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Average Continuous Control of Piecewise Deterministic Markov Processes

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Abstract— **In this paper we consider the long run average continuous control problem of piecewise-deterministic Markov processes (PDP's for short). The control variable acts on the jump rate λ and transition measure Q of the PDP. The main goal of the paper is to characterize the optimality equation of the problem in terms of integro-differential equations for the continuous-time problem as well as in terms of embedded discrete-time Markov chains associated to the PDP.**

I. INTRODUCTION

A general family of non-diffusion stochastic models suitable for formulating many optimization problems in several areas of operations research, namely piecewise-deterministic Markov processes (PDP's), were introduced in [1], [2]. These processes are determined by three local characteristics; the flow ϕ, the jump rate λ and the transition measure Q. Several results are now available on the theory of PDP's, including invariant measures [3], Poisson equation [4], continuous control [5], [6], [7], optimal stopping [8], [9], and impulse control [10], [11], [12]. Related to PDP's are the so-called Markov Decision Drift Processes, which deals with problems involving interventions (similar to impulse control) and continuous control on the rate of jumps and post-jump location measure. Such problems have been analyzed in [13], [14] via time-discretization, and in [15], [16] via Bellman inequalities.

In this paper we study the average control problem of PDP's with the control variable acting on the jump rate λ and the transition measure Q. In section II we present the main definitions and assumptions. Section III deals with the definition of the problem, the definition of the PDP, and two embedded discrete-time Markov chains associated with the PDP. The main results are presented in section IV. The optimality equation for the long run average problem is characterized in terms of integro-differential equations for the continuous-time PDP as well as in terms of embedded discrete-time Markov chains associated with the PDP. One of the embedded Markov chains is the post-jump location. However, as shown in [3], the second embedded Markov chain has some nicer stability properties than the post-jump location Markov chain, and might be more useful in studying necessary conditions for the existence of a solution for the optimality equation, as discussed in section V.

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II. PRELIMINARIES

For any Borel space $\mathbb{X}$ we denote by $\mathcal{P}(\mathbb{X})$ the set of all probability measures over $\mathbb{X}$. Let E^0 be an open non-empty subset of R^d, ∂E^0 its boundary. Let $\phi(t,x)$ be the flow of a Lipschitz continuous vector field $\mathcal{L}$, and define the sets $\partial E^0_- := \{z \in \partial E^0 : z = \phi(-t,x)$ for some $t > 0$ and for some $x \in E^0\}$, $\partial E^0_+ := \{z \in \partial E^0 : z = \phi(t,x)$ for some $t > 0$ and for some $x \in E^0\}$. The points $z \in \partial E^0_-$ are such that $z \in \partial E^0$ and starting at z the flows $\phi(t,z)$ will move forward until it reaches some $x \in E^0$ at a finite time t. If the set ∂E^0_- is empty, the flow $\phi(t,x)$ can never reach the boundary when it starts from any point $x \in E^0$ and moves backward. Similarly, the points $z \in \partial E^0_+$ are such that $z \in \partial E^0$ and starting at x the flow $\phi(t,x)$ will move forward until it reaches z at a finite time t. If the set ∂E^0_+ is empty, the flow $\phi(t,x)$ can never reach the boundary when it starts from any point $x \in E^0$ and moves forward.

Define $\partial E^0_1 = \partial E^0_- - \partial E^0_+$ and set $E = E^0 \cup \partial E^0_1, \sigma(E)$ the Borel σ-field of E, $\partial E^* = \partial E^0_+$, and for each $x \in E$, write $t^*(x) = inf\{t > 0; \phi(t,x) \in \partial E^*\}$, where $inf\{\emptyset\} := \infty$. Define $t^*(z) = 0$ for $z \in \partial E^*$, and for $t < t^*(x)$,

$$\mathcal{I}_t(x) = \begin{cases} [t,\infty) & if\ t^*(x) = \infty, \\ [t, t^*(x)] & if\ t^*(x) < \infty. \end{cases}$$

We set $\mathbb{B}(E)$ and $\mathbb{B}(\partial E^*)$ the set of all bounded Borel measurable functions from E and ∂E^* into R respectively. We define

$$\begin{aligned} \mathbb{B}^c(E) = \{ & f \in \mathbb{B}(E); f(\phi(.,x)) : [0,t^*(x)) \mapsto R \\ & \text{is continuous for each } x \in E, \text{ and} \\ & lim_{t\uparrow t^*(x)} f(\phi(t,x)) \text{ exists} \}, \\ \mathbb{B}^{ac}(E) = \{ & f \in \mathbb{B}^c(E); f(\phi(.,x)) : [0,t^*(x)) \mapsto R \\ & \text{is absolutely continuous for each } x \in E\} \end{aligned}$$

and we define for $f \in \mathbb{B}^c(E)$, $f(z) = lim_{t\uparrow t^*(x)} f(\phi(t,x))$ for all $z = lim_{t\uparrow t^*(x)} \phi(t,x) \in \partial E^*$. We consider the following parameters for our problem:

a) $U(.)$ is a Borel set-valued function such that for each $x \in E \cup \partial E^*$, $U(x) \subset \mathbb{U}$, where $\mathbb{U}$ is a non-empty Borel space, and
   - a.1) $U(x)$ is compact for each point $x \in E \cup \partial E^*$,
   - a.2) $U(\phi(.,x)) : [0, t^*(x)) \mapsto \mathbb{U}$ is right continuous for each $x \in E$.

