, we can show

$$
\begin{aligned}
\text{rank}\Phi\ =\ & \text{rank}\begin{bmatrix} \sum\limits_{i=0}^{m}J^i\otimes A_i & I_p\otimes B \end{bmatrix} \\
=\ & \text{rank}\begin{bmatrix}
\Omega(s_1) & * & 0 & \cdots & 0 \\
0 & \Omega(s_2) & * & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & \Omega(s_p) & *
\end{bmatrix}
\end{aligned} \tag{28}
$$

where

$$\Omega(s)=[A(s)\ B],$$

and the terms denoted by $*$ may be zero and $s_i,\ i=1,\cdots,p$, are the eigenvalues of matrix F (not necessarily distinct). It thus follows from the above relation that

$$\text{rank}\Phi=np \tag{29}$$

if and only if (17) holds. When (29) holds, according to linear equation theory the maximum number of free parameters of the solution (V, W) is

$$\pi=np+rp-\text{rank}\Phi=rp.$$

With this we complete the proof. ■

Using Lemma 2 and Proposition 4, we immediately have the following corollary of Theorem 6.

Corollary 7: Let $A(s)\in\mathbb{R}^{n\times n}[s]$ be given by (3), $F\in\mathbb{C}^{p\times p}$, $B\in\mathbb{R}^{n\times r}$, $\text{rank}B=r$. Then the maximum number of free parameters in the solution (V, W) to the generalized Sylvester matrix equation (4) is rp if one of the following conditions holds:

1) the high-order system (2) is R-controllable;
2) the eigenvalues of F are different from the poles of the system, i.e.,

$$\{s\mid\det A(s)=0\}\cap\sigma\left(F\right)=\varnothing.$$

B. The General Solution

Following Lemma 1, there exist polynomial matrices $N\left(s\right)\in\mathbb{R}^{n\times r}[s]$ and $D\left(s\right)\in\mathbb{R}^{r\times r}[s]$ satisfying the relation (8). If we denote $D\left(s\right)=\left[d_{ij}\left(s\right)\right]_{r\times r}$ and

$$\omega=\max\left\{\deg\left(d_{ij}\left(s\right)\right),\ i,j=1.2,\cdots,r\right\},$$

then $N\left(s\right)$ and $D\left(s\right)$ can be written in the form of (19).

Regarding the general solution to the high-order generalized Sylvester matrix equation (4), we have the following result.

Theorem 8: Let $A(s)\in\mathbb{R}^{n\times n}[s]$ be given in (3), $F\in\mathbb{C}^{p\times p}$, $B\in\mathbb{R}^{n\times r}$ with $\text{rank}B=r$, and $A(s)$ and B are F-left coprime. Further, let $N\left(s\right)$ and $D\left(s\right)$ be given by (19) and satisfy (8). Then

1) the matrices $V\in\mathbb{C}^{n\times p},\ W\in\mathbb{C}^{r\times p}$ given by

$$\left\{\begin{array}{l} V=N_{\circledast Z}(F)=N_0Z+N_1ZF+\cdots+N_{\omega}ZF^{\omega} \\ W=D_{\circledast Z}(F)=D_0Z+D_1ZF+\cdots+D_{\omega}ZF^{\omega} \end{array}\right. \tag{30}$$

satisfy the high-order Sylvester matrix equation (4) for arbitrary matrix $Z \in \mathbb{C}^{r\times p}$;

2) all the matrices $V \in \mathbb{C}^{n\times p}$, $W \in \mathbb{C}^{r\times p}$ satisfying the matrix equation (4) can be parameterized as in (30) if and only if $N(s)$ and $D(s)$ are F-right coprime, i.e., condition (15) holds.

