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A Kalman Decomposition for Robustly Unobservable Uncertain Linear Systems

Ian R. Petersen

Abstract— **This paper considers the structure of uncertain linear systems building on a concept of robust unobservability which was introduced in a previous paper. Compared to the previous paper, this paper specializes to the case of linear time invariant uncertain systems with unstructured uncertainty described by an averaged integral quadratic constraint. The paper presents a new characterization of the robustly unobservable states and shows that this leads to a Kalman type decomposition for uncertain linear systems with robustly unobservable states.**

I. INTRODUCTION

The notion of observability is one of the fundamental properties of a linear system; e.g., see [1]. One reason for considering the issue of observability for uncertain systems might be to determine if a robust state estimator can be constructed for the system; e.g., see [2]. In this case, one would be interested in the question of whether the system is "observable" for all possible values of the uncertainty. This approach to observability is considered in the paper [3].

For the case of linear systems, the issue of observability is also central to realization theory. For example, it is known that if a linear system contains unobservable states, those states can be removed in order to obtained a reduced dimension realization of the system's input-output behavior. For the case of uncertain systems, a natural extension of this notion of observability is to consider robustly unobservable states which are "unobservable" for all possible values of the uncertainty. This idea was developed in the paper [4] for the case of uncertain linear time varying systems with structured uncertainty subject to averaged integral quadratic constraints. The main result of [4] is the characterization of the robustly unobservable states of an uncertain system in terms of a certain parameter dependent Linear Quadratic optimal control problem (and in some cases in terms of a parameter dependent Riccati differential equation.)

In this paper, we consider a more restricted class of uncertain systems than those considered in [4]. In particular, we consider uncertain systems in which the nominal system is linear time invariant and for which the uncertainty is unstructured. In this case, the uncertainty is described by a single average integral quadratic constraint. For this class of uncertain systems, this paper provides a geometric characterization for the set of robustly unobservable states. This characterization implies that the set of robustly unobservable states is in fact a linear subspace. Furthermore, the characterization leads to a Kalman type decomposition for the uncertain systems under consideration.

The results presented in this paper will provide insight into the structure of uncertain systems as it relates to questions of realization theory; e.g., see [5].

The notion of robust unobservability used in this paper involves extending the definition of the observability Gramian to the case of uncertain systems; see also [6]. The framework considered is similar to that of [3] except that [3] is concerned with observability for all uncertainties whereas in this paper we are concerned with unobservability for all uncertainties.

As in the papers [7], [8], the uncertain systems considered in this paper will use an averaged integral quadratic constraint (IQC) uncertainty description. However, we consider the case of only a single averaged IQC which corresponds to the case of unstructured uncertainty.

II. PROBLEM FORMULATION

We consider the following linear time invariant uncertain system defined on the finite time interval $[0, T]$:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B\xi(t); \\ y(t) &= Cx(t) + D\xi(t); \\ z(t) &= Kx(t) \end{aligned} \tag{1}$$

where $x \in \mathbf{R}^n$ is the *state*, $y \in \mathbf{R}^l$ is the *measured output*, $z \in \mathbf{R}^h$ is the *uncertainty output*, and $\xi \in \mathbf{R}^r$ is the *uncertainty input*.

System Uncertainty. The uncertainty is required to satisfy a certain "Averaged Integral Quadratic Constraint".

Averaged Integral Quadratic Constraint. Let $d > 0$ be a given positive constant associated with the system (1). We will consider sequences of uncertainty inputs $\mathcal{S} = \{\xi^1(\cdot), \xi^2(\cdot), \ldots \xi^q(\cdot)\}$. The number of elements q in any such sequence is arbitrary. A sequence of uncertainty functions of the form $\mathcal{S} = \{\xi^1(\cdot), \xi^2(\cdot), \ldots \xi^q(\cdot)\}$ is an *admissible uncertainty sequence* for the system (1) if the following conditions hold: Given any $\xi^i(\cdot) \in \mathcal{S}$ and any corresponding solution $\{x^i(\cdot), \xi^i(\cdot)\}$ to (1) defined on $[0, T]$, then $\xi^i(\cdot) \in \mathbf{L}_2[0, T]$, and

$$\frac{1}{q} \sum_{i=1}^{q} \int_0^T \left(\|\xi^i(t)\|^2 - \|z^i(t)\|^2 \right) dt \quad \leq \quad d. \tag{2}$$

The class of all such admissible uncertainty sequences is denoted Ξ. One way in which such uncertainty could be generated is via unstructured feedback uncertainty is shown in the block diagram in Figure 1.

