

## Two Alternative Views on Control Design with Degree Constraint

Tryphon T. Georgiou and Anders Lindquist

**Abstract**—The purpose of this note is to highlight similarities and differences between two alternative methodologies for feedback control design under constraints on the McMillan degree of the feedback system. Both sets of techniques focus on uniformly optimal designs. The first is based on the work of Gahinet and Apkarian, and Skelton, Iwasaki, Grigoriades and their co-workers, while the other is based on earlier joint work of the authors with C. I. Byrnes.

### I. INTRODUCTION

The dimension of components in a feedback system is often a critical design parameter. Robustness and reliability are often adversely affected by the dimension of the components, numerics are hampered by the size of problem, and finally, high order dynamical response may be undesirable to a human interacting with the particular system. Thus, it is of interest to explore the possibility of effective control design techniques that are capable of incorporating degree constraints. The purpose of this note is to discuss and compare two alternative methodologies for feedback control design which allow a handle on the McMillan degree.

Gahinet and Apkarian [11], and Skelton, Iwasaki, and Grigoriadis [19] have introduced methodologies for controller design under degree constraints. Performance and stability are expressed, in the spirit of current control design practices, using suitable linear matrix inequalities, while a rank condition relates directly to the dimension of the controller. Controller design is treated herein as an interpolation problem in the style of  $H_\infty$  (circa 1980's), and in fact, we focus on the single input single output paradigm of sensitivity minimization (despite its limited practical appeal). Our goal is to contrast the aforementioned methods with a more direct approach based on our own earlier work on analytic interpolation with degree constraint (see e.g., [4], [5]). The authors have repeatedly been asked to explain the connection between the techniques in e.g., [4], and the aforementioned approach using linear matrix inequalities, as they both pertain to degree constraints. The purpose of this paper is to answer this question and to explain similarities and differences between the two approaches.

Consider the feedback interconnection in Figure 1 and let  $d$  represent an external disturbance whose effect on the output is to be minimized. When the dynamical system is

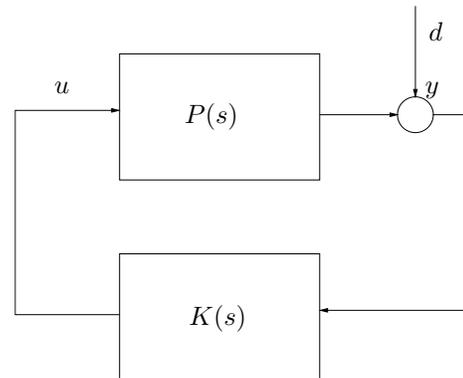


Fig. 1. Feedback loop

linear, this can be formulated as a standard  $H_\infty$ -minimization problem. The controller is chosen to ensure internal stability and minimize the gain of the sensitivity function

$$S = \frac{1}{1 - PK} \quad (1)$$

over selected frequency bands. Throughout,  $P, K$  represent the transfer functions of plant and controller and  $\deg(P), \deg(K)$  represent their respective McMillan degrees. In the standard  $H_\infty$  formalism the performance is encapsulated in a weighting function  $W(s)$  and the design specifications cast in the form of ensuring a bound on the weighted norm

$$\|W(s)S(s)\|_\infty < \gamma \quad (2)$$

subject to internal stability. Typically,  $\deg(K), \deg(S)$  depend on  $\deg(P)$  and  $\deg(W)$  (and in fact, the sum of these two McMillan degrees). Aside from the resulting “inflation” of the degree for the controller, the choice of the weight  $W(s)$  is a delicate task since it is not at all transparent how it affects feasibility of the performance specification (2). Indeed, small changes in the desired bandwidth of the system and the desired “shape” of  $S(s)$  (dictated by our choice of  $W(s)$ ) may render the performance specification unattainable. Although the task of choosing weights in  $H_\infty$ -design is somewhat intuitive and more accepted than that of choosing design parameters in, say, linear quadratic problems, it is far from straightforward and often a challenging task [23], [10].

Starting from a state-space formalism to  $H_\infty$ -control problems [8] and via a clever use of the bounded real lemma, Gahinet and Apkarian [11] (see also [19]) expressed the conditions for the existence of a controller which guarantees performance and has given McMillan degree, in the form of a linear matrix inequality (LMI) with a rank constraint.

