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Output Regulation for Linear Minimum Phase Systems with Unknown Order Exosystem

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***Abstract*— Under the assumption that the regulation error is the only feedback signal, it is shown how to design a regulator capable of rejecting and/or tracking signals made of at most m sinusoidal terms with unknown frequencies, magnitudes and phases, for linear observable and controllable minimum phase systems. The asymptotic convergence of the tracking error to zero is guaranteed provided that an upper bound is known on the number of sinusoidal terms, even though the values of the unknown frequencies may not be recovered.**

I. INTRODUCTION

The regulator problem was formulated for linear systems modeled as

$$\begin{aligned} \dot{x} &= Ax + bu + Pw, \ x \in \mathbb{R}^n, \ u \in \mathbb{R} \\ \dot{w} &= Rw, \ w \in \mathbb{R}^r \\ e &= cx - qw, \ e \in \mathbb{R} \end{aligned} \tag{1}$$

in which disturbances Pw and reference signals to be tracked qw are both generated by a linear exosystem $\dot{w} = Rw$; the problem is to design a controller feeding back only the regulation error $e = cx - qw$ so that all signals are bounded and the regulation error $e(t)$ tends to zero as t goes to infinity, from any initial condition. A fundamental result (see [1], [2], [3]) gives necessary and sufficient conditions for the solution of the problem, namely:

(A1) the pair (A, b) is stabilizable, i.e. rank $\begin{bmatrix} \lambda I - A & b \end{bmatrix} < n$ implies $\Re(\lambda) < 0$;
(A2) the pair (A, c) is detectable, i.e. rank $\begin{bmatrix} \lambda I - A \\ c \end{bmatrix} < n$ implies $\Re(\lambda) < 0$;
(A3) rank $\begin{bmatrix} \lambda I - A & b \\ c & 0 \end{bmatrix} = n + 1$ for any eigenvalue λ of the matrix R, i.e. no eigenvalue of R is a zero of the transfer function $c(sI - A)^{-1}b$.

The design of the regulator requires the knowledge of the matrices (A, b, c, P, R, q) and uses the 'internal model principle', that is the exosystem itself is incorporated in the controller. It is remarkable that regulators can be designed for non-minimum phase systems as well. It was natural to ask whether the regulator problem can be solved without knowing the matrix R since, for instance, the disturbances may be sinusoidal with unknown frequencies. It was recently shown in [4] that the knowledge of R is not required provided that:

(A4) the eigenvalues of R are distinct and on the imaginary axis of the complex plane; all of its oscillatory modes are excited by the initial conditions and observable from the open loop solution $u = \gamma w$ of the regulator problem, which is obtained by solving the regulator matrix equations

$$\begin{aligned} \Gamma R &= A\Gamma + b\gamma + P \\ c\Gamma &= q \end{aligned} \tag{2}$$

for Γ and γ (recall that a unique solution exists under Assumption (A3)).

The solution provided in [4] makes use of an adaptive internal model which exponentially converges to the unknown exosystem under Assumption (A4). However, this additional assumption does not require the knowledge of R but it requires the knowledge of its dimension r, since all oscillatory modes have to be excited by the initial conditions. This may still be an unrealistic assumption. In this paper we ask whether the regulator problem is still solvable knowing only an upper bound on r. Our answer is affirmative provided that the system is restricted to be minimum phase: the proposed regulator still guarantees asymptotic convergence to zero of the regulation error.

II. MAIN RESULT

Consider the linear single-input, single-output system (1). The matrices A, b and c are assumed to be known while the matrices P, R and q are uncertain. Moreover, we assume that:

(H1) the pair (A, b) is controllable and the pair (A, c) is observable;
(H2) the zeros of the transfer function $c(sI - A)^{-1}b$ are in the open left-hand complex plane (minimum phase);
(H3) the spectrum of R is $\pm j\omega_i, \ 1 \le i \le m$, with ω_i unknown distinct positive parameters and m a known integer, the pair (R, γ) is observable, with γ the solution of the regulator equations (2) which exists since Assumption (A3) holds.