Proof: By substituting (19) into (8), we can easily show

$$\begin{aligned} & A(s)N(s) - BD(s) \\ = & \sum_{i=0}^{m}\sum_{j=0}^{\omega} A_i N_j s^{i+j} - \sum_{k=0}^{\omega} BD_k s^k \\ = & \sum_{k=0}^{\omega}\left(\sum_{i=0}^{k} A_i N_{k-i} - BD_k\right) s^k \\ & + \sum_{k=\omega+1}^{m+\omega}\left(\sum_{i=0}^{k} A_i N_{k-i}\right) s^k \\ = & \ 0. \end{aligned}$$

Therefore, by setting the coefficients of s^k in the above equation to zero, we obtain

$$\sum_{i=0}^{k} A_i N_{k-i} = \begin{cases} BD_k, & k = 0,1,2,\cdots,\omega \\ 0, & k = \omega+1,\cdots,\omega+m \end{cases}. \tag{31}$$

Using (30) and (31), we can show

$$\sum_{i=0}^{m} A_i V F^i = \sum_{i=0}^{m} A_i \left(\sum_{j=0}^{\omega} N_j Z F^j\right) F^i = BW$$

This states that the matrices V and W given by (30) satisfy the matrix equation (4). The first conclusion is proven.

In order to prove the second conclusion, putting vec$(\cdot)$ on both sides of (30) and using the equation (7), we obtain

$$\begin{cases} \operatorname{vec}(V) = \left[\sum_{i=0}^{\omega}\left(\left(F^T\right)^i \otimes N_i\right)\right]\operatorname{vec}(Z) \\ \operatorname{vec}(W) = \left[\sum_{i=0}^{\omega}\left(\left(F^T\right)^i \otimes D_i\right)\right]\operatorname{vec}(Z) \end{cases},$$

or equivalently,

$$\begin{bmatrix} \operatorname{vec}(V) \\ \operatorname{vec}(W) \end{bmatrix} = \begin{bmatrix} \sum_{i=0}^{\omega}\left(\left(F^T\right)^i \otimes N_i\right) \\ \sum_{i=0}^{\omega}\left(\left(F^T\right)^i \otimes D_i\right) \end{bmatrix} \operatorname{vec}(Z). \tag{32}$$

According to Theorem 6, the number of free parameters of the solution (V, W) is rp. Recalling the fact that $Z \in \mathbb{C}^{r\times p}$ is an arbitrary parameter matrix, we need only to validate that each element in Z contributes independently to (V, W) if and only if the condition (15) holds. It follows from (32) that each element in Z contributes independently to (V, W) if and only if

$$\operatorname{rank}\begin{bmatrix} \sum_{i=0}^{\omega}\left(\left(F^T\right)^i \otimes N_i\right) \\ \sum_{i=0}^{\omega}\left(\left(F^T\right)^i \otimes D_i\right) \end{bmatrix} = rp. \tag{33}$$

Therefore, in the following we suffice only to show that (33) holds if and only if $N(s)$ and $D(s)$ are F-right coprime.

By the definition of Kronecker product, we have

$$\begin{aligned} & \operatorname{rank}\begin{bmatrix} \sum_{i=0}^{\omega}\left(\left(F^T\right)^i \otimes N_i\right) \\ \sum_{i=0}^{\omega}\left(\left(F^T\right)^i \otimes D_i\right) \end{bmatrix} \\ = & \operatorname{rank}\begin{bmatrix} \sum_{i=0}^{\omega}\begin{bmatrix} \left(F^T\right)_{11}^i N_i & \cdots & \left(F^T\right)_{1p}^i N_i \\ \vdots & \ddots & \vdots \\ \left(F^T\right)_{p1}^i N_i & \cdots & \left(F^T\right)_{pp}^i N_i \end{bmatrix} \\ \sum_{i=0}^{\omega}\begin{bmatrix} \left(F^T\right)_{11}^i D_i & \cdots & \left(F^T\right)_{1p}^i D_i \\ \vdots & \ddots & \vdots \\ \left(F^T\right)_{p1}^i D_i & \cdots & \left(F^T\right)_{pp}^i D_i \end{bmatrix} \end{bmatrix} \end{aligned}$$

