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# Lyapunov Adaptive Stabilization of Parabolic PDEs—Part II: Output Feedback and Other Benchmark Problems

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*Abstract*— **We deal with parametric uncertainties in boundary conditions or reaction terms involving boundary values. We show how adaptive boundary control problems can be solved using output feedback, for unstable PDEs with infinite relative degree. Boundary sensing is employed, along with a Kreisselmeier type adaptive observer. We also design adaptive boundary controllers for a reaction-advection-diffusion system with all three of the coefficients unknown. Our Lyapunov approach yields parameter estimators that do not require the measurement of any of the spatial derivatives of the controlled variable, which are needed in other approaches. The designs in this paper illustrate the requirement in the Lyapunov approach that parameter projection be used in the update laws. Projection is not used as a robustification tool but to prevent adaptation transients that would require overly conservative restrictions on the size of the adaptation gain.**

## I. INTRODUCTION

In this paper we first deal with benchmark problems that contain parametric uncertainties in boundary conditions or reaction terms involving boundary values. Both benchmark plants are unstable. These two problems are solved first with state feedback and then with output feedback, using scalar sensing at the boundary, in which case the plant relative degree is infinite. The output feedback designs employ adaptive observers which we construct as infinite-dimensional extensions of Kreisselmeier-type filters used in [6].

While in [5] we introduced the idea of Lyapunov-based adaptive boundary control on a benchmark example of a reaction-diffusion system with only the destabilizing reaction coefficient unknown, in this paper we also present an adaptive controller for a full reaction-advection-diffusion system with all three of the coefficients considered unknown. The Lyapunov estimators do not require the measurement of spatial derivatives, which are needed in other approaches.

The benchmark problems in this paper expose a limitation of the 'log-Lyapunov paradigm:' in general it requires not only a restriction on the value of the adaptation gain $\gamma$ but also the use of parameter projection. A small $\gamma$ is a tool for preventing destabilizing transients. Projection is used to make the restriction on $\gamma$ a priori verifiable.

The controllers we design are explicit functions of the parameter updates, the spatial variable, and the measured state. The update laws involve gain functions given in quadratures.

## II. PROJECTION

Let $\hat{\theta}$ denote the parameter estimate, to be kept within the interval $\left[\underline{\theta}, \bar{\theta}\right]$ and $\tau$ the nominal update law. The standard projection operator is defined as

$$\mathrm{Proj}_{[\underline{\theta},\bar{\theta}]}\{\tau\} = \tau \begin{cases} 0, & \hat{\theta} = \underline{\theta} \text{ and } \tau < 0 \\ 0, & \hat{\theta} = \bar{\theta} \text{ and } \tau > 0 \\ 1, & \text{else} \end{cases} \tag{1}$$

Unfortunately, (1) is discontinuous. This presents two problems: (1) it is possible to obtain only Filippov solutions; (2) noise may induce frequent switching of the update law. This issue is not as serious as controller switching in sliding mode control because the projection operator does not drive an actuator. Since the projection drives only the update law $\dot{\hat{\theta}}$ there are no discontinuities in $\hat{\theta}(t)$ and therefore no jumps in the control action. However, not having to deal with Filippov solutions is a good enough reason to consider continuous projection where, instead of a hard switch, a boundary layer of width $\delta > 0$ is introduced:

$$\mathrm{Proj}^{\delta}_{[\underline{\theta},\bar{\theta}]}\{\tau\} = \tau \begin{cases} \frac{\hat{\theta}-\underline{\theta}+\delta}{\delta}, & \underline{\theta} - \delta \le \hat{\theta} < \underline{\theta} \text{ and } \tau < 0 \\ \frac{\bar{\theta}+\delta-\hat{\theta}}{\delta}, & \bar{\theta} < \hat{\theta} \le \bar{\theta} + \delta \text{ and } \tau > 0 \\ 1, & \text{else} \end{cases} \tag{2}$$

where the update law $\tau$ is scaled linearly with $\theta$ in the boundary layer. With the help of [6, Lemma E.1] we get:

*Lemma 1:* The following is guarantee;

1) The operator is a locally Lipschitz function of $\hat{\theta}, \tau$ on $\left[\underline{\theta} - \delta, \bar{\theta} + \delta\right] \times \mathbb{R}$.
2) $\left(\mathrm{Proj}^{\delta}_{[\underline{\theta},\bar{\theta}]}\{\tau\}\right)^2 \le \tau^2$.
3) For $\hat{\theta}(0) \in \left[\underline{\theta} - \delta, \bar{\theta} + \delta\right]$, the solution of $\dot{\hat{\theta}} = \mathrm{Proj}^{\delta}_{[\underline{\theta},\bar{\theta}]}\{\tau\}$ remains in $\left[\underline{\theta} - \delta, \bar{\theta} + \delta\right]$.
4) $-\tilde{\theta}\mathrm{Proj}^{\delta}_{[\underline{\theta},\bar{\theta}]}\{\tau\} \le -\tilde{\theta}\tau$, $\forall \hat{\theta} \in \left[\underline{\theta} - \delta, \bar{\theta} + \delta\right], \theta \in \left[\underline{\theta}, \bar{\theta}\right]$.

The properties in Lemma 1 except 1) also hold for (1) (with $\delta = 0$). Projection (1) is preferable in implementation because it is incorporated in the integrator block in Simulink. Hence, to avoid clutter in our presentation, we employ (1).