Assumption (H1) about controllability and observability of system (1) may be relaxed and replaced by the less restrictive assumption of stabilizability and detectability as in [4]. Moreover, as in [4], the spectrum of R may also contain a zero frequency in Assumption (H3). Those

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simplifying assumptions are required for ease of exposition. Note also that since no assumption is made on the initial conditions of the exosystem, these conditions may be zero so that actually m in Assumption (H3) represents an upper bound on the order of the exosystem.

We can prove the following result.

Theorem 2.1: Consider system (1) under the assumptions (H1)-(H3). There exists an output error feedback dynamic controller such that, for any initial condition: (i) the state variables $x(t)$ of system (1) and those of the controller are bounded for any $t \geq 0$; (ii) the output regulation error $e(t)$ converges asymptotically to zero as t tends to infinity.

Proof. Defining $x_r = \Gamma w$ and $u_r = \gamma w$, we have

$$\begin{aligned} \dot{x}_r &= Ax_r + bu_r + Pw \\ cx_r + qw &= 0 \, . \end{aligned} \tag{3}$$

Introducing the state regulation error $\tilde{x} = x - x_r$, from (1) and (3) we can write

$$\begin{aligned} \dot{\tilde{x}} &= A\tilde{x} + b(u - u_r) = A\tilde{x} - b\gamma w + bu \\ e &= c\tilde{x} \\ \dot{w} &= Rw \, . \end{aligned} \tag{4}$$

Since (R, γ) is observable by Assumption (H3), to generate u_r we may equivalently consider its observer canonical form

$$\begin{aligned} \dot{\eta} &= R_c \eta \\ u_r &= \gamma_c \eta \end{aligned} \tag{5}$$

with

$$\begin{aligned} R_c &= \begin{bmatrix} -\alpha_{m-1} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -\alpha_1 & 0 & \cdots & 1 \\ -\alpha_0 & 0 & \cdots & 0 \end{bmatrix} \\ \gamma_c &= \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \end{aligned}$$

and $\det(sI - R) = s^m + \alpha_{m-1}s^{m-1} + \ldots + \alpha_0$. From (4) and (5), we obtain

$$\begin{aligned} \begin{bmatrix} \dot{\tilde{x}} \\ \dot{\eta} \end{bmatrix} &= \begin{bmatrix} A & -b\gamma_c \\ 0 & R_c \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \eta \end{bmatrix} + \begin{bmatrix} b \\ 0 \end{bmatrix} u \\ e &= \begin{bmatrix} c & 0 \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \eta \end{bmatrix} . \end{aligned} \tag{6}$$

Since the pair (A, c) is observable, by a suitable change of coordinates

$$\tilde{x}_o = T_1 \tilde{x} \tag{7}$$

we can write

$$\begin{aligned} \begin{bmatrix} \dot{\tilde{x}}_o \\ \dot{\eta} \end{bmatrix} &= \begin{bmatrix} A_o & -b_o\gamma_c \\ 0 & R_c \end{bmatrix} \begin{bmatrix} \tilde{x}_o \\ \eta \end{bmatrix} + \begin{bmatrix} b_o \\ 0 \end{bmatrix} u \\ e &= \begin{bmatrix} c_o & 0 \end{bmatrix} \begin{bmatrix} \tilde{x}_o \\ \eta \end{bmatrix} \end{aligned} \tag{8}$$

with

$$\begin{aligned} A_o &= \begin{bmatrix} -a_{n-1} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -a_1 & 0 & \cdots & 1 \\ -a_0 & 0 & \cdots & 0 \end{bmatrix}, \quad b_o = \begin{bmatrix} b_{n-1} \\ \vdots \\ b_1 \\ b_0 \end{bmatrix} \\ c_o &= \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} . \end{aligned}$$