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Typically, weighting functions are incorporated into the plant description (“inflating” the degree of the “new” plant accordingly). With that in place, a case which is particularly appealing is when, in our context,  $\deg(K) = \deg(P)$ . Then the approach in [11], [19] leads to a set of ordinary linear matrix inequalities (LMI’s). Requiring further degree reduction is a highly nontrivial problem in general.

An alternative viewpoint is to consider the totality of sensitivity functions of a given degree that meet a possibly conservative bound, and then select a particular one within this class. This hinges upon an effective parametrization of sensitivity functions that can be so achieved. Such a parametrization is in place for the precise class of sensitivity functions that do not exceed in dimension the sum of unstable plant poles and non-minimum phase plant zeros [4]. The central object of interest is therefore the sensitivity function and its dimension (or, more generally, any closed loop mapping). Provided the plant is strictly proper and we select the sensitivity function within this class, it also holds that  $\deg(K) < \deg(P)$  (see e.g., [17]). To determine such controllers by the approach in [11], [19], one needs to impose a rank condition, thus destroying the LMI structure of the problem.

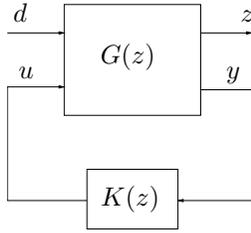


Fig. 2. Standard feedback interconnection.

## II. LMI-BASED DESIGN

We begin by explaining the pertinent formalism and key findings in [19], [11]. Assume the standard setting of a dynamical system  $G$  with two sets of inputs and outputs  $d, u$  and  $z, y$ , respectively, as in Figure 2, and transfer function

$$G(z) = \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix} + \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} (zI - A)^{-1} \begin{pmatrix} B_1 & B_2 \end{pmatrix}$$

with  $(A, B_2, C_2)$  stabilizable and detectable, and  $D_{22} = 0$ . The search for dynamic controllers

$$K(z) = D_K + C_K(zI - A_K)^{-1} B_K$$

having input  $y$ , output  $u$ , dimension  $\deg(K)$ , and ensuring an  $H_\infty$ -gain from  $d$  to  $y$  less than  $\gamma$ , proceeds as follows. Determine a pair of symmetric matrices  $X, Y$  of dimensions  $\deg(P) \times \deg(P)$  satisfying

$$\begin{pmatrix} X & \gamma I \\ \gamma I & Y \end{pmatrix} \geq 0, \quad (3)$$

and

$$\begin{pmatrix} B_2 \\ D_{12} \end{pmatrix}^\perp \left[ \begin{pmatrix} X & 0 \\ 0 & \gamma^2 I \end{pmatrix} - \right. \quad (4)$$

$$\left. - \begin{pmatrix} A & B_1 \\ C_1 & D_{11} \end{pmatrix} \begin{pmatrix} X & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} A & B_1 \\ C_1 & D_{11} \end{pmatrix}' \right] \begin{pmatrix} B_2 \\ D_{12} \end{pmatrix}^{\perp'} > 0$$

$$\begin{pmatrix} C_2' \\ D_{21}' \end{pmatrix}^\perp \left[ \begin{pmatrix} Y & 0 \\ 0 & \gamma^2 I \end{pmatrix} - \right. \quad (5)$$

$$\left. - \begin{pmatrix} A & B_1 \\ C_1 & D_{11} \end{pmatrix}' \begin{pmatrix} Y & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} A & B_1 \\ C_1 & D_{11} \end{pmatrix} \right] \begin{pmatrix} C_2' \\ D_{21}' \end{pmatrix}^{\perp'} > 0,$$

where  $M^\perp$  denotes any matrix whose rows form a basis of the left null space of a matrix  $M$ . The above conditions are linear matrix inequalities and can be easily solved by standard methods.