## III. BENCHMARK PROBLEM $g$

Consider the plant

$$u_t = u_{xx} + g u(0,t) \tag{3}$$

$$u_x(0) = 0\,, \tag{4}$$

where $g$ is a constant, unknown parameter and $u(0,t)$ is the boundary value of $u(x,t)$ at $x = 0$. This system is inspired by a model of a thermal instability in solid propellant rockets [3]. We will control this system via Dirichlet actuation, $u(1,t)$. In the absence of control, $u(1,t) \equiv 0$, the system is unstable if and only if $g > 2$. We assume that this is indeed the case, $g > 2$. Let us further assume that an upper bound $\bar{g}$ on $g$ is known to us. It is important to note that

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such an assumption was not made on $\lambda$ in [5]. We will design an adaptive controller whose update law incorporates the standard projection operator [6] to keep the parameter estimate $\hat{g}$ in the interval $[2, \bar{g}]$, while driving $u(x,t)$ to zero.

A stabilizing control formula was designed in [10] as

$$u(1) = -\int_0^1 \sqrt{\hat{g}} \sinh\left(\sqrt{\hat{g}}(1-\xi)\right) u(\xi) d\xi \,. \tag{5}$$

Consider the variable change

$$w(x) = u(x) + \int_0^x \sqrt{\hat{g}} \sinh\left(\sqrt{\hat{g}}(x-\xi)\right) u(\xi) d\xi \,. \tag{6}$$

It can be shown that [13]

$$\begin{aligned} w_t &= w_{xx} + \dot{\hat{g}} \int_0^x w(\xi) \frac{\sinh\left(\sqrt{\hat{g}}(x-\xi)\right)}{\sqrt{\hat{g}}} d\xi \\ &\quad + \tilde{g} w(0) \cosh\left(\sqrt{\hat{g}} x\right) \,, \end{aligned} \tag{7}$$

with $w_x(0) = w(1) = 0$, where $\tilde{g} = g - \hat{g}$. Consider the Lyapunov function candidate

$$V = \frac{1}{2}\log\left(1 + \|w\|^2\right) + \frac{1}{2\gamma}\tilde{g}^2 \,. \tag{8}$$

Taking its time derivative we arrive at the update law

$$\dot{\hat{g}} = \frac{\gamma}{1+\|w\|^2} \text{Proj}_{[2,\bar{g}]} \left\{ w(0) \int_0^1 w(x) \cosh\left(\sqrt{\hat{g}} x\right) dx \right\} \,. \tag{9}$$

The derivative of the Lyapunov function is

$$\dot{V} = -\frac{\|w_x\|^2}{1+\|w\|^2} + \dot{\hat{g}} \frac{\int_0^1 w(x) \int_0^x w(\xi) \frac{\sinh\left(\sqrt{\hat{g}}(x-\xi)\right)}{\sqrt{\hat{g}}} d\xi dx}{1+\|w\|^2} . \tag{10}$$

It can be shown that

$$\dot{V} \le -\left(1 - 2\gamma \mathrm{e}^{2\sqrt{\bar{g}}}\right) \frac{\|w_x\|^2}{1+\|w\|^2} \,. \tag{11}$$

Stability is thus achieved whenever

$$\gamma < \frac{1}{2}\mathrm{e}^{-2\sqrt{\bar{g}}} \,. \tag{12}$$

This condition highlights the key differences between the design for the PDE in [5] and for the PDE (3). First, the adaptation gain, which was limited by 1 in [5], needs to decrease as $g$ increases in (3). Second, the knowledge of the parameter's upper bound is needed for the plant (3). Projection is used to keep the parameter within the a priori bound, such that the condition (12) is sufficient to achieve stability. It should also be noted that stability can be achieved without projection, by selecting $\gamma$ to satisfy

$$\gamma < \frac{1}{2}\mathrm{e}^{-2\left(\sqrt{2\tilde{g}+\hat{g}(0)} + \left(\gamma \log\left(1+\|w_0\|^2\right)\right)^{1/4}\right)} \,, \tag{13}$$

where $w_0(x)$ is determined using the initial state $u_0(x)$ and the initial parameter estimate $\hat{g}(0)$. While it may be unusual to choose the adaptation gain based on the initial state $u_0$, it is consistent with the Lyapunov function (8), yielding estimates on $\|u(t)\|$ and $\tilde{g}(t)$ that depend on $\|u_0\|$ and $\tilde{g}(0)$. However, in application one would prefer projection due to its added assurance against drift.

One proves boundedness in maximum norm in a similar manner as in [5, Section III]. A lengthy calculation yields

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|w_x\|^2 &= -\|w_{xx}\|^2 - w_x(1) \Big[ \tilde{g} \cosh\left(\sqrt{\hat{g}}\right) w(0) \\ &\quad + \dot{\hat{g}} \int_0^1 w(x) \frac{\sinh\left(\sqrt{\hat{g}}(1-x)\right)}{\sqrt{\hat{g}}} dx \Big] \\ &\quad - \dot{\hat{g}}\sqrt{\hat{g}} \int_0^1 w(x) \int_0^x \sinh\left(\sqrt{\hat{g}}(x-\xi)\right) w(\xi) d\xi dx \\ &\quad - \tilde{g}\hat{g} w(0) \int_0^1 w(x) \cosh\left(\sqrt{\hat{g}} x\right) dx \,, \end{aligned} \tag{14}$$

which can be majorized by

$$\frac{1}{2}\frac{d}{dt}\|w_x\|^2 \le 8\left(\gamma^2 + \tilde{g}^2\right) \mathrm{e}^{4\sqrt{\bar{g}}} \|w_x\|^2 \,. \tag{15}$$

Integrating (11) and (15) one gets boundedness of $\|w_x\|$. Regulation is shown similar as in [5, Section III]. The results in the $u(x,t)$ variable follow from the inverse transformation

$$u(x) = w(x) + \hat{g} \int_0^x (x-\xi) w(\xi) d\xi \,. \tag{16}$$

*Theorem 2:* Suppose that the system (3)–(5), (9) has a well defined classical solution for all $t \ge 0$. Then, for any initial condition $u_0 \in H_1$ and any $\hat{g}(0) \in [2, \bar{g}]$, the solutions $u(x,t)$ and $\hat{g}(t)$ are uniformly bounded and $\lim_{t\to\infty} u(x,t) = 0$ for all $x \in [0,1]$.