b) $\lambda(.,.) : E \times \mathbb{U} \mapsto R_+$ is a bounded Borel measurable function such that for each $x \in E$, $\lambda(x,.) : U(x) \mapsto R_+$ is continuous,

c) For all $(x,u) \in \big[E \cup \partial E^*\big] \times \mathbb{U}$, $Q(.;x,u) \in \mathcal{P}(E)$ is such that for each $g \in \mathbb{B}(E)$, and $x \in E$, $z \in \partial E^*$; c.1) $Qg(x,.) : U(x) \mapsto R$ is lower semi continuous, and c.2) $Qg(z,.) : U(z) \mapsto R$ is lower semi continuous, where we use the notation, $Qg(x,.) = \int_E g(y)Q(dy;x,.)$.

d) $f(.,.) : E \times \mathbb{U} \mapsto R$ is a bounded Borel measurable function such that for each $x \in E$, $f(x,.) : U(x) \mapsto R$ is lower semi continuous,

e) $r(.,.) : \partial E^* \times \mathbb{U} \mapsto R$ is a bounded Borel measurable function and for each $z \in \partial^* E$ $r(z,.) : U(z) \mapsto R$ is lower semi continuous.

We also define

$$\mathcal{U} = \{u(.) : E \cup \partial^* E \mapsto \mathbb{U} \text{ measurable function such that } u(\phi(t,x)) \in U(\phi(t,x)) \text{ for each } x \in E,\ t \in \mathcal{I}_0(x)\}.$$

We set, for $u \in \mathcal{U}$, 1) $\lambda^u(x) = \lambda(x,u(x))$, 2) $Q^u(.;x) = Q(.;x,u(x))$, 3) $f^u(x) = f(x,u(x))$ and 4) $r^u(z) = r(z,u(z))$ for $z \in \partial E^*$, and $\Lambda^u(t,x) = \int_0^t \lambda^u(\phi(s,x))ds$.

Remark 2.1: : From the assumptions b), c.1), c.2), d), and e) we have that for $g,h \in \mathbb{B}(E)$, $r(z,.) + Qg(z,.)$ and

$$f(x,.) - \lambda(x,.) \int_E (h(x) - g(y))Q(dy;x,.)$$

are lower semi-continuous in $U(x)$. Therefore from the compactness assumption in a.1) and the results in Proposition D5) in [17], page 183, there exists a measurable selector $u^* \in \mathcal{U}$ such that for each $x \in E$, $z \in \partial E^*$

$$f^{u^*}(x) - \lambda^{u^*}(x) \int_E (h(x) - g(y))Q^{u^*}(dy;x) = \min_{a \in U(x)} \{f(x,a) - \lambda(x,a) \int_E (h(x) - g(y))Q(dy;x,a)\} \tag{1}$$

$$r^{u^*}(z) + Q^{u^*}g(z) = \min_{a \in U(z)} \{r(z,a) + Qg(z,a)\}. \tag{2}$$

Define for each $x \in E$ and Borel set $\mathcal{I} \subset [0, t^*(x)]$

$$V_{\mathcal{I}}(x) = \{v(.) : \mathcal{I} \mapsto \mathbb{U} \text{ measurable}; v(s) \in U(\phi(s,x)) \text{ for each } s \in \mathcal{I}\}$$

and we write $V_t(x) = V_{\mathcal{I}_t(x)}(x)$, $V(x) = V_0(x)$. For any $u \in \mathcal{U}$, we shall define $v(u) \in V(x)$ as

$$v(u)(t) = u(\phi(t,x)),\ t \in \mathcal{I}_0(x). \tag{3}$$

From the definition of $\mathcal{U}$ above it is clear that indeed $v(u) \in V(x)$. For any $v \in V(x)$ set: 1) $\lambda^v(\phi(s,x)) = \lambda(\phi(s,x), v(s))$, 2) $Q^v(.;\phi(s,x)) = Q(.;\phi(s,x),v(s))$, 3) $f^v(\phi(s,x)) = f(\phi(s,x),v(s))$ and 4) $r^v(\phi(t^*(x),x)) = r(\phi(t^*(x),x), v(t^*(x)))$ and, as before, $\Lambda^v(t,x) = \int_0^t \lambda^v(\phi(s,x))ds$.

III. AVERAGE CONTINUOUS CONTROL PROBLEM

A. PROBLEM FORMULATION

Let Ω denote the space of right-continuous functions $\omega(.)$ on R_+ taking values in E such that the left limit exists for all $t > 0$. Denote by x_t the coordinate function $x_t(\omega) = \omega(t)$ for all $\omega \in \Omega$. Let $\mathcal{F}_t^0 = \sigma\{x_s; 0 \le s \le t\}$ and $\mathcal{F}^0 = \bigvee_{t \ge 0} \mathcal{F}_t^0$. For $\omega \in \Omega$, set $T_0(\omega) = 0$, and for $k = 1,2,\ldots$,

$$T_k(\omega) = \begin{cases} \inf\{t > T_{k-1}(\omega); x_t(\omega) \neq x_{t^-}(\omega)\}, \\ \qquad \text{if } T_{k-1}(\omega) < \infty, \\ \infty \qquad \text{otherwise} \end{cases}$$

$$T_\infty(\omega) = \lim_{k \to \infty} T_k(\omega)$$

$$Z_k(\omega) = \begin{cases} x_{T_k(\omega)}(\omega) & if\, T_k(\omega) < \infty \\ \Delta & if\, T_k(\omega) = \infty \end{cases}$$

where Δ represents a cemetery state. We shall write for $k = 0,1,\ldots$, $\varpi_k(\omega) = (Z_0(\omega), T_1(\omega), Z_1(\omega), \ldots, T_k(\omega), Z_k(\omega))$.

