

# Consensus Control for a class of Networks of Dynamic Agents: Fixed Topology

Guangming Xie and Long Wang

**Abstract**—This paper investigates the consensus problems for networks of dynamic agents. The agent dynamics is adopted as a typical point mass model based on the Newton's law. The average-consensus problem is proposed for such class of networks, which includes two aspects, the agreement of the states of the agents, and the convergence to zero of the speeds of the agents. A linear consensus protocol for such networks is established for solving such a consensus problem that includes two parts, a local speed feedback controller and the interactions from the finite neighbors. The convergence analysis is proved and the protocol performance is discussed as well. The simulation results are presented that are consistent with our theoretical results.

## I. INTRODUCTION

In recent years, decentralized control of communicating-agent systems has emerged as a challenging new research area. It has attracted multi-disciplinary researchers in a widely range including physics, biophysics, neurobiology, systems biology, apply mathematics, mechanics, computer science and control theory. The applications of multi-agent systems are diverse, ranging from cooperative control of unmanned air vehicles, formation control of mobile robots, control of communication networks, design of sensor-network, to flocking of social insects, swarm-based computing, etc. A common characteristics of the relevant analytical techniques is that they are deeply connected with decentralized, or networked control theory.

Agreement and consensus protocol design is one of the important problems encountered in decentralized control of communicating-agent systems. It has been paid attention for a long time by computer scientists, particularly in the field of automata theory and distributed computation [1]. Agreement upon certain quantities of interest is required in many applications such as multivehicle systems, multirobot systems, groups of agents and so on.

In the past decade, quite a tremendous amount of interesting results have been addressed for agreement and consensus problems in different formulations due to different type of agent dynamics and different type of tasks of interest. In [2], the problem of cooperation among a collection of vehicles performing a shared task using intervehicle communication to coordinate their actions was considered. The agents in the group were with linear dynamics. Tools from algebraic graph theory were used to prove the formation

stability. In [3], a dynamic graph structure was provided as a convenient framework for modelling distributed dynamic systems where the topology of the interaction among its elements evolves in time. Some promising directions were highlighted as well.

Followed the pioneering work in [4], there are many researchers have worked in analysis of swarms [5]-[9],[13]-[24]. In [5], the stability analysis for swarms with continuous-time model in  $n$ -dimensional space was addressed. Following this direction, stability analysis of social foraging swarms that move in an  $n$ -dimensional space according to an attractant/repellent or a nutrient profile was addressed in [6]. The corresponding results in the case of noisy environment was given in [7].

Different from the above disciplinary, in [8] and [9], a model of coordinated dynamical swarms with physical size and asynchronous communication was introduced and analysis of stability properties of such swarms were presented with a fixed communication topology. A potential application of these theoretical results is in the field of the leader-follower formation control of multi-robot systems [10]-[12].

In [13], a simple discrete-time model of finite autonomous agents all moving in the plane with same speed but with different heading was proposed. Moreover, the concept of Neighbors of agents was introduced. Some simulation results to demonstrate the nearest neighbor rule were obtained. Based on this model, theoretical explanations were first given in [14] for the simulation results in [13]. Some sufficient conditions for coordination of the system of agents in the point of view of statistical mechanics. Another qualitative analysis for this model under certain simplifying assumption was given in [15].

In [16], a systematical framework of consensus problem in networks of dynamic agents with fixed/switching topology and communication time-delays was addressed. Under the assumption that the dynamic of the agent is a simple scalar continuous-time integrator  $\dot{x} = u$ , three consensus problems were discussed. They are directed networks with fixed topology, directed networks with switching topology and undirected networks with communication time-delays and fixed topology. Moreover, a disagreement function was introduced for disagreement dynamics of a directed network with switching topology. The undirected networks case was discussed by the same authors in [17]. Some other interesting results can be seen in [18]-[24] and the references therein.