The characteristic polynomial of R_c is given by

$$\det(sI - R_c) = s^{2m} + \theta_1 s^{2(m-1)} + \cdots + \theta_m$$

with $\theta = \begin{bmatrix} \theta_1 & \cdots & \theta_m \end{bmatrix}^T$ being a vector of positive unknown coefficients which are related to the unknown distinct frequencies ω_i, $1 \leq i \leq m$. Assumptions (H2) and (H3) guarantee that a parameter dependent change of coordinates

$$\zeta = T_2(\theta) \begin{bmatrix} \tilde{x}_o \\ \eta \end{bmatrix}, \quad \zeta \in \mathbb{R}^{n+2m} \tag{9}$$

exists such that (8) is transformed into an observer canonical form

$$\begin{aligned} \dot{\zeta} &= A_c\zeta - a[0]e + b[0]u + \sum_{i=1}^{m} \theta_i(-a[i]e + b[i]u) \\ e &= c_c\zeta \end{aligned} \tag{10}$$

in which

$$\begin{aligned} A_c &= \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \\ c_c &= \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \\ a[0] &= \begin{bmatrix} a_{n-1} & \cdots & a_0 & 0 & \cdots & 0 \end{bmatrix}^T \\ b[0] &= \begin{bmatrix} b_{n-1} & \cdots & b_0 & 0 & \cdots & 0 \end{bmatrix}^T \\ a[1] &= \begin{bmatrix} 0 & 1 & a_{n-1} & \cdots & a_0 & 0 & \cdots & 0 \end{bmatrix}^T \\ b[1] &= \begin{bmatrix} 0 & 0 & b_{n-1} & \cdots & b_0 & 0 & \cdots & 0 \end{bmatrix}^T \\ &\vdots \\ a[m] &= \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 & a_{n-1} & \cdots & a_0 \end{bmatrix}^T \\ b[m] &= \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & b_{n-1} & \cdots & b_0 \end{bmatrix}^T \end{aligned}$$

with $a[i], b[i] \in \mathbb{R}^{n+2m}$, $0 \leq i \leq m$. According to [5], [6], [7], choosing any vector $d = \begin{bmatrix} 1 & d_{n+2m-2} & \cdots & d_0 \end{bmatrix}^T$ such that all the roots of the polynomial $s^{n+2m-1} + d_{n+2m-2}s^{n+2m-2} + \cdots + d_0$ have negative real part, and defining the Hurwitz matrix

$$D = \begin{bmatrix} -d_{n+2m-2} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -d_1 & 0 & \cdots & 1 \\ -d_0 & 0 & \cdots & 0 \end{bmatrix} \tag{11}$$

the filtered transformation

$$\begin{aligned} \dot{\xi}_i &= D\xi_i + \left[\begin{array}{cc} 0 & I_{n+2m-1} \end{array} \right] (-a[i]e + b[i]u), \\ & \qquad \xi_i(0) = \xi_{i0},\ \xi_i \in \mathbb{R}^{n+2m-1} \\ \mu_i &= \left[\begin{array}{cccc} 1 & 0 & \cdots & 0 \end{array} \right] \xi_i,\ 1 \le i \le m \\ z &= \zeta - \left[\begin{array}{c} 0 \\ \sum_{i=1}^m \xi_i \theta_i \end{array} \right],\ z \in \mathbb{R}^{n+2m} \end{aligned} \tag{12}$$

transforms (10) into the adaptive observer form

$$\begin{aligned} \dot{z} &= A_c z - a[0]e + b[0]u + d\sum_{i=1}^m \mu_i\theta_i \\ e &= c_c z \end{aligned} \tag{13}$$

for which we can define the adaptive observer

$$\begin{aligned} \dot{\hat{z}} &= A_c\hat{z} - a[0]e + b[0]u + d\sum_{i=1}^m \mu_i\hat{\theta}_i + k_o(e - c_c\hat{z}) \\ \dot{\hat{\theta}}_i &= g_i\mu_i(e - c_c\hat{z}),\ 1 \le i \le m \end{aligned} \tag{14}$$

where $k_o = (A_c + \lambda I)d$ and λ, g_i are positive design parameters. The choice of k_o guarantees that the triple $(A_c - k_o c_c, d, c_c)$ is strictly positve real so that by virtue of Meyer-Kalman-Yacubovich Lemma (see [6]) two symmetric positive definite matrices P and Q exist such that

$$\begin{aligned} (A_c - k_o c_c)^T P + P(A_c - k_o c_c) &= -Q \\ Pd &= c_c^T\ . \end{aligned} \tag{15}$$

