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Redesigning a Class of Nonlinear Observers for Certainty-Equivalence Control

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***Abstract*— We study how the class of observers introduced in [1], here referred to as *circle-criterion observers*, can be incorporated in output-feedback control. Due to the absence of a controller-observer separation property for nonlinear systems, the certainty-equivalence implementation of a state-feedback design may lead to severe forms of instability. We show, on the contrary, that the state-dependent convergence properties of circle-criterion observers can prevent such instabilities. Exploiting these convergence properties we develop a modified circle-criterion observer design that guarantees global asymptotic stability for certainty-equivalence controllers.**

I. Introduction

For nonlinear systems the task of incorporating an observer in output-feedback control is hampered by the absence of a controller-observer separation property. Several examples in the literature (see *e.g.* [7]) have shown that a certainty-equivalence implementation of a globally stabilizing state-feedback controller with observer estimates can lead to finite escape time. Local stability has been established in [16], [4]. Larger regions of attraction have been achieved either with restrictions on the growth of nonlinearities [16], [2], or by systematically increasing the observer speed as in high-gain observers [6], [15], [3]. An alternative approach is to depart from certainty-equivalence, and to pursue an inter-dependent design of the controller-observer pair as in the schemes presented in [10], [9], [12], [8].

In this paper we show that, upon a slight modification, the *circle-criterion observer* of [1] achieves global asymptotic stability (GAS) in a certainty-equivalence implementation, under mild assumptions on the state-feedback controller, and without any growth restrictions on controller or observer nonlinearities. As we shall see, this stability property is due to the "state-dependent" convergence of the circle-criterion observer in which the convergence speed depends on the state trajectories of the plant. The onset of instability in a certainty-equivalence implementation stimulates faster convergence of the circle-criterion observer, which results in a recovery of stability achievable with the underlying state-feedback design.

To make this state-dependent convergence property explicit, we show that the nonlinear terms in the observer error system, despite their coupling with the states of the plant, satisfy an $\mathcal{L}_1$-bound that is independent from the behavior of those states. We then make use of a *separation theorem* recently proven in [11], which exploits such $\mathcal{L}_1$ observer terms to relax the conditions on the state-feedback controller. When adapted to our class of systems and observers, the condition on the state-feedback controller is that the closed-loop plant admit a Lyapunov function with a bounded gradient. This condition is often satisfied, possibly upon an appropriate nonlinear scaling of the Lyapunov function.

In Section II we review the basic circle-criterion observer design of [1]. In Section III we augment this observer with a new nonlinear term and study the resulting convergence properties. In Section IV we design this nonlinear term to satisfy the conditions of [11], thus recovering GAS in a certainty-equivalence implementation. In Section V we pursue a "relaxed" certainty-equivalence design, in which only the unmeasured state components are replaced with their observer estimates in the controller. With this relaxed design we remove an assumption employed in Section IV on output-dependent nonlinearities of the plant.

II. Basic Circle-Criterion Observer

We first review the basic design in [1], which is applicable to systems of the form

$$\dot{x} \;=\; Ax + G\gamma(Hx) + \beta(y, u) \qquad (1)$$

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$$y = Cx,$$

where $x \in I\!R^n$ is the state, $y \in I\!R^p$ is the measured output, $u \in I\!R^m$ is the control input, the pair (A, C) is detectable, and the functions $\gamma(\cdot)$ and $\beta(\cdot, \cdot)$ are locally Lipschitz. We assume that the row vector H is linearly independent from the rows of C because, otherwise, the argument Hx of the nonlinearity $\gamma(\cdot)$ can be expressed in terms of the output y and, hence, the term $G\gamma(Hx)$ can be incorporated in $\beta(y, u)$. For simplicity of derivations we also assume that $\gamma(\cdot)$ is a scalar nonlinearity.