Meanwhile, there are many researchers in physics, bio-

The authors are with the Center for Systems and Control, LTCS and Department of Mechanics and Engineering Science, Peking University, Beijing, 100871, China. E-mail: xiegming@mech.pku.edu.cn.

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physics who consider a closely related to consensus problems on graphs, named as synchronization of coupled oscillators where a consensus is reached regarding the frequency of oscillation of all agents [25]-[34].

In this paper, we follow the work in [16][17] and consider consensus problem for a more general class of networks. In our network model, the dynamic of the agents is a type of kinematic model, it can be viewed as an approximation of a model with point mass which moves based on the Newton's law  $ma = F$ . Such a dynamic is more general and complex than a scalar integrator in [16][17] and can be used to model more processes in reality. The main contribution in this paper is to pose and address consensus problems for undirected networks of point mass dynamic agents with fixed topology. Not only the convergence of the topology is presented, but also the performance of reaching an agreement is discussed. The case where the switching topology is used is treated separately in another paper [36].

An outline of this paper is as follows. In Section II, we recall the consensus problems on graphs. In Section III, the control protocol is given. The convergence analysis and performance discussion are presented in Sections IV and V, respectively. The simulation results are presented in Section VI. Finally, we conclude the paper in Section VII.

## II. CONSENSUS PROBLEMS ON GRAPH

In this section, we introduce networks of dynamic agents and consensus problems.

### A. Algebraic Graph Theory

Let  $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$  be a undirected graph with the set of vertices  $\mathcal{V} = \{v_1, v_2, \dots, v_M\}$ , the set of edges  $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ , and a weighted adjacency matrix  $\mathcal{A} = [a_{ij}]$  with nonnegative adjacency elements  $a_{ij}$ . The node indexes of  $\mathcal{G}$  belong to a finite index set  $\mathcal{I} = \{1, 2, \dots, M\}$ . An edge of  $\mathcal{G}$  is denoted by  $e_{ij} = (v_i, v_j)$ . The adjacency elements associated with the edges are positive, i.e.,  $e_{ij} \in \mathcal{E} \iff a_{ij} > 0$ . Moreover, we assume  $a_{ii} = 0$  for all  $i \in \mathcal{I}$ . Since the graph considered is undirected, it means once  $e_{ij}$  is an edge of  $\mathcal{G}$ ,  $e_{ji}$  is an edge of  $\mathcal{G}$  as well. As a result, the adjacency matrix  $\mathcal{A}$  is a symmetric nonnegative matrix.

The set of *neighbors* of node  $v_i$  is denoted by  $N_i = \{v_j \in \mathcal{V} : (v_i, v_j) \in \mathcal{E}\}$ . A *cluster* is any subset  $J \subseteq \mathcal{V}$  of the nodes of the graph. The set of neighbors of a cluster  $N_J$  is defined by

$$N_J = \bigcup_{v_i \in J} N_i. \quad (1)$$

The *degree* of node  $v_i$  is the number of its neighbors  $|N_i|$  and is denoted by  $\deg(v_i)$ . The *degree matrix* is an  $M \times M$  matrix define as  $\Delta = [\Delta_{ij}]$  where

$$\Delta_{ij} = \begin{cases} \deg(v_i), & i = j; \\ 0, & i \neq j. \end{cases}$$

The *Laplacian* of graph  $\mathcal{G}$  is defined by

$$L = \Delta - A \quad (2)$$

An important fact of  $L$  is that all the row sums of  $L$  are zero and thus  $\mathbf{1}_M = [1, 1, \dots, 1]^T \in \mathbb{R}^M$  is an eigenvector of  $L$  associated with the eigenvalue  $\lambda = 0$ .

A *path* between each distinct vertices  $v_i$  and  $v_j$  is meant a sequence of distinct edges of  $\mathcal{G}$  of the form  $(v_i, v_{k_1}), (v_{k_1}, v_{k_2}), \dots, (v_{k_l}, v_j)$ . A graph is called *connected* if there exist a path between any two distinct vertices of the graph.