The observer error dynamics can be written as

$$\begin{aligned} \dot{\tilde{z}} &= (A_c - k_o c_c)\tilde{z} + d\mu^T\tilde{\theta} \\ \dot{\tilde{\theta}} &= -G\mu d^T P\tilde{z} \end{aligned} \tag{16}$$

in which $\tilde{z} = z - \hat{z}$, $\tilde{\theta} = \theta - \hat{\theta}$, $G = \mathrm{diag}[g_1, \ldots, g_m]$, and $\mu = [\mu_1, \ldots, \mu_m]^T$. If we introduce the variables

$$\begin{aligned} \chi_i &= \xi_i - M_i\tilde{x},\ \chi_i \in \mathbb{R}^{n+2m-1},\ 1 \le i \le m \\ M_1 &= \left[\begin{array}{c} 0 \\ I_n \\ 0 \\ \vdots \\ 0 \end{array} \right],\ \ldots,\ M_m = \left[\begin{array}{c} 0 \\ \vdots \\ 0 \\ I_n \\ 0 \end{array} \right] \end{aligned}$$

and compute their dynamics

$$\begin{aligned} \dot{\chi}_i &= D\chi_i + \left[\begin{array}{cc} 0 & I_{n+2m-1} \end{array} \right] b[i]\eta_1 \\ \mu_i &= \left[\begin{array}{cccc} 1 & 0 & \cdots & 0 \end{array} \right] \chi_i,\ 1 \le i \le m \end{aligned} \tag{17}$$

we note that since η_1 is bounded (it coincides with u_r) and D is Hurwitz, all vectors $\chi_i(t)$ and, consequently, $\mu(t)$ and all its time derivatives are bounded. Consider the function

$$V = \tilde{z}^T P\tilde{z} + \tilde{\theta}^T G^{-1}\tilde{\theta} \tag{18}$$

whose time derivative, according to (16), is given by

$$\dot{V} = -\tilde{z}^T Q\tilde{z} \tag{19}$$

so that both $\tilde{z}(t)$ and $\tilde{\theta}(t)$ are bounded. Since μ is bounded, from (16) it follows that $\dot{\tilde{z}}(t)$ is bounded, which along with (19) and Barbalat Lemma (see [6]) imply that $\lim_{t\to\infty} \tilde{z}(t) = 0$. Moreover, from (16) we obtain

$$\ddot{\tilde{z}} = (A_c - k_o c_c)\dot{\tilde{z}} + d\dot{\mu}^T\tilde{\theta} + d\mu^T\dot{\tilde{\theta}}\ . \tag{20}$$

From (16) and (17), we see that $\dot{\tilde{z}}$, $\dot{\tilde{\theta}}$ and $\dot{\mu}$ are bounded, so that from (20) also $\ddot{\tilde{z}}$ is bounded. Therefore, $\dot{\tilde{z}}$ is uniformly continuous and, since $\int_0^t \dot{\tilde{z}}(\tau)d\tau = \tilde{z}(\tau) - \tilde{z}(0)$ and $\lim_{t\to\infty} \tilde{z}(t) = 0$, by Barbalat Lemma it follows that

$$\lim_{t\to\infty} \dot{\tilde{z}}(t) = 0$$

which, in the light of (16), implies $\lim_{t\to\infty} \mu^T(t)\tilde{\theta}(t) = 0$. Consider now system (4) and note that by choosing

$$u = \gamma w + k_c\tilde{x} \tag{21}$$

with k_c such that the matrix $(A+bk_c)$ is Hurwitz, we obtain the closed loop dynamics

$$\dot{\tilde{x}} = (A + bk_c)\tilde{x} \tag{22}$$

so that $\tilde{x}(t)$ and $e(t)$ tend to zero while $u(t)$ is bounded and tends to γw. From (12), we obtain that also $\xi_i(t)$ are bounded. From (7), (9) and (12), it follows that $z(t)$ is bounded. From (13), we have

$$\begin{aligned} \dot{z}_{n+1} &= z_{n+2} + d_{2m-1}\sum_{i=1}^m \mu_i\theta_i \\ &\vdots \\ \dot{z}_{n+2m} &= d_0\sum_{i=1}^m \mu_i\theta_i\ . \end{aligned} \tag{23}$$