## IV. Benchmark Problem $q$

Consider the plant

$$\begin{aligned} u_t &= u_{xx} & (17) \\ u_x(0) &= -q u(0,t) \,, & (18) \end{aligned}$$

where $q$ is constant but unknown. This system is also inspired by [3]. We use Dirichlet actuation via $u(1,t)$. Without control the system is unstable if and only if $q > 1$. We assume that $q > 1$ and that an upper bound $\bar{q}$ on $q$ is known. We use projection to keep $\hat{q}$ in the interval $[1, \bar{q}]$.

A stabilizing control formula for this system is

$$u(1) = -\int_0^1 \hat{q} \mathrm{e}^{\hat{q}(1-\xi)} u(\xi) d\xi \,. \tag{19}$$

Consider the variable change

$$w(x) = u(x) + \int_0^x \hat{q} \mathrm{e}^{\hat{q}(x-\xi)} u(\xi) d\xi \,. \tag{20}$$

It can be shown that [13]

$$\begin{aligned} w_t &= w_{xx} + \dot{\hat{q}} \int_0^x w(\xi) \mathrm{e}^{\hat{q}(x-\xi)} d\xi & (21) \\ w_x(0) &= -\tilde{q} w(0) & (22) \end{aligned}$$

with $w(1) = 0$, where $\tilde{q} = q - \hat{q}$. Consider the Lyapunov function candidate (8) with $\tilde{q}$ instead of $\tilde{g}$. Taking its time derivative we arrive at the update law

$$\dot{\hat{q}} = \frac{\gamma}{1+\|w\|^2} \text{Proj}_{[1,\bar{q}]} \left\{ w(0)^2 \right\} \,. \tag{23}$$

The derivative of the Lyapunov function is

$$\dot{V} = -\frac{\|w_x\|^2}{1+\|w\|^2} + \dot{\hat{q}} \frac{\int_0^1 w(x) \int_0^x w(\xi) \mathrm{e}^{\hat{q}(x-\xi)} d\xi dx}{1+\|w\|^2} . \tag{24}$$

With a lengthy, careful calculation, applying twice the Cauchy-Schwartz inequality, one can show that

$$\left|\int_0^1 w(x)\int_0^x w(\xi)\mathrm{e}^{\hat{q}(x-\xi)}d\xi dx\right| \le \frac{\mathrm{e}^{\hat{q}}}{\sqrt{2}}\|w\|^2 . \quad (25)$$

Using projection and Agmon's inequality, it then follows that

$$\dot{V} \le -\left(1-\sqrt{2}\gamma\mathrm{e}^{\bar{q}}\right)\frac{\|w_x\|^2}{1+\|w\|^2} . \quad (26)$$

Stability is thus achieved whenever $\gamma < \frac{\sqrt{2}}{2}\mathrm{e}^{-\bar{q}}$.

We have proved $L_2$ stability and square integrability in $t$ for $w$. The same holds for $u$ due to the inverse transform

$$u(x) = w(x) + \hat{q}\int_0^x w(\xi)d\xi . \quad (27)$$

Unfortunately, boundedness and regulation of $u(x)$ are hard to prove because of $\tilde{q}$ in (22). This difficulty is consistent with observations in [1], although it was overcome in [8] using a particular "nonlinear damping" feedback, which is not possible here because we do not actuate at $x = 0$.

*Theorem 3:* Suppose that the system (17)–(19), (23) has a well defined classical solution for all $t \ge 0$. Then, for any initial condition $u_0 \in L_2$ and any $\hat{q}(0) \in [1, \bar{q}]$, the spatial $L_2$ norm $\|u(t)\|$ remains bounded and the spatial $H_1$ norm $\|u_x(t)\|$ is square integrable over the infinite time interval. Moreover, the estimate $\hat{q}(t)$ is kept uniformly bounded.

The "frozen adaptation" version of (21) is exp. stable iff $\hat{q} > q - 1$, which justifies the nonnegative update law (23).

## V. Output-Feedback Designs

### A. Benchmark $g$

As in Section III, we consider the plant (3). Suppose that only $u(0,t)$ is measured, whereas $u(1,t)$ is actuated. The transfer function from $u(1,t)$ to $u(0,t)$ has infinitely many poles and no zeros (the relative degree is infinite).

We employ an adaptive observer with input filter

$$\eta_t = \eta_{xx} \quad (28)$$

$$\eta_x(0) = 0 \quad (29)$$

with $\eta(1) = 0$, the output filter

$$v_t = v_{xx} + u(0) \quad (30)$$

with $v_x(0) = v(1) = 0$ and an estimate of $u(x)$ given by $\hat{g}v(x) + \eta(x)$. Our adaptive controller employs the control

$$u(1) = -\int_0^1 \sqrt{\hat{g}}\sinh\left(\sqrt{\hat{g}}(1-\xi)\right)\left(\hat{g}v(\xi)+\eta(\xi)\right)d\xi , \quad (31)$$

and the update law

$$\dot{\hat{g}} = \frac{\gamma}{1+\|w\|^2+a\|v\|^2}\mathrm{Proj}_{[2,\bar{g}]}\left\{v(0) \times\int_0^1\left(av(x)+\hat{g}\cosh\left(\sqrt{\hat{g}}x\right)w(x)\right)dx\right\} \quad (32)$$

where $a$ and $\gamma$ are positive and sufficiently small. The variable change $(\eta, v) \mapsto w(x)$ is defined as

$$w(x) = \hat{g}v(x) + \eta(x) + \int_0^x \sqrt{\hat{g}}\sinh\left(\sqrt{\hat{g}}(x-\xi)\right) \times\left(\hat{g}v(\xi)+\eta(\xi)\right)d\xi . \quad (33)$$