By exchanging certain rows of the matrix in the above equation, we further get

$$\begin{aligned} & \operatorname{rank}\begin{bmatrix} \sum_{i=0}^{\omega}\left(\left(F^T\right)^i \otimes N_i\right) \\ \sum_{i=0}^{\omega}\left(\left(F^T\right)^i \otimes D_i\right) \end{bmatrix} \\ = & \operatorname{rank}\left[\sum_{i=0}^{\omega}\begin{bmatrix} \mu_{11}^i \begin{bmatrix} N_i \\ D_i \end{bmatrix} & \cdots & \mu_{1p}^i \begin{bmatrix} N_i \\ D_i \end{bmatrix} \\ \vdots & \ddots & \vdots \\ \mu_{p1}^i \begin{bmatrix} N_i \\ D_i \end{bmatrix} & \cdots & \mu_{pp}^i \begin{bmatrix} N_i \\ D_i \end{bmatrix} \end{bmatrix}\right] \\ = & \operatorname{rank}\left[\sum_{i=0}^{\omega}\left(\left(F^T\right)^i \otimes \begin{bmatrix} N_i \\ D_i \end{bmatrix}\right)\right], \end{aligned}$$

where $\mu_{ij} = \left(F^T\right)_{ij}$, denotes the element in the i-th row and j-th column of the matrix F^T. By Lemma 5, (33) holds if and only if

$$\operatorname{rank}\begin{bmatrix} N(\lambda) \\ D(\lambda) \end{bmatrix} = r \text{ for any } \lambda \in \sigma\left(F^T\right).$$

Since F and F^T have the same eigenvalues, the above condition is equivalent to the F-right coprimeness of $N(s)$ and $D(s)$. With this we complete the proof. ■

When a pair of polynomial matrices $N(s)$ and $D(s)$ satisfying (8) or (9) are obtained, the solution (30) to the high-order Sylvester matrix equation (4) can be immediately written out. For general numerical algorithms solving the right factorization (9), one can refer to [14]-[15]. In view of Lemma 1, we can also solve a pair of right coprime polynomial matrices $N(s)$ and $D(s)$ satisfying (8) using matrix elementary transformations.

IV. EXAMPLE

Consider a third-order generalized Sylvester matrix equation in the form of (4) with

$$A_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ A_2 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & -1 \\ 1 & -1 & 3 \end{bmatrix},$$

$$A_1 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix},\ A_0 = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix},$$

and

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

It is easy to verify that the corresponding system (2) is R-controllable. Using matrix elementary transformations, we obtain

$$N(s) = \begin{bmatrix} s^3+2s^2+s+1 & -s^2+2s+2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$D(s) = \begin{bmatrix} s^6+3s^5+5s^4+6s^3+3s^2+2s-1 & -s^5+s^4+2s^3+7s^2+4s \\ s^5+2s^4+s^3+2s+1 & -s^4+2s^3+5s^2+s+1 \end{bmatrix}.$$

So we have

$$N_0 = \begin{bmatrix} 1 & 2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix},\ N_1 = \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$N_2 = \begin{bmatrix} 2 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},\ N_3 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

and

$$D_0 = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix},\ D_1 = \begin{bmatrix} 2 & 4 \\ 2 & 1 \end{bmatrix},$$

$$D_2 = \begin{bmatrix} 3 & 7 \\ 0 & 5 \end{bmatrix},\ D_3 = \begin{bmatrix} 6 & 2 \\ 1 & 2 \end{bmatrix},$$

$$D_4 = \begin{bmatrix} 5 & 1 \\ 2 & -1 \end{bmatrix},\ D_5 = \begin{bmatrix} 3 & -1 \\ 1 & 0 \end{bmatrix},\ D_6 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

According to Theorem 8, for an arbitrary matrix $F \in \mathbb{C}^{p\times p}$, a complete analytical and explicit solution to the 3rd order Sylvester matrix equation (4) can be parameterized as

$$\begin{cases} V = \sum_{i=0}^{3} N_i Z F^i \\ W = \sum_{i=0}^{6} D_i Z F^i \end{cases},$$

where $Z \in \mathbb{R}^{3\times p}$ is an arbitrary parameter matrix. Particularly, let

$$F = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix},$$

and specially choose

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

we get a special solution

$$V = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix},\ W = \begin{bmatrix} -3 & 6 \\ -1 & 4 \end{bmatrix}.$$

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