Since (23) are not influenced by $u(t)$, it follows that $z_{n+i}(t)$, $1 \le i \le 2m$, are bounded for any input $u(t)$. Now, we distinguish two cases: $\rho = 1$ and $\rho > 1$. If $\rho = 1$, we make the change of coordinates (for simplicity, we assume that $b_{n-1} = 1$)

$$\begin{aligned} z_1 &= z_1 \\ \left[\begin{array}{c} \bar{z}_2 \\ \vdots \\ \bar{z}_n \end{array} \right] &= \left[\begin{array}{c} z_2 \\ \vdots \\ z_n \end{array} \right] - \left[\begin{array}{c} b_{n-2} \\ \vdots \\ b_0 \end{array} \right] z_1 \\ z_{n+i} &= z_{n+i},\ 1 \le i \le 2m \end{aligned} \tag{24}$$

obtaining from (13),

$$\begin{aligned} \dot{z}_1 &= \bar{z}_2 + b_{n-2}z_1 - a_{n-1}z_1 + u + \sum_{i=1}^m \mu_i\theta_i \\ \left[\begin{array}{c} \dot{\bar{z}}_2 \\ \vdots \\ \dot{\bar{z}}_n \end{array} \right] &= \left[\begin{array}{ccccc} -b_{n-2} & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ -b_1 & 0 & 0 & \cdots & 1 \\ -b_0 & 0 & 0 & \cdots & 0 \end{array} \right] \left[\begin{array}{c} \bar{z}_2 \\ \vdots \\ \bar{z}_n \end{array} \right] \\ &\quad + z_1 \left[\begin{array}{c} b_{n-3} - b_{n-2}^2 \\ \vdots \\ b_0 - b_1 b_{n-2} \\ -b_0 b_{n-2} \end{array} \right] \end{aligned}$$

$$+z_1 \begin{bmatrix} a_{n-2} - b_{n-2}a_{n-1} \\ \vdots \\ a_1 - b_1 a_{n-1} \\ a_0 - b_0 a_{n-1} \end{bmatrix}$$

$$+\sum_{i=1}^{m} \mu_i \theta_i \begin{bmatrix} d_{n+2m-2} - b_{n-2} \\ \vdots \\ d_{2m+1} - b_1 \\ d_{2m} - b_0 \end{bmatrix}$$

$$\triangleq \quad F \begin{bmatrix} \bar{z}_2 \\ \vdots \\ \bar{z}_n \end{bmatrix} + z_1 f_1 + \mu^T \theta f_2 \ . \qquad (25)$$

Define

$$u \quad = \quad a_{n-1}e - \sum_{i=1}^{m} \mu_i \hat{\theta}_i - \hat{z}_2 - ke \qquad (26)$$

so that the first equation in (25) becomes

$$\dot{z}_1 \quad = \quad -kz_1 + \tilde{z}_2 + \mu^T \tilde{\theta} \ . \qquad (27)$$

Since $\tilde{z}_2(t)$ and $\mu^T(t)\tilde{\theta}(t)$ are bounded and converge to zero, from [8] (Lemma B.8) it follows that $e(t) = z_1(t)$ is bounded and converges asymptotically to zero. From (25), since the matrix F is Hurwitz, it follows that $z_i(t)$, $2 \leq i \leq n$, are bounded so that $\tilde{z}(t)$ being bounded, from (26) also $u(t)$ is bounded. In view of (12), $\xi_i(t)$ are bounded, so that by (12), (9) and (7), $\tilde{x}(t)$ and, consequently, $x(t)$ are bounded. If $1 < \rho \leq n$, we introduce the filtered transformation (for simplicity, we assume that $b_{n-\rho} = 1$)