We shall define the set of admissible continuous control strategies Ψ in the following way. A point $\psi = (\zeta_1, \zeta_2, \ldots) \in \Psi$ if it is predictable (see [2], page 264) that is, for every $\omega \in \Omega$,

$$\psi(\omega,t) = 1_{\{0 \le t \le T_1(\omega)\}} \zeta_1(Z_0(\omega),t) + \sum_{k=2}^{\infty} 1_{\{T_{k-1}(\omega) < t \le T_k(\omega)\}} \zeta_k\Big(\varpi_{k-1}(\omega), t - T_{k-1}(\omega)\Big) + 1_{\{T_\infty(\omega) \le t\}} \Delta$$

where ζ_k is a Borel measurable function from $\big[E \cup \{\Delta\}\big] \times \prod_{i=1}^{k-1} \Big(\big[R_+ \cup \{\infty\}\big] \times \big[E \cup \{\Delta\}\big]\Big) \times R_+$ to $\mathbb{U} \cup \{\Delta\}$ satisfying, for all $s \in \mathcal{I}_0(Z_{k-1}(\omega))$ and $\omega \in \Omega$ such that $T_{k-1}(\omega) < \infty$, $\zeta_k(\varpi_{k-1}(\omega), s) \in U(\phi(s, Z_{k-1}(\omega)))$, whereas for the case in which $T_{k-1}(\omega) = \infty$, the control variable takes the value of the cemetery state Δ. It follows that $\zeta_k(\varpi_{k-1}(\omega),.) \in V(Z_{k-1}^\psi(\omega))$. We define the motion of the process $\{X_t^\psi\}$ associated to the strategy ψ, starting from a point $x \in E$, in the following way. Take a random variable T_1^ψ such that

$$P_x(T_1^\psi > t) = \begin{cases} exp\{-\Lambda^{\zeta_1}(t,x)\} & \text{for } t < t^*(x) \\ 0 & \text{for } t \ge t^*(x) \end{cases}.$$

If T_1^ψ generated according to the above probability is equal to infinity, then $X_t^\psi = \phi(t,x)$ for all $t \ge 0$. Otherwise select independently an E-valued random variable having distribution $Q^{\zeta_1}(.;\phi(T_1^\psi,x))$. The trajectory of $\{X_t^\psi\}$ starting at x, for $t \le T_1^\psi$, is given by $X_t^\psi = \phi(t,x)$ for $t < T_1^\psi$ $X_t^\psi = Z_1^\psi$ for $t = T_1^\psi$. Starting from $X_{T_1^\psi}^\psi = Z_1^\psi$, we now select the next inter-jump time $T_2^\psi - T_1^\psi$ and post-jump location $X_{T_2^\psi}^\psi = Z_2^\psi$ is a similar way, using $\zeta_2(x, T_1^\psi, Z_1^\psi, .)$ (which by definition belongs to $V(Z_1^\psi)$ for T_1^ψ and Z_1^ψ fixed) instead of $\zeta_1(x,.)$. This gives a time-variant piecewise-deterministic trajectory for the process $\{X_t^\psi\}$ with jump times $T_1^\psi, T_2^\psi, \ldots$, and post-jump location $Z_1^\psi, Z_2^\psi, \ldots$. Note that in general we can not guarantee that the process $\{X_t^\psi\}$ is a Markov process, since ζ_k may depend on the whole history of the process up to k^{th}-jump. For the case in which we have, for all $k = 0, 1, \ldots$ $\zeta_k(\varpi_{k-1}(\omega), .) = u_k(\phi(., Z_{k-1}(\omega)))$ for $u_k \in \mathcal{U}$, $k = 0, 1, \ldots$, we write $\psi = (u_1, u_2, \ldots)$ and it can be shown by arguments similar to those in [2], page 62-66, that $\{X_t^\psi\}$ is a strong Markov process (although not necessarily homogeneous). The particular case in which $u_k = u$ for all $k = 0, 1, \ldots$, for some $u \in \mathcal{U}$, corresponds to the situation of a time homogeneous PDP, to be further detailed in the next subsection.

The procedure above defines a family of measures $\{P_x^\psi; x \in E\}$ on $(\Omega, \mathcal{F}^0)$. The final assumption on ψ to be an admissible control is that for every $x \in E$,

$$E_x^\psi\left(\sum_{k=1}^{\infty} 1_{\{T_k^\psi \le t\}}\right) < \infty \tag{4}$$

for all $t \in R_+$. In particular (4) implies that $T_k^\psi \to \infty$ as $k \to \infty$ almost surely. For any $\mu \in \mathcal{P}(E)$, define P_μ^ψ on $(\Omega, \mathcal{F}^0)$ as $P_\mu^\psi(A) = \int_E P_x^\psi(A)\mu(dx)$. Let $\mathcal{F}_t^{\psi,\mu}$ be the completion of $\mathcal{F}_t^0$ with respect to all P_μ^ψ-null set of $\mathcal{F}^0$, and $\mathcal{F}_t^\psi = \bigcup_{\mu \in \mathcal{P}(E)} \mathcal{F}_t^{\psi,\mu}$. By the same arguments as in [2], Theorem 25.3, page 63, it follows that $\mathcal{F}_t^\psi$ is right-continuous. Define $p^{\psi*}(t) = \sum_{i=1}^{\infty} 1_{\{t \ge T_i^\psi\}} 1_{\{X^\psi_{T_i(\omega)^-} \in \partial^* E\}}$ which is a measure that counts the number of jumps from the boundary. Associated to an admissible control strategy $\psi = (\zeta_1, \zeta_2, \ldots) \in \Psi$ we have the following cost for $\tau > 0$:

$$\mathcal{J}^\psi(\nu, \tau) = E_\nu^\psi\left(\int_0^\tau f^\psi(X_s^\psi)ds + \int_0^\tau r^\psi(X_{s^-}^\psi)dp^{\psi*}(s)\right), \tag{3.2}$$

where for $T_{i-1}^\psi(\omega) \le s < T_i^\psi(\omega)$,

$$f^\psi(X_s^\psi(\omega)) = f^{\zeta_i(\varpi_{i-1}(\omega),.)}(\phi(s - T_{i-1}^\psi(\omega), Z_{i-1}(\omega)))$$

and for $s = T_i^\psi(\omega)$ and $X^\psi_{T_i(\omega)^-}(\omega) \in \partial^* E$, $r^\psi(X^\psi_{s^-}(\omega)) = r^{\zeta_i(\varpi_{i-1}(\omega),.)}(X^\psi_{T_i(\omega)^-}(\omega))$. We consider the following long run average cost problem:

$$\rho(\nu) = \inf_{\psi \in \Psi} \varliminf_{\tau \to \infty} \frac{\mathcal{J}^\psi(\nu, \tau)}{\tau}.$$