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School of Information Technology and Electrical Engineering, Australian Defence Force Academy, Canberra ACT 2600, Australia, irp@ee.adfa.edu.au.

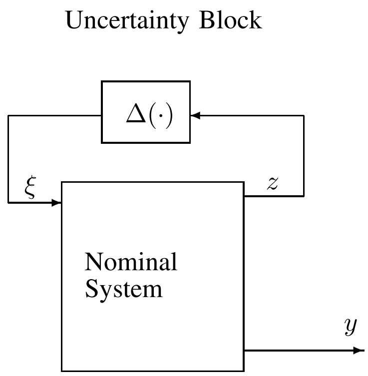

Fig. 1. An uncertain system with unstructured feedback uncertainty.

Definition 1: The *robust unobservability function* for the uncertain system (1), (2) is defined as

$$L_o(x_0) \triangleq \sup_{\mathcal{S}\in\Xi} \frac{1}{q}\sum_{i=1}^{q}\int_0^T \|y(t)\|^2 dt \tag{3}$$

where $x(0) = x_0$ in (1).

This definition extends the standard definition of the observability Gramian for linear systems.

Notation.

$$\mathcal{D} \triangleq \{d : d > 0\}.$$

Definition 2: A non-zero state $x_0 \in \mathbf{R}^n$ is said to be *robustly unobservable* for the uncertain system (1), (2) if

$$\inf_{d\in\mathcal{D}} L_o(x_0) = 0.$$

The set of all robustly unobservable states for the uncertain system (1), (2) is referred to as the *robustly unobservable set* $\mathcal{U}$; i.e.,

$$\mathcal{U} \triangleq \left\{ x \in \mathbf{R}^n : \inf_{d\in\mathcal{D}} L_o(x) = 0 \right\}.$$

III. Preliminary Results

In this section, we will recall the main result of [4] specialized to the class of uncertain systems considered in this paper. We also establish a number of other preliminary results which are required in order to establish our main results.

A. An Unconstrained Optimization Problem.

For the uncertain system (1), (2), we define a function $V_\tau(x_0)$ as follows:

$$V_\tau(x_0) \triangleq \inf_{\xi(\cdot)\in\mathbf{L}_2[0,T]} \int_0^T \left(-\|y\|^2 + \tau\|\xi\|^2 - \tau\|z\|^2 \right) dt. \tag{4}$$

Here $\tau \geq 0$ is a given constant.

Observation 1: Note that by setting $\xi(\cdot) = 0$, we can see that

$$V_\tau(x_0) \leq 0$$

for all $\tau \geq 0$.

Also note that since $V_\tau(x_0)$ is the infimum of a collection of functions which are affine linear in τ, then $V_\tau(x_0)$ must be a concave function of τ.

Observation 2: We can write

$$\begin{aligned} V_\tau(x_0) &= \inf_{\xi(\cdot)\in\mathbf{L}_2[0,T]} \int_0^T \begin{pmatrix} -\|y\|^2 + \tau\|\xi\|^2 \\ -\tau\|z\|^2 \end{pmatrix} dt \\ &= \tau\tilde{V}_\tau(x_0) \end{aligned}$$

where

$$\tilde{V}_\tau(x_0) \triangleq \inf_{\xi(\cdot)\in\mathbf{L}_2[0,T]} \int_0^T \begin{pmatrix} -\frac{\|y\|^2}{\tau} + \|\xi\|^2 \\ -\|z\|^2 \end{pmatrix} dt.$$

Now it follows from the definition of $\tilde{V}_\tau(x_0)$ that if $\tilde{V}_{\tau_0}(x_0) > -\infty$ for $\tau_0 > 0$ then $\tilde{V}_\tau(x_0) > -\infty$ for all $\tau \geq \tau_0$. Hence, if $V_{\tau_0}(x_0) > -\infty$ for $\tau_0 > 0$ then $V_\tau(x_0) > -\infty$ for all $\tau \geq \tau_0$.