*Theorem 4:* Suppose that (3), (31), (32), (28), (30) has a well defined classical solution for all $t \ge 0$. Then, there exists $a^* > 0$, such that for all $a \in (0, a^*)$ there exists $\gamma^*(a) > 0$ [where both $a^*$ and $\gamma^*(a)$ can be a priori estimated by the designer], such that for all $\gamma \in (0, \gamma^*)$ the following holds: For any initial condition $u_0, \eta_0, v_0 \in H_1$ and any $\hat{g}(0) \in [2, \bar{g}]$, the solutions $u(x,t), \eta(x,t), v(x,t)$ and $\hat{g}(t)$ are uniformly bounded and $\lim_{t\to\infty} u(x,t) = \lim_{t\to\infty}\eta(x,t) = \lim_{t\to\infty} v(x,t) = 0$ for all $x \in [0,1]$.

*Proof:* We start by showing that [13]

$$w_t = w_{xx} + \dot{\hat{g}}Q + \tilde{g}\cosh\left(\sqrt{\hat{g}}x\right)(e(0)+w(0)) \quad (34)$$

and $w_x(0) = w(1) = 0$, where $\tilde{g} = g - \hat{g}$, signal $Q$ is

$$Q(x) = v(x) - \int_0^x\left(\hat{g}v(\xi)+w(\xi)\right)\frac{\sinh\left(\sqrt{\hat{g}}(x-\xi)\right)}{\sqrt{\hat{g}}}d\xi , \quad (35)$$

and $e(x,t)$ is an observer error defined as

$$e = u - gv - \eta , \quad (36)$$

and governed by $e_t = e_{xx}, e_x(0) = e(1) = 0$. Consider

$$V = \frac{1}{2}\log\left(1+\|w\|^2+a\|v\|^2\right) + \frac{b}{2}\|e\|^2 + \frac{1}{2\gamma}\tilde{g}^2 , \quad (37)$$

where $a \in (0,1)$ and $b$ are positive constants yet to be defined. We note that

$$\frac{1}{2}\frac{d}{dt}\|e\|^2 = -\|e_x\|^2 \quad (38)$$

and, with (36), (33), and (30), that

$$\frac{1}{2}\frac{d}{dt}\|v\|^2 = -\|v_x\|^2 + \left(w(0)+e(0)+\tilde{g}v(0)\right)\int_0^1 v(\xi)d\xi . \quad (39)$$

With (37), (38), (39), and (34), we get

$$\dot{V} = \frac{1}{1+\|w\|^2+a\|v\|^2}\left\{-\|w_x\|^2 - a\|v_x\|^2 + e(0)\int_0^1\left(av(x)+\hat{g}\cosh\left(\sqrt{\hat{g}}x\right)w(x)\right)dx + aw(0)\int_0^1 v(x)dx + \dot{\hat{g}}\int_0^1 w(x)Q(x)dx\right\} - b\|e_x\|^2$$

which can be majorized by

$$\dot{V} \le \frac{1}{1+\|w\|^2+a\|v\|^2}\left\{-(1-8a)\|w_x\|^2 - \frac{a}{2}\|v_x\|^2 - b\|e_x\|^2 + e(0)\int_0^1 \hat{g}\cosh\left(\sqrt{\hat{g}}x\right)w(x)dx + ae(0)\int_0^1 v(x)dx + \dot{\hat{g}}\int_0^1 w(x)Q(x)dx\right\} . \quad (40)$$

By applying Young's inequality to $e(0)$ terms, we get

$$\dot{V} \le \frac{1}{1+\|w\|^2+a\|v\|^2}\left\{-\left(1-8a-\frac{2}{\mu_1}\right)\|w_x\|^2 - \left(\frac{a}{2}-\frac{2}{\mu_2}\right)\|v_x\|^2 - \left(b-2\mu_2 a^2 - 2\mu_1\bar{g}^2\mathrm{e}^{2\sqrt{\bar{g}}}\right) \times\|e_x\|^2 + \dot{\hat{g}}\int_0^1 w(x)Q(x)dx\right\} , \quad (41)$$

where $\mu_1$ and $\mu_2$ are positive constants that we can arbitrarily choose in our analysis. It can be shown that

$$\left|\int_0^1 w(x)Q(x)dx\right| \leq 2\mathrm{e}^{2\sqrt{\bar{g}}}\left(\|w\|^2+\|v\|^2\right), \quad (42)$$

which can then be used to prove that

$$\left|\dot{\hat{g}}\int_0^1 w(x)Q(x)dx\right| \leq 2\frac{\gamma}{a}\mathrm{e}^{2\sqrt{\bar{g}}}|v(0)|\left(a\|v\|+\mathrm{e}^{2\sqrt{\bar{g}}}\|w\|\right) \quad (43)$$

With further calculations involving Young's, Poincare's, and Agmon's inequalities, and using that fact that $a,\gamma\in(0,1)$, one arrives at a conservative bound

$$\left|\dot{\hat{g}}\int_0^1 w(x)Q(x)dx\right| \leq 80\frac{\gamma}{a^2}\mathrm{e}^{8\sqrt{\bar{g}}}\|v_x\|^2+\frac{1}{4}\|w_x\|^2 \quad (44)$$