*Lemma 1:* [35] The graph  $\mathcal{G}$  is connected if and only if  $\text{rank}(L) = M - 1$ .

By Lemma 1, for a connected graph, there is only one zero eigenvalue of  $L$ , all the other ones are positive and real.

### B. Consensus Problem on Network

Given a graph  $\mathcal{G}$ , let  $x_i \in \mathbb{R}$  denote the state or value of node  $v_i$ . We refer to  $\mathcal{G}_x = (\mathcal{G}, x)$  with  $x = (x_{1,2}, \dots, x_M)^T$  as a *network* with value  $x \in \mathbb{R}^M$  and topology  $\mathcal{G}$ . Suppose each node of a graph is a dynamic agent with dynamics

$$\begin{aligned} \dot{x}_i &= v_i \\ m_i \dot{v}_i &= u_i \end{aligned} \quad (3)$$

where  $x_i$  is aforementioned state of node  $v_i$ ,  $v_i$  is the speed,  $m_i$  is the mass, and  $u_i$  is the control input that will be used for consensus problem. Moreover, we assume  $m_1 = m_2 = \dots = m_M = 1$ .

Let  $\chi : \mathbb{R}^M \rightarrow \mathbb{R}$  be a function of  $M$  variables  $x_1, x_2, \dots, x_M$  and  $x_0 = x(0)$ ,  $v_0 = v(0) = [v_1(0), \dots, v_M(0)]^T$  denote the initial state and the initial speed of the system, respectively. The  $\chi$ -*consensus problem* in a dynamic graph is distributed way to calculated  $\chi(x_0)$  by applying inputs  $u_i$  that only depend on the states of node  $v_i$  and its neighbors. We say a state feedback

$$u_i = k_i(x_{j1}, x_{j2}, \dots, x_{jl_i}) \quad (4)$$

is a *protocol* with topology  $\mathcal{G}$  if the cluster  $J_i = \{v_{j1}, v_{j2}, \dots, v_{jl_i}\}$  of nodes with indexes  $j1, j2, \dots, jl_i \in \mathcal{I}$  satisfies the property  $J_i \subseteq \{v_i\} \cup N_i$ . In addition, if  $|J_i| < M$  for all  $i \in \mathcal{I}$ , (4) is called a *distributed protocol*.

We say protocol (4) asymptotically solves the  $\chi$ -consensus problem if and only if there exists an asymptotically stable equilibrium  $x^*$  of the network satisfying  $x_i^* = \chi(x_0)$  for all  $i \in \mathcal{I}$ , and meanwhile, the speed of each agent satisfying  $\lim_{t \rightarrow \infty} v_i = 0$ ,  $i \in \mathcal{I}$ . Whenever the nodes of a network are all in consensus, the common value of all nodes is called the *group decision value*.

In this paper, we are interested in distributed solutions of the special case with  $\chi(x) = \text{Ave}(x) = 1/M(\sum_{i=1}^M x_i)$  which is called *average-consensus*. This is a very representative case with broad applications in distributed decision-making for multi-agent system.

To solve such a average-consensus problem is a challenging task. It needs one to find suitable distributed state feedback controller for each agent not only to solve the agreement of the state of network but also to stabilize the speed of the network.

### III. CONTROL PROTOCOL AND NETWORK DYNAMICS

In this section, we present the control protocol that solve the aforementioned average-consensus problem. We will use a linear protocol with fixed topology and no communication time-delays:

$$u_i = u_{i1} + u_{i2} \quad (5)$$

where

$$u_{i1} = kv_i \quad (6)$$

is the local speed feedback with feedback gain  $k$  to be designed later, and

$$u_{i2} = \sum_{j \in N_i} a_{ij}(x_j - x_i) \quad (7)$$

is the part corresponding to the neighbors of node  $v_i$  which is constant in networks with fixed topology.