B. *PIECEWISE-DETERMINISTIC MARKOV PROCESSES*

As mentioned above, the particular case in which $\psi = (u, u, \ldots)$ for some $u \in \mathcal{U}$ corresponds to the definition of a piecewise-deterministic Markov process, as presented in [2]. We assume that (4) is satisfied for every $\psi = (u, u, \ldots)$ for $u \in \mathcal{U}$. For this case we shall just write u instead of ψ for all the definitions of the previous section. As proved in [2] $\{X_t^u\}$ is a homogeneous strong Markov process, characterized by the following parameters: a) the flow $\phi(.,.)$, b) the jump rate $\lambda^u(.)$, c) the transition measure $Q^u(.;.)$. Associated with the PDP we can define a multivalued operator (see chapter 1 in [18]) $\widetilde{\mathcal{A}}^u$. This multivalued operator $\widetilde{\mathcal{A}}^u$ is a subset of $\mathbb{B}(E) \times \mathbb{B}(E)$ with domain $\mathcal{D}(\widetilde{\mathcal{A}}^u) \subset \mathbb{B}(E)$ defined as follows: $g \in \mathcal{D}(\widetilde{\mathcal{A}}^u)$ if the following conditions are satisfied:

a) $(\exists \mathcal{X} g \in \mathbb{B}(E))$ such that $(\forall x \in E)$, $(\forall t \in [0, t^*(x)))$,

$$g(\phi(t, x)) = g(x) + \int_0^t \mathcal{X} g(\phi(s, x))ds.$$

b) $\lim_{t \uparrow t^*(x)} g(\phi(t, x))$ exists whenever $t^*(x) < \infty$.

The range of $\widetilde{\mathcal{A}}^u$ is given by

$$\mathcal{R}(\widetilde{\mathcal{A}}^u) = \Big\{h \in \mathbb{B}(E) : \text{ there exists } g \in \mathcal{D}(\widetilde{\mathcal{A}}^u) \text{ such that } h = \mathcal{X} g + (Q^u g - g)\lambda^u\Big\}.$$

For $g \in \mathcal{D}(\widetilde{\mathcal{A}}^u)$, $\mathcal{A}^u g$ will denote a function in $\mathcal{R}(\widetilde{\mathcal{A}}^u)$ such that $(g, \mathcal{A}^u g) \in \widetilde{\mathcal{A}}^u$. Moreover, for $g \in \mathcal{D}(\widetilde{\mathcal{A}}^u)$, we define

$$g(z) = \lim_{t \uparrow t^*(x)} g(\phi(t, x)) \text{ for all } z = \lim_{t \uparrow t^*(x)} \phi(t, x) \in \partial^* E.$$

Notice that the limit exists from b) of the definition of $\mathcal{D}(\widetilde{\mathcal{A}}^u)$.

C. *EMBEDDED MARKOV CHAINS*

We define the following stochastic kernels: for all $x \in E \cup \partial^* E$, $v \in V(x)$, and $A \in \mathcal{B}(E)$

$$L^v(x, A) \doteq \int_0^{t^*(x)} e^{-s-\Lambda^v(s,x)} I_A(\phi(s, x))ds \tag{5}$$

$$\widetilde{L}^v(x, A) \doteq \int_0^{t^*(x)} e^{-\Lambda^v(s,x)} I_A(\phi(s, x))ds \tag{6}$$

$$K^v(x, A) \doteq \int_0^{t^*(x)} e^{-s-\Lambda^v(s,x)} \lambda^v(\phi(s, x)) Q^v(\phi(s, x), A)ds + e^{-t^*(x)-\Lambda^v(t^*(x),x)} Q^v(\phi(t^*(x), x), A) \tag{7}$$

$$\widetilde{K}^v(x, A) \doteq \int_0^{t^*(x)} e^{-\Lambda^v(s,x)} \lambda^v(\phi(s, x)) Q^v(\phi(s, x), A)ds + e^{-\Lambda^v(t^*(x),x)} Q^v(\phi(t^*(x), x), A) \tag{8}$$

$$G^v(x, A) \doteq L^v(x, A) + K^v(x, A). \tag{9}$$

$$\widetilde{G}^v(x, A) \doteq \widetilde{L}^v(x, A) + \widetilde{K}^v(x, A). \tag{10}$$

It will be useful in the sequel to define the functions $\mathcal{L}^v(x)$ and $\widetilde{\mathcal{L}}^v(x)$ as follows:

$$\mathcal{L}^v(x) := L^v(x, E), \tag{11}$$

$$\widetilde{\mathcal{L}}^v(x) := \widetilde{L}^v(x, E). \tag{12}$$

We need to define the following kernels acting on the boundary: for all $x \in E \cup \partial^* E$ and $A \in \mathcal{B}(\partial^* E)$,

$$H^v(x, A) \doteq e^{-t^*(x)-\Lambda^v(t^*(x),x)} I_A(\phi(t^*(x), x)), \tag{13}$$

$$\widetilde{H}^v(x, A) \doteq e^{-\Lambda^v(t^*(x),x)} I_A(\phi(t^*(x), x)). \tag{14}$$

Moreover, it is easy to see from the definitions of the kernels L^v, H^v and G^v (see (5), (13), (9)) that for $z \in \partial^* E$ and for any function $g \in \mathbb{B}(E)$ and $\vartheta \in \mathbb{B}(\partial^* E)$ we have

$$L^v g(z) = 0, \ H^v \vartheta(z) = \vartheta(z) \text{ and } \ G^v g(z) = Q^v g(z). \tag{15}$$

Similar results hold for the kernels $\widetilde{L}^v$, $\widetilde{H}^v$ and $\widetilde{G}^v$ (see (6), (14), (10)).