$$\begin{aligned} \dot{\phi} \quad &= \quad \begin{bmatrix} -\lambda_1 & 1 & 0 & \cdots & 0 \\ 0 & -\lambda_2 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & -\lambda_{\rho-1} \end{bmatrix} \phi + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} u \\ z_1 \quad &= \quad z_1 \\ \bar{z}_i \quad &= \quad z_i - \sum_{j=2}^{\rho} \bar{d}_i[j]\phi_{j-1} - \bar{d}_i z_1, \ 2 \leq i \leq n \\ z_{n+i} \quad &= \quad z_{n+i}, \ 1 \leq i \leq 2m \end{aligned} \qquad (28)$$

in which λ_i, $1 \leq i \leq \rho - 1$, are arbitrary positive reals, $\bar{d}_i$, $2 \leq i \leq m$, are solution of

$$s^n + \bar{d}_2 s^{n-1} + \cdots + \bar{d}_n = \prod_{i=1}^{\rho-1}(s + \lambda_i)(s^{n-\rho} + b_{n-\rho-1}s^{n-\rho-1} + \cdots + b_0) \ . \qquad (29)$$

Notice that since system (1) is minimum phase by Assumption (H2) all the roots of the polynomial (29) have negative real part. The vectors $\bar{d}[j] = [\bar{d}_1[j], \ldots, \bar{d}_n[j]]^T$, $2 \leq j \leq \rho$, are given by

$$\begin{aligned} \bar{d}[\rho] \quad &= \quad b_O \\ \bar{d}[j-1] \quad &= \quad A_c \bar{d}[j] + \lambda_{j-1}\bar{d}[j], \ \rho \geq j \geq 2 \ . \end{aligned}$$

From (13) and (28), we have

$$\dot{z}_1 \quad = \quad \bar{z}_2 + \bar{d}_2 z_1 - a_{n-1}z_1 + \phi_1 + \sum_{i=1}^{m} \mu_i \theta_i$$

$$\begin{bmatrix} \dot{\bar{z}}_2 \\ \vdots \\ \dot{\bar{z}}_n \end{bmatrix} = \begin{bmatrix} -\bar{d}_2 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ -\bar{d}_{n-1} & 0 & 0 & \cdots & 1 \\ -\bar{d}_n & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} \bar{z}_2 \\ \vdots \\ \bar{z}_n \end{bmatrix} + z_1 \begin{bmatrix} \bar{d}_3 - \bar{d}_2^2 \\ \vdots \\ \bar{d}_n - \bar{d}_{n-1}\bar{d}_2 \\ -\bar{d}_n \bar{d}_2 \end{bmatrix} + z_1 \begin{bmatrix} a_{n-2} - \bar{d}_2 a_{n-1} \\ \vdots \\ a_1 - \bar{d}_{n-1}a_{n-1} \\ a_0 - \bar{d}_n a_{n-1} \end{bmatrix}$$

$$+\sum_{i=1}^{m} \mu_i \theta_i \begin{bmatrix} d_{n+2m-2} - \bar{d}_2 \\ \vdots \\ d_{2m+1} - \bar{d}_{n-1} \\ d_{2m} - \bar{d}_n \end{bmatrix}$$

$$\triangleq \quad F \begin{bmatrix} \bar{z}_2 \\ \vdots \\ \bar{z}_n \end{bmatrix} + z_1 f_1 + \mu^T \theta f_2 \ . \qquad (30)$$

Define

$$\begin{aligned} \phi_1 \quad &= \quad \tilde{\phi}_1 + \phi_1^* \\ \phi_1^* \quad &= \quad -kz_1 - \mu^T \hat{\theta} - \hat{z}_2 + \sum_{j=2}^{\rho} \bar{d}_i[j]\phi_{j-1} + a_{n-1}z_1 \\ &= \quad -kz_1 - \mu^T \hat{\theta} - (\bar{z}_2 + \bar{d}_i z_1) + \tilde{z}_2 \end{aligned} \qquad (31)$$

so that the first equation in (30) becomes

$$\dot{z}_1 \quad = \quad -kz_1 + \tilde{z}_2 + \mu^T \tilde{\theta} + \tilde{\phi}_1 \ . \qquad (32)$$