By using the above protocol (5), the agent dynamic is given as follows:

$$\begin{aligned} \dot{x}_i &= v_i \\ m_i \dot{v}_i &= kv_i + \sum_{j \in N_i} a_{ij}(x_j - x_i) \end{aligned} \quad (8)$$

Denote

$$\xi_i = [x_i, v_i]^T, \quad i \in \mathcal{I},$$

we have

$$\dot{\xi}_i = A\xi_i + BK\xi_i + BF \sum_{j \in N_i} a_{ij}(\xi_j - \xi_i) \quad (9)$$

where

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$K = [0 \quad k], F = [1 \quad 0].$$

Furthermore, denote

$$\xi = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_M \end{bmatrix},$$

the network dynamic is summarized as follows:

$$\dot{\xi} = \Phi\xi \quad (10)$$

where

$$\Phi = I_M \otimes (A + BK) - L \otimes BF$$

with  $L$  the aforementioned Laplacian associate with the graph  $\mathcal{G}$ .

### IV. NETWORK WITH FIXED TOPOLOGY

In this section, we provide the convergence analysis of the average-consensus problem for networks with fixed topology.

*Theorem 1:* Consider a network with a fixed topology  $\mathcal{G} = (\mathcal{V}, E, A)$  that is a connected graph, then for any negative gain  $k < 0$ , the protocol (5) globally asymptotically solves the average-consensus problem.

Before giving the proof of Theorem 1, we first consider the solution of  $\lim_{t \rightarrow \infty} \exp(\Phi t)$ .

*Lemma 2:* Assume the graph  $\mathcal{G}$  is connected, then for any  $k < 0$ ,

$$\lim_{t \rightarrow \infty} \exp(\Phi t) = w_r w_l^T. \quad (11)$$

where  $w_l, w_r$  are the left and right eigenvector of  $\Phi$  associated with the eigenvalue zero, respectively. Furthermore,

$$w_r = w_l = \frac{1}{\sqrt{M}} \mathbf{1}_M \otimes [1 \quad 0]^T, \quad (12)$$

and  $w_r^T w_l = 1$ .

*Proof.*

First, we have

$$\begin{aligned} & \Phi \frac{1}{\sqrt{M}} \mathbf{1}_M \otimes [1 \quad 0]^T \\ &= \frac{1}{\sqrt{M}} (I_M \otimes (A + BK) - L \otimes BF) \mathbf{1}_M \otimes [1 \quad 0]^T \\ &= \frac{1}{\sqrt{M}} (I_M \mathbf{1}_M) \otimes ((A + BK)[1 \quad 0]^T) \\ & \quad - \frac{1}{\sqrt{M}} (L \mathbf{1}_M) \otimes (BF[1 \quad 0]^T) \\ &= \frac{1}{\sqrt{M}} \mathbf{1}_M \otimes \mathbf{0}_2 - \frac{1}{\sqrt{M}} \mathbf{0}_M \otimes [0 \quad 1]^T \\ &= \mathbf{0}_{2M}. \end{aligned}$$

This means that  $w_r = \frac{1}{\sqrt{M}} \mathbf{1}_M \otimes [1 \quad 0]^T$ .

Similarly, we have

$$\begin{aligned} & \frac{1}{\sqrt{M}} \mathbf{1}_M^T \otimes [1 \quad 0] \Phi \\ &= \frac{1}{\sqrt{M}} \mathbf{1}_M^T \otimes [1 \quad 0] (I_M \otimes (A + BK) - L \otimes BF) \\ &= \frac{1}{\sqrt{M}} (\mathbf{1}_M^T I_M) \otimes ([1 \quad 0] (A + BK)) \\ & \quad - (\mathbf{1}_M^T L) \otimes ([1 \quad 0] BF) \\ &= \frac{1}{\sqrt{M}} \mathbf{1}_M^T \otimes \mathbf{0}_2^T - \frac{1}{\sqrt{M}} \mathbf{0}_M^T \otimes [1 \quad 0] \\ &= \mathbf{0}_{2M}^T. \end{aligned}$$

This means that  $w_l = \frac{1}{\sqrt{M}} \mathbf{1}_M \otimes [1 \quad 0]^T$  as well.