For any such solution  $(X, Y)$ , we have  $X > 0, Y > 0$  and hence

$$\text{rank} \begin{pmatrix} X & \gamma I \\ \gamma I & Y \end{pmatrix} = \text{rank}(Y - \gamma^2 X^{-1}) + \deg(P). \quad (6)$$

Now compute a factorization

$$NM^{-1}N' = Y - \gamma^2 X^{-1}$$

with  $M$  is a  $k \times k$  invertible matrix, and form the positive definite matrices

$$\hat{Y} := \begin{pmatrix} Y & N \\ N' & M \end{pmatrix}, \quad \hat{X} = \gamma^2 \hat{Y}^{-1}.$$

For each such  $\hat{X}$  there is a ball of controllers defined by

$$\begin{pmatrix} D_K & C_K \\ B_K & A_K \end{pmatrix} = K_0 + R_{\text{left}}^{1/2} L R_{\text{right}}^{1/2}$$

with  $L$  any matrix having norm  $\|L\| < 1$ . The center  $K_0$  and the radii  $R_{\text{left}}$  and  $R_{\text{right}}$  can be computed as in [19, page 174] and the dimension of the controller is  $\deg(K) = k$ . Clearly,

$$\deg(K) \geq \text{rank}(Y - \gamma^2 X^{-1}); \quad (7)$$

i.e., generically,  $\deg(K) \geq \deg(P)$ . If one desires a controller of lower dimension, one needs to choose  $(X, Y)$  so that  $Y - \gamma^2 X^{-1}$  has lower rank, which destroys the LMI structure of the solution set.

The class of all controllers of dimension  $\deg(K)$  is the union of all (possibly overlapping) controller balls obtained by varying  $X, Y$  over the solution set of the earlier linear matrix inequalities together with the rank constraint. We illustrate this with a simple example, to which we return in the next section using a different approach.

Consider once again the sensitivity minimization problem with

$$P(z) = \frac{1}{z-2}$$

and the feedback loop redrawn in Figure 3 in the standard form. Then, the parameters of the nominal system  $G(z)$  are

$$\left( \begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{22} \\ C_2 & D_{21} & D_{22} \end{array} \right) = \left( \begin{array}{c|cc} 2 & 0 & 1 \\ \hline 1 & 1 & 0 \\ 1 & 1 & 0 \end{array} \right).$$

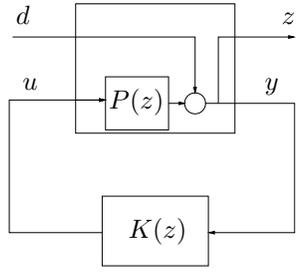


Fig. 3. The feedback loop of Fig. 1 in the standard form.

The linear matrix inequalities (4)–(5) become

$$\begin{aligned} (0 \quad 1) \left[ \begin{pmatrix} x & 0 \\ 0 & \gamma^2 \end{pmatrix} - \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} \right] \begin{pmatrix} 0 \\ 1 \end{pmatrix} &> 0, \\ (1 \quad -1) \left[ \begin{pmatrix} y & 0 \\ 0 & \gamma^2 \end{pmatrix} - \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} \right] \begin{pmatrix} 1 \\ -1 \end{pmatrix} &> 0, \end{aligned}$$

yielding, together with the positivity condition,

$$0 < x < \gamma^2 - 1 \text{ and } 0 < y < \frac{\gamma^2}{3}.$$

Inequality (3) now implies that  $\gamma > 2$ . Let us choose

$$\gamma = \frac{5}{2}.$$

At this point  $x, y$  are independent of each other, and in general (7) will require a controller of degree  $\deg(K) = 1$ . If we want controllers of degree  $\deg(K) = 0$ , i.e., constant gain, we need to take the lower bound in (7) to be zero; i.e.,

$$xy = \gamma^2 = \frac{25}{4}.$$

This gives a range of values for  $x$

$$3 < x < \gamma^2 - 1 = \frac{21}{4}.$$

For each value of  $x$  in this range we compute

$$K_0 = -\frac{2x}{1+x}, \quad R_{\text{left}} = \frac{x^2 - 3x}{x+1}, \quad R_{\text{right}} = \frac{\gamma^2 - (x+1)}{x+1}.$$

Hence the range of controller gains for a given  $x$  is

$$K_0 - R < K < K_0 + R$$

where

$$R := \frac{\sqrt{x(x-3)(\gamma^2 - (x+1))}}{\gamma(x+1)} = \frac{\frac{2}{5}\sqrt{x(x-3)(\frac{21}{4} - x)}}{x+1}.$$