The dynamics of $\tilde{\phi}_1$ are given by

$$\begin{aligned} \dot{\tilde{\phi}}_1 \quad &= \quad \dot{\phi}_1 - \dot{\phi}_1^* \\ &= \quad -\lambda_1\phi_1 + \phi_2 + (k + \bar{d}_i)\dot{z}_1 + \mu^T \dot{\hat{\theta}} + \dot{\mu}^T \hat{\theta} + \dot{\hat{z}}_2 \\ &= \quad -\lambda_1\phi_1 + \phi_2 + (k + \bar{d}_i)(\tilde{z}_2 - kz_1 + \mu^T \tilde{\theta} + \tilde{\phi}_1) \\ &\quad + \dot{\hat{z}}_2 + \mu^T G \mu \tilde{z}_1 + \sum_{i=1}^{m} \hat{\theta}_i(-d_{n+2m-2}\xi_{i1} + \xi_{i2}). \end{aligned} \qquad (33)$$

Let

$$\begin{aligned} \phi_2 \quad &= \quad \tilde{\phi}_2 + \phi_2^* \\ \phi_2^* \quad &= \quad \lambda_1\phi_1^* - z_1 - (k + \bar{d}_i)(-kz_1 + \tilde{\phi}_1) \\ &\quad - \sum_{i=1}^{m} \hat{\theta}_i(-dn + 2m - 2\xi_{i1} + \xi_{i2}) \\ &= \quad \lambda_1\phi_1^* - z_1 - (k + \bar{d}_i)(-kz_1 + \tilde{\phi}_1) \end{aligned}$$

$$-\sum_{i=1}^{m}\hat{\theta}_i\dot{\mu}_i \quad (34)$$

so that (33) becomes

$$\begin{aligned}\dot{\tilde{\phi}}_1 &= -\lambda_1\tilde{\phi}_1 - z_1 + \tilde{\phi}_2 + (k+\bar{d}_i)(\tilde{z}_2 + \mu^T\tilde{\theta}) \\ &\quad + \dot{\tilde{z}}_2 + \mu^T G\mu\tilde{z}_1 \ . \end{aligned} \quad (35)$$

If $\rho = 2$, $\phi_2 = \phi_2^* = u$ and $\tilde{\phi}_2 = 0$, so that since system (32), (35) is a linear exponentially stable system perturbed by bounded inputs $\tilde{z}_1$, $\tilde{z}_2$, $\mu^T\tilde{\theta}$, $\dot{\tilde{z}}_2$ converging to zero, by virtue of [8] (Lemma B.8) $z_1(t)$ and $\tilde{\phi}_1(t)$ converge asymptotically to zero. Moreover, from (34), since $\dot{\mu}$ is bounded [see (17)] it follows that $u = \phi_2^*$ is bounded as well and, consequently, $\xi_i(t)$ in (12) and $\phi(t)$ in (28) are bounded. From (30), since F is Hurwitz we obtain that $\bar{z}_2(t), \ldots, \bar{z}_n(t)$ and, in turn, $z_2(t), \ldots, z_n(t)$ are bounded. From (18), (9) and (7), it follows that $x(t)$ is bounded. If $\rho > 2$, the proof can be concluded by iterating the previous steps. □

III. EXAMPLE

Consider the system

$$\begin{aligned}\dot{x}_1 &= x_2 + u + w_1 \\ \dot{x}_2 &= u \\ e &= x_1 \\ \dot{w}_1 &= w_2 \\ \dot{w}_2 &= -\theta w_1 \end{aligned} \quad (36)$$

in which $\theta > 0$ is an unknown parameter. By making the change of coordinates

$$\zeta_1 = x_1,\ \zeta_2 = x_2 + w_1,\ \zeta_3 = w_2 + \theta x_1,\ \zeta_4 = \theta x_2$$

we obtain

$$\begin{aligned}\dot{\zeta}_1 &= \zeta_2 + u \\ \dot{\zeta}_2 &= \zeta_3 - \theta x_1 + u \\ \dot{\zeta}_3 &= \zeta_4 + \theta u \\ \dot{\zeta}_4 &= \theta u \ . \end{aligned} \quad (37)$$