Next, denote the eigenvalues of  $L$  are  $0 = \lambda_1 < \lambda_2 \leq \dots \leq \lambda_M$ , there exists a orthogonal matrix  $W$  such that  $W^{-1} L W = \text{diag}\{\lambda_1, \lambda_2, \dots, \lambda_M\}$ . It follows that

$$\begin{aligned} & (W^{-1} \otimes I_2) \Phi (W \otimes I_2) \\ &= I_M \otimes (A + BK) - \text{diag}\{\lambda_1, \lambda_2, \dots, \lambda_M\} \otimes BF \\ &= \text{diag}\{A + BK, A + BK - \lambda_2 BF, \\ & \quad \dots, A + BK - \lambda_M BF\} \end{aligned}$$

Since

$$\det(A + BK - \lambda_i BF) = \det \begin{pmatrix} 0 & 1 \\ -\lambda_i & k \end{pmatrix} = \lambda_i \neq 0,$$

we have  $\text{rank}(A + BK - \lambda_i BF) = 2$ , it follows that

$$\begin{aligned} & \text{rank}((W^{-1} \otimes I_2) \Phi (W \otimes I_2)) \\ &= \text{rank}(A + BK) + \sum_{i=2}^M \text{rank}(A + BK - \lambda_i BF) \\ &= 1 + 2(M - 1) = 2M - 1. \end{aligned}$$

This implies that  $\Phi$  has only one eigenvalue at zero. Consider the eigenpolynomial of  $A + BK - \lambda_i BF$

$$f_i(s) = s^2 - ks + \lambda_i,$$

the corresponding eigenvalues are

$$\gamma_{i1} = (k + \sqrt{k^2 - 4\lambda_i})/2, \quad \gamma_{i2} = (k - \sqrt{k^2 - 4\lambda_i})/2$$

Since  $\lambda_i > 0$ , for any  $k < 0$ , we have

$$\mathcal{R}e(\gamma_{i2}) \leq \mathcal{R}e(\gamma_{i1}) < 0,$$

for  $i = 2, \dots, M$ . This means that all the matrices  $A + BK - \lambda_2 BF, \dots, A + BK - \lambda_M BF$  are Hurwitz stable.

Let  $J$  be the Jordan form associated with  $\Phi$ , there exists another nonsingular matrix  $S \in \mathbb{R}^{2M \times 2M}$  such that

$$\begin{aligned} J &= S^{-1}(W^{-1} \otimes I_2)\Phi(W \otimes I_2)S \\ &= \text{diag}\{0, -k, \gamma_{21}, \gamma_{22}, \dots, \gamma_{M1}, \gamma_{M2}\}, \end{aligned}$$

We have

$$\exp(\Phi t) = (W \otimes I_2)S \exp(Jt)S^{-1}(W^{-1} \otimes I_2).$$

It follow that

$$\begin{aligned} &\lim_{t \rightarrow \infty} \exp(\Phi t) \\ &= (W \otimes I_2)S \lim_{t \rightarrow \infty} \exp(Jt)S^{-1}(W^{-1} \otimes I_2) \\ &= (W \otimes I_2)SQS^{-1}(W^{-1} \otimes I_2), \end{aligned}$$

where  $Q = [q_{ij}]$  with a single nonzero element  $q_{11} = 1$ .

Since  $S^{-1}(W^{-1} \otimes I_2)\Phi = S^{-1}(W^{-1} \otimes I_2)J$ , the first row of  $S^{-1}(W^{-1} \otimes I_2)$  is  $w_l^T$ . Due to the fact that  $S^{-1}(W^{-1} \otimes I_2)(W \otimes I_2)S = I_{2M}$ ,  $w_l$  satisfies  $w_l^T w_r = 1$ .