B. A Formula for the Robust Unobservability Function.

We first introduce the following notation:

$$\Gamma(x_0) \triangleq \{ \ \tau : \tau > 0 \text{ and } V_\tau(x_0) > -\infty \ \},$$

and

$$\bar{\Gamma}(x_0) \triangleq \{ \ \tau : \tau \geq 0 \text{ and } V_\tau(x_0) > -\infty \ \}.$$

Assumption 1: The set $\bar{\Gamma}(x_0)$ defined above is non-empty for all $x_0 \in \mathbf{R}^n$.

Remark: The above assumption is a technical assumption required to establish the main results of this paper. It represents an assumption on the size of the uncertainty in the system relative to the time interval $[0, \ T]$ under consideration. In general, this assumption can always be satisfied by choosing a sufficiently small $T > 0$.

Theorem 1: (See [4] for proof). Consider the uncertain system (1), (2) and corresponding robust unobservability function (1). Then for any initial condition $x(0) = x_0$,

$$L_o(x_0) = \inf_{\tau\in\bar{\Gamma}(x_0)} \{-V_\tau(x_0) + \tau d\}. \tag{5}$$

Corollary 1: (See [4] for proof). If we define

$$\tilde{L}_o(x_0) \triangleq \inf_{d\in\mathcal{D}} L_o(x_0)$$

then

$$\begin{aligned} \tilde{L}_o(x_0) &= \inf_{\tau\in\bar{\Gamma}(x_0)} \{-V_\tau(x_0)\} \\ &= -\sup_{\tau\in\bar{\Gamma}(x_0)} V_\tau(x_0). \end{aligned} \tag{6}$$

Observation 3: From the above corollary, it follows immediately that the unobservable set $\mathcal{U}$ can be written in the form:

$$\mathcal{U} = \left\{ x \in \mathbf{R}^n : \sup_{\tau\in\bar{\Gamma}(x)} \{V_\tau(x)\} = 0 \right\}.$$

Lemma 1: Suppose the non-zero state $x_0 \in \mathbf{R}^n$ is an observable state for the pair (K, A). Then $V_\tau(x_0) \to -\infty$ as $\tau \to \infty$.

Proof. Since, x_0 is observable from the output z, it follows that the solution to the system (1) with $\xi(\cdot) \equiv 0$ is such that

$$\int_0^T \|z\|^2 dt > 0$$

and hence,

$$\int_0^T \left(-\|y\|^2 - \tau\|z\|^2 \right) dt \to -\infty$$

as $\tau \to \infty$. From this, the lemma follows using the definition of $V_\tau(x_0)$.
□

Lemma 2: Suppose that the uncertain system (1), (2) is such that Assumption 1 is satisfied. Then the following statements are equivalent:

1) There exists an $x_0 \in \mathbf{R}^n$ such that the supremum in (6) is achieved at $\tau = 0$.
2) The system (1) is such that the transfer function from input ξ to output y is zero; i.e.,

$$G(s) \triangleq C(sI - A)^{-1}B + D \equiv 0.$$

3) For all $x_0 \in \mathbf{R}^n$, the supremum in (6) is achieved at $\tau = 0$.

Proof 1) $\Rightarrow$ 2). Suppose $x_0 \in \mathbf{R}^n$ is such that the supremum in (6) is achieved at $\tau = 0$ and consider

$$V_0(x_0) = \inf_{\xi(\cdot) \in \mathbf{L}_2[0,T]} \int_0^T \left(-\|y\|^2\right) dt. \quad (7)$$

This quantity is simply the value of $V_\tau(x_0)$ corresponding to $\tau = 0$.