Note that for every $x \in E$, $0 < L^v(x, E) < 1$, and for every $x \in E \cup \partial^* E$, $G^v(x, E) = 1$. Thus $G^v(.,.)$ is a stochastic kernel. For the case in which $v = v(u)$ for some $u \in \mathcal{U}$, we have that G^u is the stochastic kernel of an embedded Markov chain associated with the PDP $\{X_t^u\}$, defined in Section III-B, which we shall denote by $\{Y_n\}$. Similar comments hold for the stochastic kernel $\widetilde{G}^v(.,.)$ and, in this case, the embedded Markov chain will be denoted by $\{\widetilde{Y}_n\}$. The stochastic kernel $\widetilde{G}^v(.,.)$ is associated to the post-jump location of the PDP, so that $\widetilde{Y}_n = Z_n^u$. It was shown in [3] that there is some nice ergodic correspondence between the PDP $\{X_t^u\}$ and the Markov chain $\{Y_n\}$.

IV. MAIN RESULTS

In this section we shall present the main results of the paper. Theorem 4.3 characterizes the optimality equation for the long run average cost problem in terms of integro-differential equations for the continuous-time problem as well as in terms of embedded discrete-time Markov chains associated with the PDP. Before presenting this theorem, we shall need the following auxiliary result:

Proposition 4.1: Suppose that there exists a real number β and $w \in \mathbb{B}^{ac}(E)$ such that

$$w(x) = \inf_{v\in V(x)}\{-\beta\mathcal{L}^v(x)+L^v f^v(x)+H^v r^v(x)+G^v w(x)\} \tag{16}$$

for all $x \in E\cup\partial E^*$. Then the following equation is satisfied for all $x \in E$, and $t \in [0, t^*(x))$:

$$\begin{aligned} w(x) = \inf_{v\in V_{[0,t)}(x)} \bigg\{ \int_0^t e^{-(s+\Lambda^v(s,x))}\big(f^v(\phi(s,x)) \\ + \lambda^v(\phi(s,x))Q^v w(\phi(s,x)) + w(\phi(s,x)) - \beta\big)ds \\ + e^{-(t+\Lambda^v(t,x))}w(\phi(t,x))\bigg\} \end{aligned} \tag{17}$$

Proof: For all $x \in E$, $t \in [0, t^*(x))$, and for any $v \in V(x)$, define $v_t \in V(\phi(t,x))$ and $p_t \in V_{[0,t)}(x)$ as:

$$v_t(s) = v(t+s),\ \ s\in[0,t^*(x)-t), \tag{18}$$

$$p_t(s) = v(s),\ \ s\in\mathcal{I}_{[0,t)}(x). \tag{19}$$

We have that

$$\mathcal{L}^v(x) = \int_0^t e^{-(s+\Lambda^{p_t}(s,x))}ds + e^{-(t+\Lambda^{p_t}(t,x))}\int_0^{t^*(\phi(t,x))} e^{-(s+\Lambda^{v_t}(s,\phi(t,x)))}ds, \tag{20}$$

$$\begin{aligned} L^v f^v(x) = \int_0^t e^{-(s+\Lambda^{p_t}(s,x))}f^{p_t}(\phi(s,x))ds \\ + \int_0^{t^*(\phi(t,x))} e^{-(s+\Lambda^{v_t}(s,\phi(t,x)))}f^{v_t}(\phi(s,\phi(t,x)))ds \\ \times e^{-(t+\Lambda^{p_t}(t,x))}, \end{aligned} \tag{21}$$

$$H^v r^v(x) = e^{-(t+\Lambda^{p_t}(t,\phi(t,x)))}H^{v_t}r^{v_t}(\phi(t,x)), \tag{22}$$

$$\begin{aligned} G^v w(x) = \int_0^t e^{-(s+\Lambda^{p_t}(s,x))}(\lambda^{p_t}(\phi(s,x))Q^{p_t}w(\phi(s,x)) \\ + w(\phi(s,x)))ds + e^{-(t+\Lambda^{p_t}(t,x))}G^{v_t}w((\phi(t,x))). \end{aligned} \tag{23}$$