This completes the proof.  $\square$

Now we give the proof of Theorem 1.

*Proof of Theorem 1.* By Lemma 2, we have

$$\xi(t) = \exp(\Phi t)\xi(0).$$

It follows that

$$\begin{aligned} &\lim_{t \rightarrow \infty} \xi(t) \\ &= \lim_{t \rightarrow \infty} \exp(\Phi t)\xi(0) \\ &= w_r w_l^T \xi(0) \\ &= (w_l^T \xi(0))w_r \\ &= \frac{1}{\sqrt{M}}(w_l^T \xi(0))\mathbf{1}_M \otimes [1 \ 0]^T. \end{aligned}$$

Since

$$\begin{aligned} &\frac{1}{\sqrt{M}}w_l^T \xi(0) \\ &= \frac{1}{M}(\mathbf{1}_M^T \otimes [1 \ 0])[x_1(0), v_1(0), \dots, x_M(0), v_M(0)] \\ &= \frac{1}{M} \sum_{i=1}^M x_i(0) \\ &= \text{Ave}(x_0) \end{aligned}$$

and it is obvious that

$$\lim_{t \rightarrow \infty} v_i(t) = 0, \quad i = 1, 2, \dots, M.$$

This implies the protocol (5) globally asymptotically solves the average-consensus problem.

This completes the proof.  $\square$

## V. PERFORMANCE DISCUSSION

In this section, we discuss performance issues of protocol (5). We write  $\xi$  as

$$\xi = \text{Ave}(x_0)\mathbf{1}_M \otimes [1 \ 0]^T + \delta \quad (13)$$

where  $\delta$  is called the *disagreement vector*. It is easy to verify that  $\delta$  satisfies the following *disagreement dynamics*

$$\dot{\delta} = \Phi \delta \quad (14)$$

Defining the *group disagreement function* as

$$V(\delta) = \delta^T P \delta \quad (15)$$

where  $P$  is a positive semi-definite matrix to be designed in the following. We get

$$\dot{V}(\delta) = \delta^T (\Phi^T P + P \Phi) \delta \quad (16)$$

In the following, we will determine such a positive semi-definite matrix  $P$  such that

$$\delta^T P \delta > 0,$$

$$\delta^T (\Phi^T P + P \Phi) \delta < 0,$$

for any nonzero  $\delta$  satisfying  $\delta^T \mathbf{1}_M \otimes [1 \ 0]^T = 0$ .

In fact, noticing that the matrix

$$\begin{aligned} &\text{diag}\{k, A + BK - \lambda_2 BF, \dots, A + BK - \lambda_M BF\} \\ &\in \mathbb{R}^{(2M-1) \times (2M-1)} \end{aligned}$$

is Hurwitz stable, there exists a positive definite matrix  $\hat{P} \in \mathbb{R}^{(2M-1) \times (2M-1)}$  such that the matrix

$$\begin{aligned} &\text{diag}\{k, A + BK - \lambda_2 BF, \dots, A + BK - \lambda_M BF\}^T \hat{P} \\ &+ \hat{P} \text{diag}\{k, A + BK - \lambda_2 BF, \dots, A + BK - \lambda_M BF\} \end{aligned}$$

is negative definite. It follows that

$$\eta^T \text{diag}\{0, \hat{P}\} \eta > 0,$$

$$\begin{aligned} &\eta^T \left( \text{diag}\{A + BK, A + BK - \lambda_2 BF, \dots, A + BK - \lambda_M BF\}^T \text{diag}\{0, \hat{P}\} \right. \\ &\left. + \text{diag}\{0, \hat{P}\} \text{diag}\{k, A + BK - \lambda_2 BF, \dots, A + BK - \lambda_M BF\} \right) \eta < 0, \end{aligned}$$

for any nonzero  $\eta \in \mathbb{R}^{2M}$  satisfying  $[1, 0, \dots, 0]\eta = 0$ . Let