We now proceed by contradiction and suppose that $G(s) \not\equiv 0$. It follows that there exists an input $\xi^*(\cdot) \in \mathbf{L}_2[0,T]$ such that $\|\xi^*(\cdot)\|_2 = 1$ and the corresponding output y of the system (1) with initial condition $x(0) = 0$ is non-zero. Here $\|\xi^*(\cdot)\|_2$ denotes the $\mathbf{L}_2[0,T]$ norm of the input $\xi^*(\cdot)$. Also, let $y_0^*(\cdot)$ denote the output of the system (1) corresponding to the input $\xi^*(\cdot)$ and initial condition $x(0) = 0$. We write

$$\alpha = \|y_0^*(\cdot)\|_2 > 0.$$

Furthermore, if $y^*(t)$ is the output of the system (1) with input $\xi^*(t)$ and initial condition $x(0) = x_0$, then we can write

$$y^*(t) = Ce^{At}x_0 + y_0^*(t)$$

for all $t \in [0,\ T]$.

Now suppose that the input $\xi^*(\cdot)$ is scaled by a factor $\mu > 0$. Let $y_\mu(t)$ denote output of the system (1) with input $\xi(t) = \mu\xi^*(t)$ and initial condition $x(0) = x_0$. Then, we can write

$$y_\mu(t) = Ce^{At}x_0 + \mu y_0^*(t)$$

for all $t \in [0,\ T]$. From this, it follows from the Cauchy-Schwartz inequality that

$$\begin{aligned} \|y_\mu(\cdot)\|_2 &\geq \mu\|y_0^*(\cdot)\|_2 - \|Ce^{At}x_0\|_2 \\ &= \mu\alpha - \|Ce^{At}x_0\|_2. \end{aligned}$$

Hence, $\|y_\mu(\cdot)\|_2 \to \infty$ as $\mu \to \infty$. Therefore

$$\int_0^T \left(-\|y\|^2\right) dt \to -\infty$$

as $\mu \to \infty$. Thus using (7), we must have $V_0(x_0) = -\infty$. However, we have assumed that the supremum in (6) is achieved at $\tau = 0$. Thus, we must have $V_\tau(x_0) = -\infty$ for all $\tau \geq 0$. This contradicts Assumption 1. Thus, we must have $G(s) \equiv 0$.

2) $\Rightarrow$ 3). Suppose $G(s) \equiv 0$. It follows that given any input $\xi(\cdot) \in \mathbf{L}_2[0,T]$ and any initial condition $x(0) = x_0$, the corresponding output y of the system (1) satisfies

$$y(t) = Ce^{At}x_0. \quad (8)$$

Hence, in this case we have

$$\begin{aligned} &\int_0^T \left(-\|y\|^2 + \tau\|\xi\|^2 - \tau\|z\|^2\right) dt \\ &\quad = \int_0^T \left(-\|Ce^{At}x_0\|^2\right) dt + \tau\int_0^T \left(\|\xi\|^2 - \|z\|^2\right) dt \end{aligned}$$

Therefore,

$$\begin{aligned} V_\tau(x_0) &= \int_0^T \left(-\|Ce^{At}x_0\|^2\right) dt \\ &\quad +\tau \inf_{\xi(\cdot)\in\mathbf{L}_2[0,T]} \int_0^T \left(\|\xi\|^2 - \|z\|^2\right) dt \\ &= \int_0^T \left(-\|Ce^{At}x_0\|^2\right) dt + \tau\bar{V}(x_0) \end{aligned}$$

where

$$\bar{V}(x_0) = \inf_{\xi(\cdot)\in\mathbf{L}_2[0,T]} \int_0^T \left(\|\xi\|^2 - \|z\|^2\right) dt \leq 0.$$

Hence,

$$\sup_{\tau\geq 0} V_\tau(x_0) = \int_0^T \left(-\|Ce^{At}x_0\|^2\right) dt$$

and the supremum is achieved at $\tau = 0$. From this, condition 3) of the lemma follows.

3) $\Rightarrow$ 1). This part of the lemma follows by definition.
□

IV. The Main Results

In this section we present the main results of this paper which provide a geometric characterization of the robustly unobservable states of the uncertain system (1), (2). We first consider the case in which $G(s) \equiv 0$.