Defing the vector $d = \begin{bmatrix} 1 & d_2 & d_1 & d_0 \end{bmatrix}^T$, the filtered transformation

$$\begin{aligned}\dot{\xi} &= \begin{bmatrix} -d_2 & 1 & 0 \\ -d_1 & 0 & 1 \\ -d_0 & 0 & 0 \end{bmatrix}\xi \\ &\quad + \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\left(\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}u + \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}y\right) \\ \mu &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T\xi \\ z &= \zeta - \begin{bmatrix} 0 \\ \xi \end{bmatrix}\theta \end{aligned}$$

transforms (37) into the adaptive observer form (13)

$$\begin{aligned}\dot{z} &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}z + \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}u + \begin{bmatrix} 1 \\ d_2 \\ d_1 \\ d_0 \end{bmatrix}\mu\theta \\ e &= \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}z \end{aligned}$$

with $a[0] = 0$, $b[0] = \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix}^T$, for which we can define the adaptive observer

$$\begin{aligned}\dot{\hat{z}} &= A_c\hat{z} + b[0]u + d\mu\hat{\theta} + k_o(e - c_c\hat{z}) \\ \dot{\hat{\theta}} &= g\mu(e - c_c\hat{z}) \end{aligned}$$

with $k_o = (A_c + \lambda I)d$ and $\lambda > 0$. Finally, we define the control u as in (26)

$$u = -\hat{z}_2 - \mu\hat{\theta} - ke \ . \quad (38)$$

Some simulations have been carried out for system (36) controlled by (38). The task of the control algorithm was to regulate the output to zero against the action of a sinusoidal disturbance generated by the exosystem in (36). Note that the parameter $\theta > 0$ is unknown while the initial conditions of the exosystem may be zero, so that actually the disturbance may be either a sinusoid with unknown frequency, amplitude and phase or a null signal. In the simulation, the disturbance is equal to $\sin(2t)$ for $0 \le t \le 30$ s and equal to zero for $t > 30$ s, while the control parameters are chosen as: $k = 5$, $\lambda = 3$, $d = \begin{bmatrix} 1 & 6 & 11 & 6 \end{bmatrix}^T$, $g = 1000$. The results are illustrated in Fig. 1 where are reported the time histories of the regulation error $e(t)$, the control input $u(t)$, the disturbance $w_1(t)$ and the estimate of the square of the frequency $\hat{\theta}(t)$. Since, after 30 s the persistency of excitation condition is not satisfied, the estimate of the frequency is not correct after that time interval.

IV. CONCLUSIONS

An output error feedback dynamic regulator has been presented for linear observable and controllable minimum phase systems. The regulator guarantees asymptotic tracking and/or rejection of sinusoidal signals with unknown frequencies, magnitudes and phases, provided that an upper bound is known on the number of sinuosoids, even though persistency of excitation conditions may not be satisfied. An example has been included and simulated to show the regulator performance.

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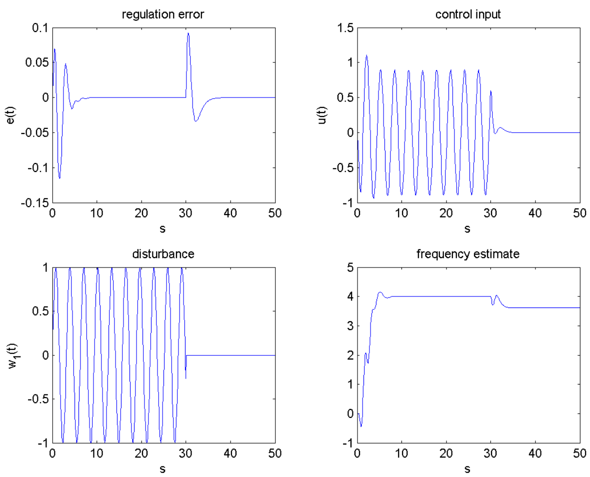

Fig. 1. Simulation results.

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