$$P = (W^{-T} \otimes I_2) \text{diag}\{0, \hat{P}\} (W^T \otimes I_2),$$

since

$$\begin{aligned} &(W^{-1} \otimes I_2)\Phi(W \otimes I_2) \\ &= \text{diag}\{A + BK, A + BK - \lambda_2 BF, \dots, A + BK - \lambda_M BF\}, \end{aligned}$$

we get

$$\left( (W \otimes I_2) \eta \right)^T P \left( (W \otimes I_2) \eta \right) > 0,$$

$$\left( (W \otimes I_2) \eta \right)^T \left( \Phi^T P + P \Phi \right) \left( (W \otimes I_2) \eta \right) < 0,$$

for any nonzero  $\eta \in \mathbb{R}^{2M}$  satisfying  $[1, 0, \dots, 0]\eta = 0$ . Then the thing left is to prove that

$$\{\delta : \mathbf{1}_M^T \otimes [1 \ 0]\delta = 0\} = (W \otimes I_2)\{\eta : [1, 0, \dots, 0]\eta\} \quad (17)$$

This statement is formulated as the following lemma.

*Lemma 3:* Give a connected graph  $\mathcal{G}$ , then (17) holds.

*Proof.* See Appendix A.  $\square$

As a result, we summarize the above analysis as the following theorem.

*Theorem 2:* Consider a network with a fixed topology  $\mathcal{G} = (\mathcal{V}, E, A)$  that is a connected graph, given protocol (5), there exists a positive semi-definite matrix  $P$  such that the smooth, positive definite and proper function (15) is a valid Lyapunov function for the disagreement dynamics. Furthermore, there exists a positive scalar  $\kappa$  such that

$$V(\delta(t)) \leq V(\delta(0)) \exp(-\kappa t). \quad (18)$$

*Proof.*

The existence of  $P$  has been shown as above. The existence of  $\kappa$  is obvious. In fact, just let

$$\kappa = - \min_{\delta \neq 0, \mathbf{1}_M^T \otimes [1 \ 0]\delta = 0} \frac{\delta^T (\Phi^T P + P \Phi) \delta}{\delta^T P \delta}. \quad (19)$$

which satisfies (18).  $\square$

## VI. SIMULATIONS

In this section, we consider solving average-consensus problem for graphs  $\mathcal{G}_a$  and  $\mathcal{G}_b$  shown in Fig.1 and the adjacency matrices are limited to 0, 1 matrices. Fig. 2 and Fig. 3 show the simulation results for the consensus protocol (5) for a network with graph  $\mathcal{G}_a$  with random set of initial conditions. Fig.4 and Fig. 5 show the simulation results for the consensus protocol (5) for a network with graph  $\mathcal{G}_b$  with random set of initial conditions.

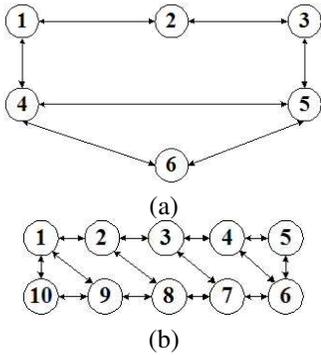


Fig. 1. Undirected graphs used for consensus problems: a)  $\mathcal{G}_a$  with  $M = 6$  nodes, and b)  $\mathcal{G}_b$  with  $M = 10$  nodes.

## VII. CONCLUSION

In this paper, convergence analysis of a consensus protocol for a class of networks of dynamic agents with fixed topology was presented. The agent dynamics is adopted as a typical motion equation. The protocol contains two part,

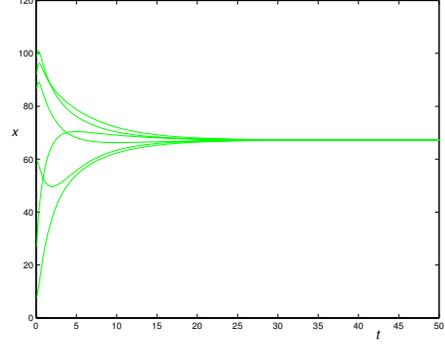


Fig. 2. State trajectories of the network on graph  $\mathcal{G}_a$ .