Theorem 2: Consider the uncertain system (1), (2) and suppose that Assumption 1 is satisfied. Also, suppose that $G(s) \equiv 0$. Then a state x_0 is robustly unobservable if and only if it is an unobservable state for the pair (C, A).

Proof. We first suppose x_0 is a robustly unobservable state for the uncertain system (1), (2). It follows from the assumptions of the theorem and Lemma 2 that the supremum in (6) is achieved at $\tau = 0$. Hence, using Observation 3 it follows that $V_0(x_0) = 0$. That is,

$$\sup_{\xi(\cdot)\in\mathbf{L}_2[0,T]} \int_0^T \left(\|y\|^2\right) dt = 0.$$

Setting $\xi(t) \equiv 0$, it follows that

$$\int_0^T \left(\|Ce^{At}x_0\|^2\right) dt = 0.$$

From this, we can conclude that x_0 is an unobservable state for the pair (C, A).

Conversely, suppose that x_0 is an unobservable state for the pair (C, A). Then using the fact that $G(s) \equiv 0$, it follows from Lemma 2 that the supremum in (6) is achieved at $\tau = 0$ and hence, we can write

$$\sup_{\tau \in \bar{\Gamma}(x_0)} V_\tau(x_0) = V_0(x_0) = \inf_{\xi(\cdot) \in \mathbf{L}_2[0,T]} \int_0^T \left(-\|y\|^2\right) dt.$$

For the system (1) with initial condition $x(0) = x_0$ and any input $\xi(\cdot)$, the corresponding output y satisfies (see (8))

$$y(t) = Ce^{At}x_0 \equiv 0$$

since x_0 is an unobservable state for the pair (C, A). Hence, we must have $\sup_{\tau \in \bar{\Gamma}(x_0)} V_\tau(x_0) = 0$. Therefore Observation 3 implies that x_0 is a robustly unobservable state. This completes the proof of the theorem.
□

Observation 4: The above theorem implies that when $G(s) \equiv 0$ (and Assumption 1 is satisfied), the robustly unobservable set is a linear space equal to the unobservable subspace of the pair (C, A). From this, it follows that we can apply the standard Kalman decomposition (e.g., see [1]) to represent the uncertain system as show in Figure 2.

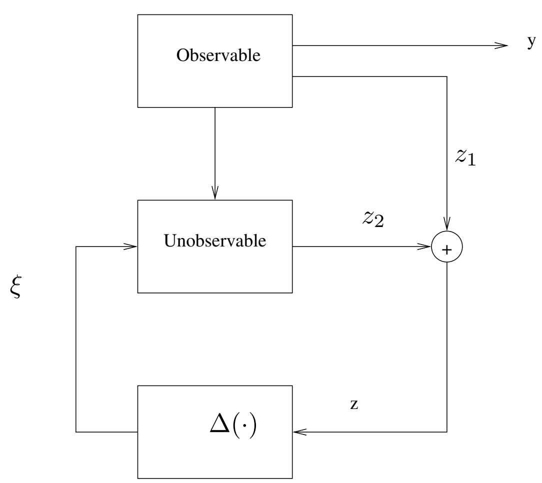

Fig. 2. Kalman Decomposition for the Uncertain System in Theorem 2.

In particular, note that in this case, all of the uncertainty is in the unobservable subsystem.

We now consider the case in which $G(s) \not\equiv 0$.

Theorem 3: Consider the uncertain system (1), (2) and suppose that Assumption 1 is satisfied. Also, suppose that $G(s) \not\equiv 0$. Then a state x_0 is robustly unobservable if and only if it is an unobservable state for the pair $(\begin{bmatrix} C \\ K \end{bmatrix}, A)$.

Proof. We first suppose x_0 is a robustly unobservable state for the uncertain system (1), (2). It follows from Observation 3 that $\sup_{\tau \in \bar{\Gamma}(x_0)} V_\tau(x_0) = 0$. Furthermore, it follows from Lemma 2 that this supremum is not achieved at $\tau = 0$. Moreover, it follows from Observations 1 and 2 that either this supremum is achieved at some $\tau^* \in (0, \infty)$ or

$$\sup_{\tau \in \bar{\Gamma}(x_0)} V_\tau(x_0) = \lim_{\tau \to \infty} V_\tau(x_0).$$

We now consider each of these cases separately.