From equations (20), (21), (22) and (23) we get that for any $v \in V(x)$,

$$\begin{aligned} &-\beta\mathcal{L}^v(x) + L^v f^v(x) + H^v r^v(x) + G^v w(x) = \\ &-\beta\int_0^t e^{-(s+\Lambda^{p_t}(s,x))}ds + \int_0^t e^{-(s+\Lambda^{p_t}(s,x))}f^{p_t}(\phi(s,x))ds \\ &+ \int_0^t e^{-(s+\Lambda^{p_t}(s,x))}(\lambda^{p_t}(\phi(s,x))Q^{p_t}w(\phi(s,x)) \\ &+ w(\phi(s,x)))ds + e^{-(t+\Lambda^{p_t}(t,x))}[-\beta\mathcal{L}^{v_t}(\phi(t,x)) \\ &+ L^{v_t}f^{v_t}(\phi(t,x)) + H^{v_t}r^{v_t}(\phi(t,x)) + G^{v_t}w(\phi(t,x))] \end{aligned} \tag{24}$$

and thus using equation (16), it follows that

$$\begin{aligned} &-\beta\mathcal{L}^v(x) + L^v f^v(x) + H^v r^v(x) + G^v w(x) \geq \\ &-\beta\int_0^t e^{-(s+\Lambda^{p_t}(s,x))}ds + \int_0^t e^{-(s+\Lambda^{p_t}(s,x))}f^{p_t}(\phi(s,x))ds \\ &+ \int_0^t e^{-(s+\Lambda^{p_t}(s,x))}(\lambda^{p_t}(\phi(s,x))Q^{p_t}w(\phi(s,x)) \\ &+ w(\phi(s,x)))ds + e^{-(t+\Lambda^{p_t}(t,x))}w(\phi(t,x)). \end{aligned} \tag{25}$$

Taking the infimum over $v \in V(x)$ in (25), which is equivalent to take the infimum over $p_t \in V_{[0,t)}(x)$ and $v_t \in V_0(\phi(t,x))$, we have from (16) that

$$\begin{aligned} w(x) \geq \inf_{p\in V_{[0,t)}(x)} \bigg\{ -\beta\int_0^t e^{-(s+\Lambda^p(s,x))}ds \\ + \int_0^t e^{-(s+\Lambda^p(s,x))}f^p(\phi(s,x))ds \\ + \int_0^t e^{-(s+\Lambda^p(s,x))}(\lambda^p(\phi(s,x))Q^p w(\phi(s,x)) \\ + w(\phi(s,x)))ds + e^{-(t+\Lambda^p(t,x))}w(\phi(t,x))\bigg\}. \end{aligned}$$

By using equation (16), we have that for all $x \in E$, $t \in [t, t^*(x)))$ and $\epsilon > 0$, there exists $v_\epsilon \in V(\phi(t,x))$ such that $w(\phi(t,x)) + \epsilon \geq -\beta\mathcal{L}^{v_\epsilon}(\phi(t,x)) + L^{v_\epsilon}f^{v_\epsilon}(\phi(t,x)) + H^{v_\epsilon}r^{v_\epsilon}(\phi(t,x)) + G^{v_\epsilon}w(\phi(t,x))$. From (24) and considering $v \in V(x)$ such that $v(t+s) = v_\epsilon(s)$, we get that

$$\begin{aligned} &w(x) \leq -\beta\mathcal{L}^v(x) + L^v f^v(x) + H^v r^v(x) + G^v w(x) \leq \\ &-\beta\int_0^t e^{-(s+\Lambda^{p_t}(s,x))}ds + \int_0^t e^{-(s+\Lambda^{p_t}(s,x))}f^{p_t}(\phi(s,x))ds \\ &+ \int_0^t e^{-(s+\Lambda^{p_t}(s,x))}(\lambda^{p_t}(\phi(s,x))Q^{p_t}w(\phi(s,x)) \\ &+ w(\phi(s,x)))ds + e^{-(t+\Lambda^{p_t}(t,x))}w(\phi(t,x)) + \epsilon. \end{aligned} \tag{26}$$

Taking the infimum over $p_t \in V_{[0,t)}(x)$, we get from (26) that

$$\begin{aligned} &w(x) \leq \inf_{p\in V_{[0,t)}(x)} \bigg\{ \int_0^t e^{-(s+\Lambda^p(s,x))}\big(-\beta + \\ &f^p(\phi(s,x))\big)ds + \int_0^t e^{-(s+\Lambda^p(s,x))}(\lambda^p(\phi(s,x))Q^p w(\phi(s,x)) \\ &+ w(\phi(s,x)))ds + e^{-(t+\Lambda^p(t,x))}w(\phi(t,x))\bigg\} + \epsilon. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, the result follows. ∎

Similarly we have the following result:

Proposition 4.2: Suppose that there exists a real number $\widetilde{\beta}$ and $\widetilde{w} \in \mathbb{B}^{ac}(E)$ such that

$$\widetilde{w}(x) = \inf_{v \in V(x)} \{-\widetilde{\beta}\widetilde{\mathcal{L}}^v(x) + \widetilde{L}^v f^v(x) + \widetilde{H}^v r^v(x) + \widetilde{G}^v w(x)\} \tag{27}$$

for all $x \in E \cup \partial E^*$. Then the following equation is satisfied for every $t \in \mathcal{I}(x)$:

$$\widetilde{w}(x) = \inf_{v \in V_{[0,t)}(x)} \left\{ \int_0^t e^{-\Lambda^v(s,x)}(f^v(\phi(s,x)) + \lambda^v(\phi(s,x))Q^v w(\phi(s,x)) - \widetilde{\beta})ds + e^{-\Lambda^v(t,x)} w(\phi(t,x)) \right\}. \tag{28}$$

Next we present the main result of this paper.