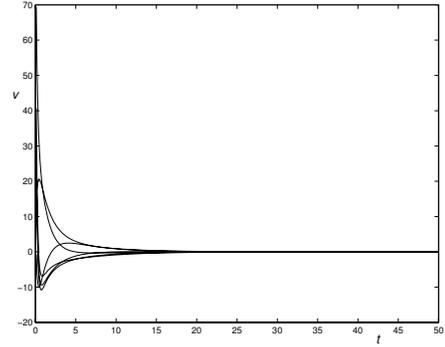


Fig. 3. Velocity trajectories of the network on graph  $\mathcal{G}_a$ .

one part is the local speed feedback controller, the other is the distributed state feedback from the neighbors. The case where the switching topology is treated separately in [36].

## APPENDIX A

*Proof of Lemma 3.*

Give a connected graph  $\mathcal{G}$ , then we have

$$W^{-1}LW = \text{diag}\{0, \lambda_1, \dots, \lambda_M\}$$

and

$$L\mathbf{1}_M = 0$$

It follows that

$$W \text{diag}\{0, \lambda_1, \dots, \lambda_M\} W^{-1} \mathbf{1}_M = 0$$

Since  $W$  is orthogonal, this implies that

$$W^{-1} \mathbf{1}_M = c[1, 0, \dots, 0]^T \in \mathbb{R}^M$$

where  $c$  is nonzero constant. It follows that

$$(W^{-1} \otimes I_2) \mathbf{1}_M \otimes [1 \ 0]^T = c[1, 0, \dots, 0]^T \in \mathbb{R}^{2M}$$

This implies that

$$(W^{-1} \otimes I_2) \text{span}\{\mathbf{1}_M \otimes [1 \ 0]^T\} = \text{span}\{[1, 0, \dots, 0]^T\}$$

We can rewrite it as

$$\text{span}\{\mathbf{1}_M \otimes [1 \ 0]^T\} = (W \otimes I_2) \text{span}\{[1, 0, \dots, 0]^T\}$$

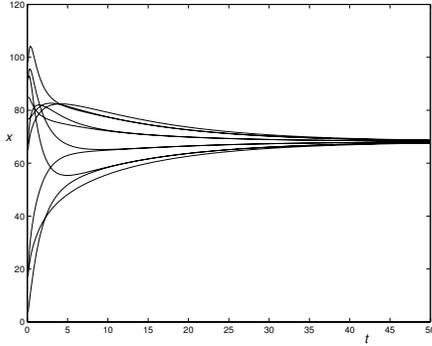


Fig. 4. State trajectories of the network on graph  $\mathcal{G}_b$ .

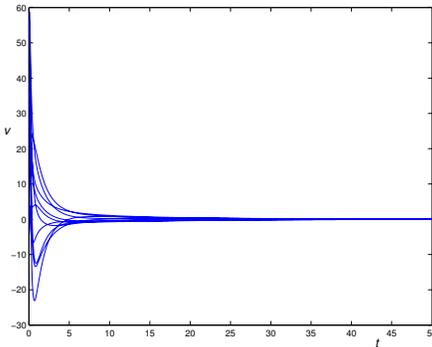


Fig. 5. Velocity trajectories of the network on graph  $\mathcal{G}_b$ .

Since  $W \otimes I_2$  is orthogonal as well, it follows that

$$\text{span}\{\mathbf{1}_M \otimes [1 \ 0]^T\}_\perp = (W \otimes I_2)\text{span}\{[1, 0, \dots, 0]^T\}_\perp,$$

which is nothing but (17).  $\square$

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