Case 1: The supremum in (6) is achieved at $\tau^* \in (0, \infty)$. In this case, we have

$$\begin{aligned} 0 &= V_{\tau^*}(x_0) \\ &= \inf_{\xi(\cdot) \in \mathbf{L}_2[0,T]} \int_0^T \left(\begin{array}{c} -\|y\|^2 + \tau^*\|\xi\|^2 \\ -\tau^*\|z\|^2 \end{array} \right) dt. \end{aligned}$$

Hence

$$\int_0^T \left(\begin{array}{c} -\|y\|^2 + \tau^*\|\xi\|^2 \\ -\tau^*\|z\|^2 \end{array} \right) dt \geq 0 \quad \forall \xi(\cdot) \in \mathbf{L}_2[0,T].$$

Setting $\xi(t) \equiv 0$, it follows that the corresponding solution to (1) with initial condition $x(0) = x_0$ satisfies

$$-\int_0^T \left(\|y\|^2 + \tau^*\|z\|^2\right) dt \geq 0$$

and hence, we can conclude that $y(t) \equiv 0$ and $z(t) \equiv 0$. Thus, x_0 is an unobservable state for the pair $(\begin{bmatrix} C \\ K \end{bmatrix}, A)$.

Case 2: $\sup_{\tau \in \bar{\Gamma}(x_0)} V_\tau(x_0) = \lim_{\tau \to \infty} V_\tau(x_0)$. In this case, we have

$$\lim_{\tau \to \infty} V_\tau(x_0) = 0. \tag{9}$$

Hence, given any $\epsilon > 0$, there exists an $M \geq 0$ such that $\tau \geq M$ implies $V_\tau(x_0) \geq -\epsilon$. That is,

$$\int_0^T \left(\begin{array}{c} -\|y\|^2 + \tau\|\xi\|^2 \\ -\tau\|z\|^2 \end{array} \right) dt \geq -\epsilon \quad \forall \xi(\cdot) \in \mathbf{L}_2[0,T].$$

We now set $\xi(t) \equiv 0$, and consider the corresponding solution to (1) with initial condition $x(0) = x_0$. It follows that

$$\int_0^T \left(\|y\|^2 + \tau\|z\|^2\right) dt \leq \epsilon. \tag{10}$$

We now recall Lemma 1. Combining this lemma with equation (9) it follows that x_0 must be an unobservable state for the pair (K, A). Hence, $z(t) \equiv 0$. Therefore, it follows from (10) that

$$\int_0^T \left(\|y\|^2\right) dt \leq \epsilon.$$

However, the constant $\epsilon > 0$ was arbitrary. Thus, we must conclude that $y(t) \equiv 0$. Thus, we have established that $z(t) \equiv 0$ and $y(t) \equiv 0$. Therefore, x_0 is an unobservable state for the pair $(\begin{bmatrix} C \\ K \end{bmatrix}, A)$.

We have now shown that x_0 is an unobservable state for the pair $(\begin{bmatrix} C \\ K \end{bmatrix}, A)$ for both of the possible cases. This completes the proof of the first part of the theorem.

To establish the converse part of the theorem, we suppose that x_0 is an unobservable state for the pair $(\left[\begin{array}{c} C \\ K \end{array}\right], A)$. Hence, given any $\xi(\cdot) \in \mathbf{L}_2[0,T]$ and $t \in [0,\ T]$

$$\begin{aligned}\left[\begin{array}{c} y(t) \\ z(t) \end{array}\right] &= \left[\begin{array}{c} C \\ K \end{array}\right] e^{At} x_0 \\ &\quad + \int_0^t \left[\begin{array}{c} C \\ K \end{array}\right] e^{A(t-s)} B\xi(s)ds + \left[\begin{array}{c} D \\ 0 \end{array}\right] \xi(t) \\ &= \int_0^t \left[\begin{array}{c} C \\ K \end{array}\right] e^{A(t-s)} B\xi(s)ds + \left[\begin{array}{c} D \\ 0 \end{array}\right] \xi(t). \end{aligned} \tag{11}$$

We now consider any $\tau \in \bar{\Gamma}(x_0)$.