Figure 4 displays the range of controller gains over each admissible value of  $x$ . The dash-dotted curve represents  $K_0$  and the solid curves represent  $K_0 \pm R$  as functions of  $x$ . A typical interval of admissible controller gains for  $x = 3.2$  is highlighted with a thick vertical line (the center indicated with a circle and the end points with an asterisk). Then, the union of all control-gain intervals over the admissible range of  $x$  is

$$-1.8 < K < -1.4.$$

Figures 5 and 6, display the range of sensitivity functions (by showing their respective Bode plots) that can be obtained by choosing  $K$  in the admissible ranges corresponding to values  $x = 3.2$  and  $x = 5.2$ , respectively, for comparison.

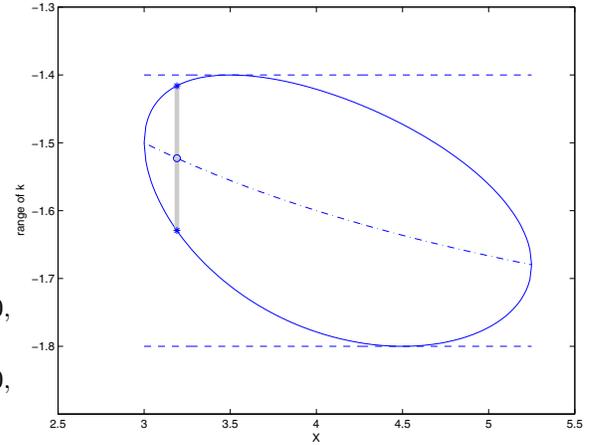


Fig. 4. The range of admissible values for the pair  $(x, K)$ .

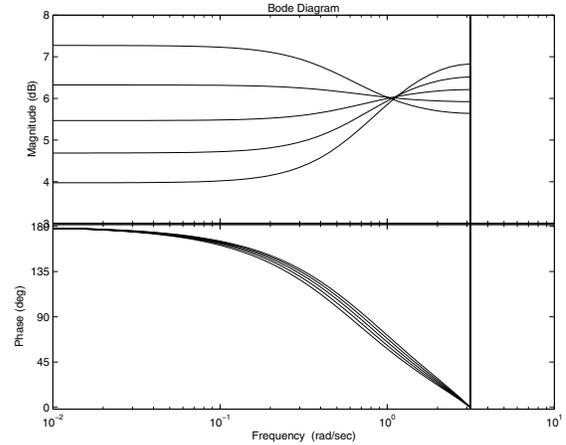


Fig. 5. Range of sensitivity shapes for  $x = 3.2$

### III. SPECTRAL-ZERO BASED DESIGN

We continue by explaining a formalism based on analytic interpolation with degree constraint developed over the last decade by the authors together with several co-workers; see e.g., [4], [5], and the reference therein.

It is well known that  $H_\infty$ -control problems in the most general form discussed in Section II, can be cast as analytic interpolation problems [18], [1]. In fact, using the Youla-Kucera parametrization of stabilizing controllers for  $(A, B_2, C_2)$  (see [9]), the standard control problem in Figure 2 can be brought into the form a so-called 4-block interpolation problem of selecting  $Q$  so as to minimize or bound

$$\|T_1 - T_2 Q T_3\|_\infty,$$

where  $T_i$ ,  $i = 1, 2, 3$ , and  $Q$  are  $H_\infty$ -matrix functions of compatible dimension,  $T_i$ ,  $i = 1, 2, 3$ , obtained from the