Substituting this bound into (41), we get

$$\begin{aligned}\dot{V} \leq & \frac{1}{1+\|w\|^2+a\|v\|^2}\left\{-\left(\frac{3}{4}-8a-\frac{2}{\mu_1}\right)\|w_x\|^2\right.\\ &-\left(\frac{a}{2}-\frac{2}{\mu_2}-80\frac{\gamma}{a^2}\mathrm{e}^{8\sqrt{\bar{g}}}\right)\|v_x\|^2\\ &\left.-\left(b-2\mu_2a^2-2\mu_1\bar{g}^2\mathrm{e}^{2\sqrt{\bar{g}}}\right)\|e_x\|^2\right\}. \quad (45)\end{aligned}$$

Selecting $a^*=\frac{1}{16},\gamma^*=\frac{a^3}{320}\mathrm{e}^{-8\sqrt{\bar{g}}},\mu_1=16,\mu_2=\frac{16}{a},b=64\left(a+\bar{g}^2\mathrm{e}^{2\sqrt{\bar{g}}}\right)$, for $a\in(0,a^*]$ and $\gamma\in(0,\gamma^*]$, we obtain

$$\dot{V} \leq -\frac{1}{8}\frac{\|w_x\|^2+a\|v_x\|^2+4b\|e_x\|^2}{1+\|w\|^2+a\|v\|^2}. \quad (46)$$

W conclude the boundedness of $\|w\|,\|v\|$ and integrability of $\|w_x\|^2,\|v_x\|^2$. It follows that $\|Q\|$ is bounded and, with Agmon's inequality, that $\dot{\hat{g}}$ is square integrable, which implies that $\dot{\hat{g}}\|Q\|$ is square integrable. Agmon's inequality also guarantees that $\tilde{g}\cosh\left(\sqrt{\hat{g}}x\right)(e(0)+w(0))$, which appears in (34), is square integrable. These properties can be used to show that $\|w_x\|$ is bounded. With a similar argument, showing that $u(0)=w(0)+e(0)+\tilde{g}v(0)$ is square integrable, one concludes that $\|v_x\|$ is bounded. One can show next that $\dot{\hat{g}}Q+\tilde{g}\cosh\left(\sqrt{\hat{g}}x\right)(e(0)+w(0))$ and $u(0)$ are bounded and use that to prove that the time derivatives of $\|w\|^2,\|v\|^2$ are bounded. By Barbalat's lemma this implies the regulation of $\|w\|,\|v\|$, and, by Agmon's inequality, the regulation of $w(x),v(x)$ for all $x\in[0,1]$. To obtain boundedness and regulation results for $u$, we first use the inverse

$$\eta(x)=w(x)-\hat{g}v(x)+\hat{g}\int_0^x(x-\xi)w(\xi)d\xi \quad (47)$$

and then invoke (36). ∎

The conservative values of $a^*$ and $\gamma^*$ are for the proof only. In implementation one would use higher values.

### *B. Benchmark q*

As in Section IV, we consider the plant (17)–(18), where only $u(0,t)$ is measured. The transfer function from $u(1,t)$ to $u(0,t)$ is unstable and of infinite relative degree.

Our output feedback adaptive controller uses the same input filter (28), but with an output filter

$$\begin{aligned}v_t &= v_{xx} \quad (48)\\ v_x(0) &= -u(0) \quad (49)\\ v(1) &= 0, \quad (50)\end{aligned}$$

a control law

$$u(1)=-\int_0^1\hat{q}\mathrm{e}^{\hat{q}(1-\xi)}\left(\hat{q}v(\xi)+\eta(\xi)\right)d\xi, \quad (51)$$

and an update law

$$\begin{aligned}\dot{\hat{q}}=&\frac{\gamma}{1+\|w\|^2+a\|v\|^2}\mathrm{Proj}_{[1,\bar{q}]}\left\{v(0)\right.\\ &\left.\times\left(av(0)+\hat{q}\left(w(0)+\hat{q}\int_0^1\mathrm{e}^{\hat{q}x}w(x)dx\right)\right)\right\}. \quad (52)\end{aligned}$$

The variable change $(\eta,v)\mapsto w(x)$ is defined as

$$w(x)=\hat{q}v(x)+\eta(x)+\int_0^x\hat{q}\mathrm{e}^{\hat{q}(x-\xi)}\left(\hat{q}v(\xi)+\eta(\xi)\right)d\xi. \quad (53)$$

*Theorem 5:* Suppose that the system (17), (51), (52), (28), has a well defined classical solution for all $t\geq 0$. Then, there exists $a^*>0$, such that for all $a\in(0,a^*)$ there exists $\gamma^*(a)>0$ [where both $a^*$ and $\gamma^*(a)$ can be a priori estimated by the designer], such that for all $\gamma\in(0,\gamma^*)$ the following holds: For any initial condition $u_0,\eta_0,v_0\in L_2$ and any $\hat{q}(0)\in[1,\bar{q}]$, the spatial $L_2$ norms $\|u(t)\|,\|\eta\|,\|v\|$ remain bounded and the spatial $H_1$ norms $\|u_x(t)\|,\|\eta_x(t)\|,\|v_x(t)\|$ are square integrable over the infinite time interval. Moreover, $\hat{q}(t)$ is bounded.

To prove this result we first show that [13]

$$\begin{aligned}w_t &= w_{xx}+\dot{\hat{q}}\left\{v+\int_0^x\mathrm{e}^{\hat{q}(x-\xi)}\left(\hat{q}v(\xi)+w(\xi)\right)d\xi\right\}\\ &\quad+\hat{q}^2\mathrm{e}^{\hat{q}x}\left(e(0)+\tilde{q}v(0)\right) \quad (54)\\ w_x(0) &= -\hat{q}\left(e(0)+\tilde{q}v(0)\right) \quad (55)\end{aligned}$$

with $w(1)=0$, and proceed with (37), with $\tilde{q}$ instead of $\tilde{g}$, going through inequalities as in Section V-A. The inverse transform for deducing the properties of $\eta$, $u$ is

$$\eta(x)=w(x)-\hat{q}v(x)+\hat{q}\int_0^x w(\xi)d\xi. \quad (56)$$

## VI. Design for Systems with Unknown Diffusion and Advection Coefficients

Consider the system

$$u_t=\varepsilon u_{xx}+bu_x+\lambda u \quad (57)$$

with $u(0)=0$, where $\epsilon,b,\lambda$ are unknown constants.