Theorem 4.3: The following assertions are equivalent:

i) There exists a real number ρ and $h \in \mathbb{B}^{ac}(E)$ such that

$$\inf_{a \in U(x)} \{\mathcal{X}h(x) + f^a(x) - \lambda^a(x) \int_E (h(x) - h(y))Q^a(dy;x)\} = \rho \tag{29}$$

for all $x \in E$, and

$$h(z) = \inf_{a \in U(z)} \{r(z,a) + Qh(z,a)\}, \tag{30}$$

for all $z \in \partial E^*$.

ii) There exists a real number β and $w \in \mathbb{B}^{ac}(E)$ such that

$$w(x) = \inf_{v \in V(x)} \{ - \beta\mathcal{L}^v(x) + L^v f^v(x) + H^v r^v(x,v) + G^v w(x)\} \tag{31}$$

for all $x \in E \cup \partial E^*$. For each $x \in E \cup \partial E^*$ the infimum in (31) is reached for some $\widehat{v}_x \in V(x)$.

iii) There exists a real number $\widetilde{\beta}$ and $\widetilde{w} \in \mathbb{B}^{ac}(E)$ such that

$$\widetilde{w}(x) = \inf_{v \in V(x)} \{ - \widetilde{\beta}\widetilde{\mathcal{L}}^v(x) + \widetilde{L}^v f^v(x) + \widetilde{H}^v r^v(x) + \widetilde{G}^v w(x)\} \tag{32}$$

for all $x \in E \cup \partial E^*$. For each $x \in E \cup \partial E^*$ the infimum in (32) is reached for some $\widehat{v}_x \in V(x)$.

Moreover, if i) ii), or iii) holds, there exists $\widehat{u} \in \mathcal{U}$ such that the infimum in (29) and (30) is reached at $\widehat{u}(x) \in U(x)$ and $\widehat{u}(z) \in U(z)$ for each $x \in E$ and $z \in \partial E^*$ respectively, $\widehat{v}_x = v(\widehat{u})$, $\rho(\nu) = \beta = \widetilde{\beta} = \rho$, and the optimal strategy is $\psi = (\widehat{u}, \widehat{u}, \ldots)$.

Proof: Suppose that ii) holds. For $x \in E$ fixed, consider any $v \in V(x)$, and write $u = v(0)$. From (17) we have that

$$e^{-(t+\Lambda^{p_t}(t,x))} w(\phi(t,x)) - w(x) \geq \int_0^t e^{-(s+\Lambda^{p_t}(s,x))}(-f^{p_t}(\phi(s,x)) - \lambda^{p_t}(\phi(s,x))Q^{p_t} w(\phi(s,x)) - w(\phi(s,x)) + \beta)ds. \tag{33}$$

Dividing by t and taking the limit as t goes to 0 we obtain from (33) that

$$\mathcal{X}w(x) - (1 + \lambda^u(x))w(x) \geq - f^u(x) - w(x) - \lambda^u(x)Q^u w(x) + \beta.$$

Since u can be chosen arbitrarily in $U(x)$, we have that

$$\beta \leq \inf_{u \in U(x)} \{\mathcal{X}w(x) + f^u(x) - \lambda^u(x)(w(x) - Q^u w(x))\}. \tag{34}$$

Considering $v = \widehat{v}_x$, we have equality in (33), and thus we have equality in (34), showing (29). Finally we have that for $z \in \partial E^*$, equation (31) reduces to $h(z) = \inf_{a \in U(z)}\{r(z,a) + Qh(z,a)\}$, showing (30).

Suppose that i) holds and let $\widehat{u}$ be a measurable selector as in (1) and (2). Let $p = v(\widehat{u}) \in V(x)$ as in (3), that is, $p(t) = \widehat{u}(\phi(t,x))$. Recall that the multivalued operator $\widetilde{\mathcal{A}}^{\widehat{u}}$, as in sub-section III-B, is such that for any $g \in \mathcal{D}(\widetilde{\mathcal{A}}^{\widehat{u}})$,

$$\mathcal{A}^{\widehat{u}} g(x) = \mathcal{X}g(x) + \lambda^{\widehat{u}}(x)(Q^{\widehat{u}} g(x) - g(x))$$
$$g(z) = \lim_{t \to t^*(x)} g(\phi(t,x)).$$

It is clear that $h \in \mathcal{D}(\widetilde{\mathcal{A}}^{\widehat{u}})$ and that for $x \in E$ and $z \in \partial E^*$, $\mathcal{A}^{\widehat{u}} h(x) - \rho + f^{\widehat{u}}(x) = 0$, and $h(z) = r^{\widehat{u}}(z) + Q^{\widehat{u}} h(z)$. For any $v \in V(x)$, set $\nu(\phi(t,x)) = v(t)$, so that $\nu \in U(\phi(t,x))$ for each $t \in [0, t^*(x))$. It follows that $h \in \mathcal{D}(\widetilde{\mathcal{A}}^\nu)$, and for $x \in E$ and $z \in \partial E^*$,

$$\mathcal{A}^\nu h(x) - \rho + f^\nu(x) \geq \mathcal{A}^{\widehat{u}} h(x) - \rho + f^{\widehat{u}}(x) = 0,$$
$$r^\nu(z) + Q^\nu h(z) - h(z) \geq r^{\widehat{u}}(z) + Q^{\widehat{u}} h(z) - h(z).$$

From Theorem 3.1 in [4], we have that $L^v \mathcal{A}^\nu h(x) = G^v h(x) - h(x) - H^v(Q^\nu - I)h(x)$. Therefore, since $\mathcal{A}^\nu h(x) \geq \rho - f^\nu(x)$ it follows that (notice that L^v is a positive operator)

$$\begin{aligned} L^v \mathcal{A}^\nu h(x) &= G^v h(x) - h(x) - H^v(Q^\nu - I)h(x) \\ &\geq L^v(\rho - f^\nu)(x) \\ &= \rho L^v(1)(x) - L^v(f^\nu)(x). \end{aligned}$$

Notice also that since $Q^\nu h(z) \geq h(z) - r^\nu(z)$, we have that (again, notice that H^v is a positive operator)

$$\begin{aligned} H^v(Q^\nu - I)h(x) &= H^v Q^\nu h(x) - H^v h(x) \\ &\geq H^v h(x) - H^v r^\nu(z) - H^v h(x) \\ &= -H^v r^\nu(z). \end{aligned}$$