Claim: $V_\tau(x_0) = 0$.

We establish this claim by contradiction. Indeed suppose that $V_\tau(x_0) < 0$. It follows that there exists a $\bar{\xi}(\cdot) \in \mathbf{L}_2[0,T]$ such that the corresponding response to the system satisfies

$$\int_0^T \left(-\|\bar{y}\|^2 + \tau\|\bar{\xi}\|^2 - \tau\|\bar{z}\|^2\right) dt = -\mu < 0. \tag{12}$$

Here $\left[\begin{array}{c} \bar{y}(t) \\ \bar{z}(t) \end{array}\right]$ is the response of system (1) with input $\bar{\xi}(t)$ and initial condition $x(0) = x_0$. Now if the input $\bar{\xi}(t)$ is replaced by the input $\xi(t) = \alpha\bar{\xi}(t)$ where $\alpha > 0$, it follows from (11) that the corresponding response of the system (1) with input $\xi(t)$ and initial condition $x(0) = x_0$ will satisfy

$$\left[\begin{array}{c} y(t) \\ z(t) \end{array}\right] = \alpha \left[\begin{array}{c} \bar{y}(t) \\ \bar{z}(t) \end{array}\right].$$

Thus, it follows from (12) that

$$\int_0^T \left(-\|\bar{y}\|^2 + \tau\|\bar{\xi}\|^2 - \tau\|\bar{z}\|^2\right) dt = -\alpha^2\mu.$$

Setting $\alpha \to \infty$, it follows that

$$V_\tau(x_0) = \inf_{\xi(\cdot)\in\mathbf{L}_2[0,T]} \int_0^T \left(\begin{array}{c} -\|y\|^2 \\ +\tau\|\xi\|^2 - \tau\|z\|^2 \end{array}\right) dt = -\infty$$

which contradicts the assumption that $\tau \in \bar{\Gamma}(x_0)$. Thus, we must have $V_\tau(x_0) = 0$. This completes the proof of the claim.

Using the above claim, we have shown that $V_\tau(x_0) = 0$ for all $\tau \in \bar{\Gamma}(x_0)$. Thus, we must have

$$\sup_{\tau\in\bar{\Gamma}(x_0)} V_\tau(x_0) = 0.$$

Therefore, using Observation 3, it follows that x_0 is a robustly unobservable state for the uncertain system (1), (2). This completes the proof of the theorem.

□

Observation 5: The above theorem implies that when $G(s) \not\equiv 0$ (and Assumption 1 is satisfied), the robustly unobservable set is a linear space equal to the unobservable subspace of the pair $(\left[\begin{array}{c} C \\ K \end{array}\right], A)$. From this, it follows that we can apply the standard Kalman decomposition (e.g., see [1]) to represent the uncertain system as show in Figure 3. In particular, note that in this case, all of the uncertainty is in the observable subsystem or in the coupling between the two subsystems.

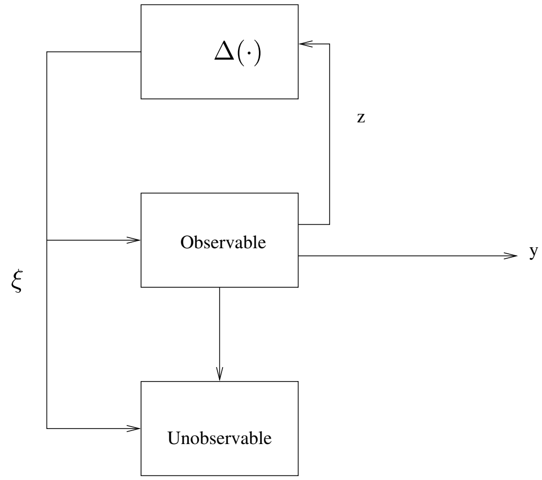

Fig. 3. Kalman Decomposition for the Uncertain System in Theorem 3.

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