The control law for this system is [10]

$$u(1)=-\int_0^1\frac{\hat{\lambda}+c}{\hat{\varepsilon}}\xi e^{-\frac{\hat{b}}{2\hat{\varepsilon}}(1-\xi)}\frac{I_1\left(\sqrt{\frac{\hat{\lambda}+c}{\hat{\varepsilon}}(1-\xi^2)}\right)}{\sqrt{\frac{\hat{\lambda}+c}{\hat{\varepsilon}}(1-\xi^2)}}u(\xi)d\xi, \quad (58)$$

where $\hat{\varepsilon},\hat{b},\hat{\lambda}$ are the estimates of $\epsilon,b,\lambda$ and $c\geq 0$ is a design gain. Using the transformation

$$w(x)=u(x)-\int_0^x k(x,\xi)u(\xi)\,d\xi \quad (59)$$

$$k(x,\xi)=-\frac{\hat{\lambda}+c}{\hat{\varepsilon}}\xi e^{-\frac{\hat{b}}{2\hat{\varepsilon}}(x-\xi)}\frac{I_1\left(\sqrt{\frac{\hat{\lambda}+c}{\hat{\varepsilon}}(x^2-\xi^2)}\right)}{\sqrt{\frac{\hat{\lambda}+c}{\hat{\varepsilon}}(x^2-\xi^2)}}, \quad (60)$$

and its inverse

$$u(x) = w(x) + \int_0^x l(x,\xi) w(\xi)\, d\xi \quad (61)$$

$$l(x,\xi) = -\frac{\hat\lambda + c}{\hat\varepsilon} \xi e^{-\frac{\hat b}{2\hat\varepsilon}(x-\xi)} \frac{J_1\left(\sqrt{\frac{\hat\lambda+c}{\hat\varepsilon}(x^2-\xi^2)}\right)}{\sqrt{\frac{\hat\lambda+c}{\hat\varepsilon}(x^2-\xi^2)}}, \quad (62)$$

we get [13]

$$\begin{aligned} w_t &= \varepsilon w_{xx} + b w_x - c w + \dot{\hat\varepsilon} \int_0^x \varphi_1 w d\xi + \dot{\hat b} \int_0^x \varphi_2 w\, d\xi \\ &\quad + \dot{\hat\lambda} \int_0^x \varphi_3 w d\xi - \tilde\varepsilon \left( \frac{\hat\lambda + c}{\hat\varepsilon} w + \frac{\hat b}{\hat\varepsilon} \int_0^x \varphi_4 w d\xi \right) \\ &\quad + \tilde b \int_0^x \varphi_4(x,\xi) w(\xi)\, d\xi + \tilde\lambda w \end{aligned} \quad (63)$$

with $w(0) = w(1) = 0$, where

$$\varphi_1(x,\xi) = -\frac{\hat\lambda + c}{\hat\varepsilon} \varphi_3(x,\xi) - \frac{\hat b}{\hat\varepsilon} \varphi_2(x,\xi) \quad (64)$$

$$\varphi_2(x,\xi) = \frac{x-\xi}{2\hat\varepsilon} k(x,\xi) + \frac{1}{2\hat\varepsilon} \int_\xi^x (x-\sigma) k(x,\sigma) l(\sigma,\xi)\, d\sigma \quad (65)$$

$$\varphi_3(x,\xi) = \frac{\xi}{2\hat\varepsilon} e^{-\frac{\hat b}{2\hat\varepsilon}(x-\xi)} \quad (66)$$

$$\varphi_4(x,\xi) = \mathrm{div} k(x,\xi) + \int_\xi^x (\mathrm{div} k(x,\sigma)) l(\sigma,\xi)\, d\sigma \quad (67)$$

$$\mathrm{div} k(x,\xi) = \frac{1}{\xi} k(x,\xi) + \frac{\hat\lambda + c}{\hat\varepsilon} e^{-\frac{\hat b}{2\hat\varepsilon}(x-\xi)} \frac{\xi}{x+\xi} \times I_2\left( \sqrt{\frac{\hat\lambda + c}{\hat\varepsilon}(x^2-\xi^2)} \right) \quad (68)$$

Based on (63) and the Lyapunov function

$$V = \frac{1}{2}\left( \log(1+\|w\|^2) + \frac{\tilde\varepsilon^2 + \tilde b^2 + \tilde\lambda^2}{\gamma} \right) \quad (69)$$

we choose the update laws

$$\dot{\hat\lambda} = \gamma \mathrm{Proj}_{[\underline{\lambda},\bar\lambda]}\{\tau_\lambda\} \quad (70)$$

$$\dot{\hat b} = \gamma \mathrm{Proj}_{[\underline{b},\bar b]}\{\tau_b\} \quad (71)$$

$$\dot{\hat\varepsilon} = -\gamma \mathrm{Proj}_{[\underline{\varepsilon},\bar\varepsilon]}\left\{ \frac{\left(\hat\lambda + c\right)\tau_\lambda + \hat b \tau_b}{\hat\varepsilon} \right\}, \quad (72)$$

where

$$\tau_\lambda = \frac{\|w\|^2}{1+\|w\|^2} \quad (73)$$

$$\tau_b = \gamma \frac{\int_0^1 w(x) \int_0^x \varphi_4(x,\xi) w(\xi)\, d\xi dx}{1+\|w\|^2} \quad (74)$$

and the projection, defined in the Appendix, is used to keep the parameter estimates within a priori bounds $[\underline{\lambda},\bar\lambda]$, $[\underline{b},\bar b]$, and $[\underline{\varepsilon},\bar\varepsilon]$, where $\underline{\varepsilon} > 0$. As in the earlier problems, $\gamma$ is limited by an upper bound which can be a priori computed.