Thus,

$$\begin{aligned} G^v h(x) - h(x) &\geq -H^v(Q^\nu - I)h(x) \\ &\quad + \rho L^v(1)(x) - L^v(f^\nu)(x) \\ &\geq -H^v r^\nu(z) + \rho L^v(1)(x) - L^v(f^\nu)(x) \end{aligned}$$

that is,

$$h(x) \leq G^v h(x) + H^v r^\nu(z) + L^v(f^\nu)(x) - \rho L^v(1)(x)$$

with equality for $v = p$. Thus

$$h(x) = \min_{v \in V(x)} \{-\beta\mathcal{L}^v(x) + L^v f^v(x) + H^v r^v(x) + G^v w(x)\}$$

for all $x \in E$. For $z \in \partial E^*$ the result is immediate since $L^v(z,1) = 0$, $L^v(z,f^v) = 0$, $H^v r^v(z) = r^v(z)$ and $G^v h(z) = Q^v h(z)$. The equivalence between i) and iii) can be derived in a similar way, and shall be omitted.

Suppose now that i) holds. For any admissible strategy $\psi = (\zeta_1, \zeta_2, \dots) \in \Psi$, let us define

$$\mathcal{S}^\psi_m(t,x) := E^\psi_x\bigg(\int_0^{T^\psi_m \wedge t}(f^\psi(X^\psi_s) - \rho)ds$$
$$\int_0^{T^\psi_m \wedge t} r^\psi(X^\psi_{s^-})dp^{\psi *}(s) + h\big(X^\psi_{T^\psi_m \wedge t}\big)\bigg).$$

Let us show by induction on m that $\mathcal{S}^\psi_m(t,x) \geq h(x)$ for every $m = 0,1,\dots$, every $\psi = (\zeta_1,\zeta_2,\dots) \in \Psi$, every $x \in E \cup \partial E^*$ and every $t \in \mathcal{I}(x)$. For $m = 0$ the result is immediate, since in this case $\mathcal{S}^\psi_0(t,x) = h(x)$. Suppose the result holds for m. Let us write $\psi' = (\zeta_2(x,T^\psi_1,Z^\psi_1), \zeta_3(x,T^\psi_1,Z^\psi_1),\dots)$. Then writing $v(.) = \zeta_1(x,.)$, we have from the induction hypothesis and (30) that,

$$\begin{aligned}
\mathcal{S}^\psi_{m+1}(t,x) &= E^\psi_x(\int_0^{T^\psi_1 \wedge t}(f^\psi(X^\psi_s) - \rho)ds \\
&+ h(\phi(t,x))1_{\{T^\psi_1 > t\}} + (r^\psi(\phi(t^*(x),x))1_{\{T^\psi_1 = t^*(x)\}} \\
&+ E^{\psi'}_x(\mathcal{S}^\psi_m(t - T^\psi_1, Z^\psi_1)|T^\psi_1, Z^\psi_1))1_{\{T^\psi_1 \leq t\}}) \\
&\geq E^\psi_x(\int_0^{T^\psi_1 \wedge t}(f^\psi(X^\psi_s) - \rho)ds + h(\phi(t,x))1_{\{T^\psi_1 > t\}} \\
&+ (r^\psi(\phi(t^*(x),x))1_{\{T^\psi_1 = t^*(x)\}} + h(Z^\psi_1))1_{\{T^\psi_1 \leq t\}}) \\
&= \int_0^t e^{-\Lambda^v(s,x)}(f^v(\phi(s,x)) \\
&+ \lambda^v(\phi(s,x))Q^v h(\phi(s,x)) - \rho)ds \\
&+ e^{-\Lambda^v(t,x)}h(\phi(t,x))\}. \qquad (35)
\end{aligned}$$

From (28) and (35) we have that $\mathcal{S}^\psi_{m+1}(t,x) \geq h(x)$ with equality for $\psi = (\widehat{u},\dots)$. Taking the limit as m goes to infinity, and recalling that for any admissible strategy we have $T^\psi_m \to \infty$ as $m \to \infty$ almost sure, we have from the bounded convergence theorem that

$$E^\psi_x(\int_0^t(f^\psi(X^\psi_s) - \rho)ds + \int_0^t r^\psi(X^\psi_{s^-})dp^{\psi *}(s)$$
$$+ h((X^\psi_t))) \geq h(x)$$

with equality for $\psi = (\widehat{u},\dots)$. Therefore,

$$\rho \leq \frac{E^\psi_x(\int_0^t f^\psi(X^\psi_s)ds + \int_0^t r^\psi(X_{s-})dp^{\psi *}(s))}{t}$$
$$+ \frac{E^\psi_x(h(X^\psi_t)) - h(x)}{t}$$

and, recalling that h is bounded, we obtain, by taking the liminf as $t \to \infty$, that

$$\rho \leq \varliminf_{t \to \infty} \frac{\mathcal{J}^\psi(x,t)}{t}$$

with equality when $\psi = (\widehat{u},...)$. ■

V. CONCLUSION

In this paper we studied the long run average control problem for piecewise deterministic Markov processes. The control acts on the jump rate λ and the transition measure Q, which specifies the post-jump location. The main result of the paper was presented in Theorem 4.3, which characterizes the optimality equation of the problem in terms of integro-differential equations for the continuous-time problem as well as in terms of embedded discrete-time Markov chains associated with the PDP. One of the embedded discrete-time Markov chain is related to the post-jump location of the PDP. The other one was shown, in [3], to have important ergodic proprieties with respect to the PDP. Further researches are being carried out at the moment to obtain sufficient conditions for the existence of a solution for the optimality equations. We believe that the general theory of the long run average cost problem for discrete-time Markov chains (see, for instance, [17]) in conjunction with the equivalence results obtained in Theorem 4.3 will be important for achieving this goal.

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