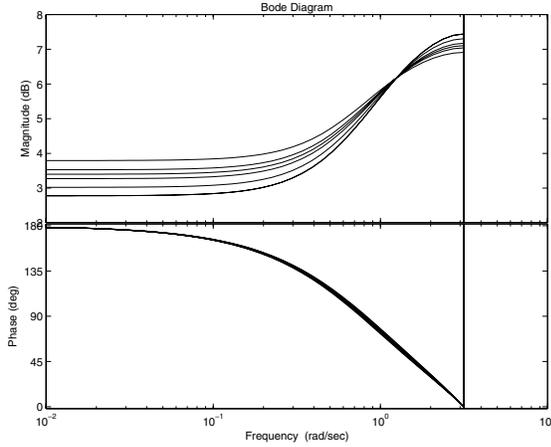


Fig. 6. Range of sensitivity shapes for  $x = 5.2$

problem data, while  $Q$  specifies the controller. The class of possible functions  $T = T_1 - T_2QT_3$  is constrained at the singularities of the  $T_k$ 's,  $k = 2, 3$ , where the value of  $T$  is independent of  $Q$  and agrees with  $T_1$ , giving rise to “tangential interpolation” constraints. In turn, analytic interpolation constraints of a most general nature can be cast as moment constraints [15], [1], and the theory of analytic interpolation with degree constraint can be extended to cover the generalized moment problem [7]. While most of the theory can be carried out in considerable generality (incorporating degree constraints, e.g., see [3], and also [13], [14]), we restrict our attention to the scalar sensitivity shaping problem.

For disturbance attenuation one needs to prescribe the gain of the sensitivity function of the feedback system over different frequency bands according to specification. This is to be achieved by a suitable selection of a (internally) stabilizing controller. Zames [22] formulated the problem of shaping the sensitivity function as an optimization problem. A stable and stably invertible transfer function  $W_1$  is selected, and the controller  $K$  is chosen to minimize the  $H^\infty$  norm of the “weighted sensitivity”  $W_1S$ . This choice of controller, namely

$$K_{\text{opt}} := \operatorname{arginf}\{\|W_1S\|_\infty : K \text{ stabilizing}\},$$

yields a lower bound

$$\gamma_{\text{opt}} = \inf_{K \text{ stabilizing}} \|W_1S\|_\infty$$

on the norm of the weighted sensitivity, where  $S$  is given by (1). The optimal weighted sensitivity, whenever it is attained (e.g., see [22], [9]), turns out to be an all-pass function. In fact, because internal stability can be expressed in the form of the interpolation conditions

$$S(z_i) = 1, \quad i = 1, \dots, \nu \text{ and } S(p_j) = 0, \quad j = 1, \dots, \mu, \quad (8)$$

where  $z_1, \dots, z_\nu$  are the non-minimum phase zeros of the plant  $P(z)$  and  $p_1, \dots, p_\mu$  are the unstable poles [20], it can

be shown that

$$W_1S_{\text{opt}} = \gamma_{\text{opt}} \frac{\beta}{\alpha},$$

where  $\alpha$  is a Schur polynomial of degree  $n = \nu + \mu - 1$  and  $\beta(z) = z^n \alpha(z^{-1})$ .

Such optimization problems are often very sensitive to the problem data [21], and therefore one could focus instead on stabilizing controllers in the bigger class

$$\mathcal{K}_\gamma := \{K \text{ stabilizing} : \|W_1S\|_\infty < \gamma\}$$

with  $\gamma > \gamma_{\text{opt}}$ . The so-called “central solution” in this class of suboptimal controllers,

$$K_{\text{ME}} = \operatorname{argmax}_{K \in \mathcal{K}_\gamma} \int_{-\pi}^{\pi} \log(\gamma^2 - |W_1(e^{i\theta})S(e^{i\theta})|^2) d\theta$$

is easily computed in a state-space formalism [16]. This is known as the maximum entropy solution, and the corresponding weighed sensitivity function again takes the form

$$W_1S_{\text{ME}} = \gamma \frac{\beta}{\alpha}$$

where  $\alpha$  is a Schur polynomial of degree  $n$  and  $\beta$  is a polynomial of degree at most  $n$ .

It turns out that there is an efficient characterization of all admissible sensitivity functions of degree not exceeding  $n$ , which of course includes the maximum entropy one, in terms of a weighted entropy functional

$$\int_{-\pi}^{\pi} W_2(e^{i\theta}) \log(\gamma^2 - |W_1(e^{i\theta})S(e^{i\theta})|^2) d\theta, \quad (9)$$

with

$$W_2 = \left| \frac{\rho}{\tau} \right|^2, \quad (10)$$

where

$$\tau(z) = \prod_{j=1}^{\nu} (z - \bar{z}_j^{-1}) \prod_{j=1}^{\mu} (z - \bar{p}_j^{-1}),$$

and  $\rho$  ranges over the class  $\mathcal{S}_n$  of all monic Schur polynomials of degree at most  $n$  [5]. For each  $\rho \in \mathcal{S}_n$ , the corresponding optimization problem has a unique solution

$$K_\rho = \operatorname{argmax}_{K \in \mathcal{K}_\gamma} \int_{-\pi}^{\pi} W_2(e^{i\theta}) \log(\gamma^2 - |W_1(e^{i\theta})S(e^{i\theta})|^2) d\theta$$

and the weighed sensitivity function again takes the form

$$W_1S_\rho = \gamma \frac{\beta}{\alpha}$$

where  $\alpha \in \mathcal{S}_n$  and  $\deg \beta$  is also bounded by  $n$ .