*Theorem 6:* Suppose that the system (57)–(58), (70)–(72) has a well defined classical solution for all $t \geq 0$. Then, there exists $\gamma^* > 0$ such that, for all $\gamma \in (0,\gamma^*)$, for any initial condition $u_0 \in H_1$ and any $\hat\lambda(0) \in [\underline{\lambda},\bar\lambda]$, $\hat b(0) \in [\underline{b},\bar b]$, and $\hat\varepsilon(0) \in [\underline{\varepsilon},\bar\varepsilon]$, the solutions $u(x,t)$ and $\hat\lambda(t), \hat b(t), \hat\varepsilon(t)$ are uniformly bounded and $\lim_{t\to\infty} u(x,t) = 0$ for all $x \in [0,1]$.

*Proof:* It can be shown that

$$\dot V = \frac{1}{1+\|w\|^2}\left( -\varepsilon\|w_x\|^2 - c\|w\|^2 + \dot{\hat\varepsilon} F_1 + \dot{\hat b} F_2 + \dot{\hat\lambda} F_3 \right), \quad (75)$$

where $F_i(x) = \int_0^1 w(x) \int_0^x \varphi_i(x,\xi) w(\xi)\, d\xi dx$ for $i = 1,2,3,4$. By applying the Cauchy-Schwartz inequality twice, we get $|F_i| \leq \|w\|^2 \left( \int_0^1 \int_0^x \varphi_i(x,\xi)^2 d\xi dx \right)^{1/2}$. Because the functions $\varphi_i(x,\xi)$ are continuous in $x, \xi, \hat\varepsilon, \hat b, \hat\lambda$ over the domain of their definition given by $\mathcal{T} \times [\underline{\varepsilon},\bar\varepsilon] \times [\underline{b},\bar b] \times [\underline{\lambda},\bar\lambda]$, where $\underline{\varepsilon} > 0$ and $\mathcal{T} = \{x,\xi \in \mathbb{R} | 0 \leq \xi \leq x \leq 1\}$, it can be shown that there exist continuous, nonnegative-valued, nondecreasing functions $M_i : \mathbb{R}_+^5 \to \mathbb{R}_+$ such that

$$\int_0^1 w(x) \int_0^x \varphi_i(x,\xi) w(\xi)\, d\xi dx \leq M_i\left( \frac{1}{\underline{\varepsilon}}, |\underline{b}|, |\bar b|, |\underline{\lambda}|, |\bar\lambda| \right) \quad (76)$$

The simplest one among these functions is $M_3 = \frac{1}{4\sqrt{3}\underline{\varepsilon}} e^{\frac{1}{\underline{\varepsilon}} \max\{|\underline{b}|,|\bar b|\}}$. From (75)–(76), it follows that

$$\begin{aligned} \dot V &\leq \frac{1}{1+\|w\|^2}\left[ -\varepsilon\|w_x\|^2 - c\|w\|^2 + \frac{\gamma\|w\|^4}{1+\|w\|^2} \right. \\ &\quad \left. \times \left( \frac{|\bar\lambda| + c}{\underline{\varepsilon}} M_1 + \frac{|\bar b|}{\underline{\varepsilon}} M_4 M_1 + M_4 M_2 + M_3 \right) \right] \end{aligned} \quad (77)$$

where we emphasize the emergence of the fourth power of $\|w\|$ in the last term of the first line of (77). By applying Poincare's inequality we obtain

$$\dot V \leq -\frac{\varepsilon\left(1-\gamma/\gamma^*\right)\|w_x\|^2 + c\|w\|^2}{1+\|w\|^2}, \quad (78)$$

where $\gamma^* = \frac{\varepsilon}{4}\left( \frac{|\bar\lambda|+c}{\underline{\varepsilon}} M_1 + \frac{|\bar b|}{\underline{\varepsilon}} M_4 M_1 + M_4 M_2 + M_3 \right)^{-1}$. This establishes the boundedness of $\|w\|$ for $\gamma < \gamma^*$.

To prove the boundedness of $\|w_x\|^2$, we show that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|w_x\|^2 &= -\varepsilon\|w_{xx}\|^2 - \frac{\varepsilon c + \varepsilon\tilde\lambda + \lambda\tilde\varepsilon}{\hat\varepsilon}\|w_x\|^2 \\ &\quad - \int_0^1 w_{xx}(x) G(x) dx, \end{aligned} \quad (79)$$

where

$$\begin{aligned} G(x) &= b w_x + \dot{\hat\varepsilon} \int_0^x \varphi_1 w d\xi + \dot{\hat b} \int_0^x \varphi_2 w d\xi \\ &\quad + \dot{\hat\lambda} \int_0^x \varphi_3 w d\xi + \frac{\varepsilon\hat b - \hat\varepsilon b}{\hat\varepsilon} \int_0^x \varphi_4 w d\xi. \end{aligned} \quad (80)$$