These polynomials can be computed via convex optimization [4], [5]. In fact the map

$$\varphi : \mathcal{S}_n \rightarrow \mathcal{S}_n : \rho \mapsto \alpha$$

is a homeomorphism onto its image  $\mathcal{A} := \varphi(\mathcal{S}_n)$  [6], and  $\beta$  can be computed from the interpolation conditions once  $\alpha$  is determined. Furthermore,

$$|\alpha|^2 - |\beta|^2 = \lambda|\rho|^2 \quad (11)$$

on the unit circle, for some  $\lambda > 0$ . The correspondence  $\varphi$  provides a complete and smooth parametrization of all solutions in terms of  $\rho \in \mathcal{S}_n$ . The roots of  $\rho$  can be given the interpretation as being either transmission zeros of  $\frac{1}{\gamma}S_\rho$  thought as the scattering function of a passive circuit or, as being spectral zeros of an associated spectral density [4].

The weighted entropy functional in (9) suggests that the polynomial  $\rho$  in (10) can be thought of as a “tuning parameter” in controller design. In fact,  $\rho$  can be chosen to yield large values for  $|W_2|$  in a frequency range where low sensitivity is desired. This added degree of freedom does not increase the degree of  $W_1S_\rho$ . This may often permit a choice of  $W_1$  of low degree, or simply the choice  $W_1 \equiv 1$ . The McMillan degree of the controller  $K_\rho$  is bounded by

$$\deg K_\rho \leq \deg P + \deg W_1 - 1,$$

provided the plant  $P$  is strictly proper.

We illustrate this approach on the elementary example discussed in Section II, where

$$P(z) = \frac{1}{z-2}.$$

Of course, all computations can be done by hand and the full power of convex optimization is not needed. Nevertheless, this example highlights the differences with the approach of Section II.

Since  $P$  has a non-minimum-phase zero at  $z = \infty$  and an unstable pole at  $z = 2$ , the interpolation conditions (8) are  $S(\infty) = 1$  and  $S(z) = 0$ . Consequently, the sensitivity function must take the form

$$S(z) = \frac{z-2}{z-a},$$

where we must have  $-1 < a < 1$  for  $S$  to be analytic in  $|z| > 1$ . It is easy to see that

$$\gamma_{\text{opt}} = \inf_{-1 < a < -1} \|S\|_\infty = 2.$$

We select  $\gamma = \frac{5}{2}$  as before, and write

$$S_\rho(z) = \gamma \frac{\beta(z)}{\alpha(z)} = \gamma \frac{\frac{2}{5}z - \frac{4}{5}}{z-a}.$$

As  $\rho(z)$  ranges over

$$\mathcal{S}_1 = \{z-r : -1 \leq r \leq 1\},$$

the polynomial  $\alpha(z)$  ranges over

$$\mathcal{A} = \left\{ z-a : \frac{1}{5} \leq a \leq \frac{3}{5} \right\}.$$

This can be readily verified from (11) without the need to solve the optimization problem. Indeed, substituting  $\alpha, \beta$  and  $\rho$  in (11) and eliminating  $\lambda$ , we obtain that

$$\frac{r}{1+r^2}a^2 - a + \left( \frac{1}{5} \frac{r}{1+r^2} + \frac{8}{25} \right) = 0.$$

The value  $a$  as a function of  $r$  is plotted in Figure 7 and represents a smooth and complete parametrization of all sensitivity functions of degree 1 (and controllers of degree