Next we note that $\left|\dot{\hat\lambda}\right| \leq \gamma$, $\left|\dot{\hat b}\right| \leq \gamma M_4$, $\left|\dot{\hat\varepsilon}\right| \leq \gamma M_5$, $\left|\frac{\varepsilon\hat b - \hat\varepsilon b}{\hat\varepsilon}\right| \leq M_6$, where

$$M_5 = \frac{\max\{|\underline{\lambda}|, |\bar\lambda|\} + c + M_4 \max\{|\underline{b}|, |\bar b|\}}{\underline{\varepsilon}} \quad (81)$$

$$M_6 = 2\frac{\bar\varepsilon}{\underline{\varepsilon}} \max\{|\underline{b}|, |\bar b|\}. \quad (82)$$

With Young's inequality we get

$$-\int_0^1 w_{xx}(x)G(x)dx \leq \varepsilon\|w_{xx}\|^2 + \frac{1}{4\varepsilon}\|G\|^2 . \quad (83)$$

Let us denote

$$H_i(x) = \int_0^x \varphi_i(x,\xi)w(\xi)\,d\xi \quad (84)$$

for $i = 1, 2, 3, 4$, for which, with the Cauchy-Schwartz inequality, we get

$$\|H_i\| \leq M_i\|w\| . \quad (85)$$

Then, from (80)–(85), with the triangle inequality and Poincare's inequality we obtain

$$\begin{aligned}\|G\|^2 \quad &\leq \quad 8\left[b + \gamma\left(M_5M_1^2 + M_4M_2^2 + M_3^2\right)\right. \\ &\left. +M_6M_4^2\right]\|w_x\|^2 . \quad (86)\end{aligned}$$

Substituting (86) into (83) and then into (79), we get

$$\frac{1}{2}\frac{d}{dt}\|w_x\|^2 \leq N\|w_x\|^2 , \quad (87)$$

where

$$\begin{aligned}N(t) \quad &= \quad \frac{2}{\varepsilon}\left[b + \gamma\left(M_5M_1^2 + M_4M_2^2 + M_3^2\right) + M_6M_4^2\right] \\ &-\frac{\varepsilon c + \varepsilon\tilde{\lambda} + \lambda\tilde{\varepsilon}}{\hat{\varepsilon}} \quad (88)\end{aligned}$$

is bounded. With $\|w\|$ bounded, from (78) we get that $\|w_x\|^2$ is integrable over infinite time. By integrating (87), it follows that $\|w_x\|$ is bounded. By Agmon's inequality, $w(x,t)$ is also bounded for all $t \geq 0$ and for all $x \in [0,1]$.

To show regulation, we calculate

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|w\|^2 \quad &= \quad -\varepsilon\|w_x\|^2 - c\|w\|^2 + \dot{\hat{\varepsilon}}F_1 + \dot{\hat{b}}F_2 + \dot{\hat{\lambda}}F_3 \\ &-\tilde{\varepsilon}\left(\frac{\hat{\lambda}+c}{\hat{\varepsilon}}\|w\|^2 + \frac{\hat{b}}{\hat{\varepsilon}}F_4\right) + \tilde{b}F_4 + \tilde{\lambda}\|w\|^2 . \quad (89)\end{aligned}$$

All the terms on the right hand side have been proved bounded. Therefore $\frac{d}{dt}\|w\|^2$ is bounded. Since $\|w\|^2$ is also integrable over infinite time, by Barbalat's lemma $\|w(t)\| \to 0$ as $t \to \infty$. Regulation in maximum norm follows from Agmon's inequality and the boundedness of $\|w_x\|$.

To infer the results for the original variable $u(x,t)$ from those for $w(x,t)$, we recall the inverse transformation (61)–(62), which is a bounded operator in both $L_2$ and $H_1$. ■

While the Lyapunov design requires the use of projection and a low adaptation gain, one of its remarkable properties is that, even though the plant has parametric uncertainties multiplying $u_x$ and $u_{xx}$, the adaptive scheme does not require the measurement of neither $u_x$ nor $u_{xx}$. The update laws (70)–(72) employ only the measurement of $u$. This is in contrast with adaptive controllers in [1], [2], [4], [14] for reaction-advection-diffusion systems which require the measurement of $u_{xx}$ to estimate the unknown diffusion coefficient $\varepsilon$.

The update laws employ $\varphi_4(x,\xi)$ which is given in quadratures. The integral in (67) would be calculated numerically, just like the other integrals appearing in the update laws and depending on the measured state $u(x,t)$.

## VII. Future Work

The need for projection and a bound on the adaptation gain are the peculiarities of the Lyapunov approach. In another paper on "estimation-based" approaches to adaptive control of PDEs [13] we present methods without projection or limits on the adaptation gain. The methods employ 'passivity/observer-based' and 'swapping-based' identifiers presented for finite-dimensional systems in [6]. However, in the case of uncertain diffusion and advection coefficients, these schemes require the measurement of $u_x(x,t)$ (and in some cases of $u_{xx}(x,t)$), like in [1], [2], [4], [14]. The Lyapunov schemes in Section VI require only $u(x,t)$.

While, for the sake of clarity, we chose to present our design tools through benchmark problems, it is possible to develop an adaptive controller for the class of systems

$$\begin{aligned}u_t \quad &= \quad \varepsilon u_{xx} + bu_x + \lambda u + gu(0) \quad (90) \\ u_x(0) \quad &= \quad -qu(0) , \quad (91)\end{aligned}$$

where $\varepsilon, b, \lambda, g, q$ are unknown.

We have not worked out yet an extension of Section VI to output-feedback. Although boundary observers for this class of systems were developed in [11] for $\varepsilon, b, \lambda$ known, the adaptive observer design will be more complex than for the systems in Sections V-A and V-B.

It is possible to extend the results of this paper to arbitrary dimension. For example, in 3D we can extend them to rectangular parallelepiped domains with $u_{xx}$ in (90) replaced by $\Delta u$ and $bu_x$ replaced by $b_1u_x + b_2u_y + b_3u_z$.

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