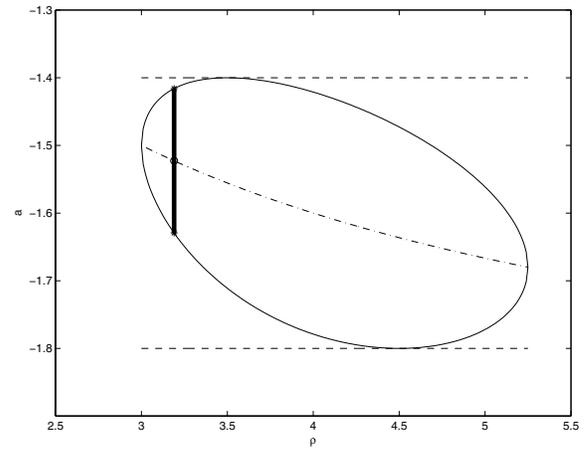


Fig. 7.  $a$  as a function of  $r$

0). In general, (11) represent quadratic equations which are difficult to solve directly. In view of (1),

$$K = (1 - S^{-1})P^{-1} = a - 2$$

is of degree zero. Consequently, the range of constant gains that satisfy  $\|S_\rho\|_\infty < \gamma$  lie in the interval

$$-1.8 \leq K \leq -1.4 \quad (12)$$

in bijective correspondence with elements in  $\rho \in \mathcal{S}_1$ . (This can also be verified directly, e.g., by computing  $\|(z-2)/(z-2-K)\|_\infty$  over the range of  $K$ ). Figure 8 shows Bode plots of  $S_\rho$  for  $\rho = z-r$  and  $r \in [-1, 1]$  at intervals of 0.25 apart.

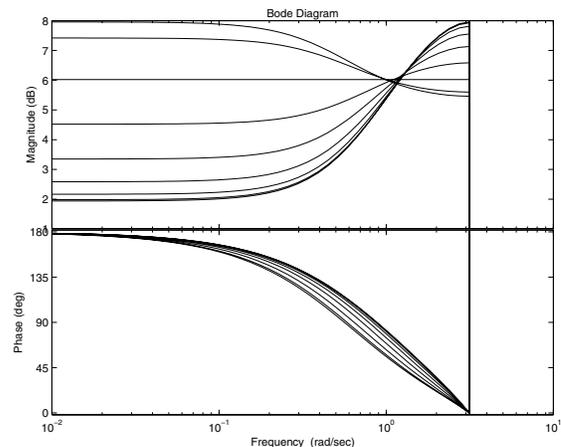


Fig. 8. The complete range of sensitivity shapes under the degree constraint.

A comparison of Figures 5, 6 and 8 reveals a fundamental difference between the approach in Section II and that of Section III. In Section III we parameterize *all* solutions of degree one in terms of the tuning parameter  $r$ , and consequently the whole range of possible sensitivity functions are depicted in Figure 8. In the approach of Section II, the range of sensitivity functions will depend on the particular solution

of the LMIs (4)–(5) and the rank condition (7). The choice  $x = 5.2$  (Figure 6) yields a very narrow subclass of possible sensitivity functions and controllers and the choice  $x = 3.2$  (Figure 5) a somewhat wider. However, no choice of  $x$  will yield the complete class of sensitivity functions depicted in Figure 8 and the corresponding interval of controllers given in (12). In this respect, note the difference of scale in Figures 5 and 8.

#### IV. CONCLUDING REMARKS

It is important to point out that, in Section III, the range of values for the controller and the closed-loop sensitivity function is provided at the outset via the parameterizing set  $\mathcal{S}_n$  and the smooth mapping  $\varphi$ . By way of contrast, in Section II, each particular solution of the linear matrix inequalities (4)–(5) yields only a subset of possible controllers. Moreover, for the sensitivity shaping problem in particular, the controller degree is larger than that obtained in Section III if one wants to avoid imposing the rank condition (7); i.e., insisting on solving only LMIs. The framework of Section III handles in the same way interior as well as boundary points and provides a complete parametrization. This is not the case for the LMI-based approach and in fact, the radii for the balls of controller gains shrink to zero as the solutions of the LMIs tend to boundary values.

The basic framework in [4], [5] extends to the matricial setting, e.g., see [3], [14]. Detailed studies of various robust control problems using this formalism have been carried out in [17], [2] and a multivariable framework is currently under development.

#### V. ACKNOWLEDGMENTS

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