For observer design, our main restriction is that $\gamma(\cdot)$ be a nondecreasing function; that is, for all $v, w \in I\!R$, it satisfies

$$(v - w)[\gamma(v) - \gamma(w)] \geq 0\,. \tag{2}$$

Under this restriction, our observer is

$$\dot{\hat{x}} = A\hat{x} + L(C\hat{x} - y) + \beta(y, u) + G\,\gamma(H\hat{x} + K(C\hat{x} - y)) \tag{3}$$

and the design task is to determine the matrices K and L to guarantee observer convergence. From (1) and (3), the dynamics of the state estimation error $e = x - \hat{x}$ are given by

$$\dot{e} = (A + LC)e + G\,[\gamma(v) - \gamma(w)]\,, \tag{4}$$

where

$$v := Hx \quad \text{and} \quad w := H\hat{x} + K(C\hat{x} - y)\,. \tag{5}$$

To design K and L, we view $\gamma(v) - \gamma(w)$ as a function of v and

$$z_1 := v - w = (H + KC)e\,; \tag{6}$$

that is

$$\varphi_1(t, z_1) := \gamma(v) - \gamma(w)\,, \tag{7}$$

where the time-dependence is due to the other argument $v = Hx(t)$, which acts as an exogenous input on the observer error system. Rewriting (4) as

$$\begin{aligned} \dot{e} &= (A + LC)e + G\varphi_1(t, z_1) \qquad (8)\\ z_1 &= (H + KC)e\,, \end{aligned}$$

and noting from (2) that $\varphi_1(t, z_1)$ satisfies the sector property

$$z_1\varphi_1(t, z_1) \geq 0\,, \tag{9}$$

we employ the Circle Criterion and derive a *strict positive realness* (SPR) condition that guarantees exponential decay for the observer error $e(t)$:

Proposition 1: **([1])** Consider the plant (1) and let $[0, t_f)$ denote the maximal interval of definition for the solution $x(t)$. If there exists a matrix $P = P^T > 0$, and a constant $\epsilon > 0$, such that

$$(A + LC)^T P + P(A + LC) + \epsilon I \leq 0 \tag{10}$$

$$PG + (H + KC)^T = 0 \tag{11}$$

then the estimation error $e = x - \hat{x}$ of the observer (3) satisfies, for all $t \in [0, t_f)$,

$$|e(t)| \leq \kappa |e(0)| \exp(-\nu t)\,, \tag{12}$$

where $\kappa = \sqrt{\frac{\lambda_{max}(P)}{\lambda_{min}(P)}}$, $\nu = \frac{\epsilon}{2\lambda_{max}(P)}$. □

III. An Augmented Design of Circle-Criterion Observers

We now exploit an additional design freedom in (3) and introduce the augmented observer

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + L(C\hat{x} - y) + G\,\gamma(H\hat{x} + K(C\hat{x} - y))\\ &\quad -\vartheta(y, \hat{x}) + \beta(y, u) \qquad (13) \end{aligned}$$

in which the new term is

$$\begin{aligned} \vartheta(\hat{x}, y) &= P^{-1}C^T(Q - S)^T \times \qquad (14)\\ &\quad [\phi(R\hat{x} + Q(C\hat{x} - y)) - \phi(R\hat{x} + S(C\hat{x} - y))], \end{aligned}$$

where $\phi(\cdot)$ is an arbitrary nondecreasing function, P is as in Proposition 1, and Q, R and S are row vectors of compatible dimensions. As we prove in Lemma 1 below, the addition of a $\vartheta(y, \hat{x})$ of this form increases the dissipation rate of the observer Lyapunov function $V_{\text{obs}} = \frac{1}{2}e^T Pe$ and, thus, preserves the exponential convergence estimate (12) achieved by the original design. It further strengthens the convergence in the sense of Lemma 2 below, which will be used in the next section to ensure stability for certainty-equivalence control.

Lemma 1: If the conditions of Proposition 1 hold, then for any nondecreasing function $\phi(\cdot)$, and row vectors Q, R, S, the observer (13)-(14) guarantees the convergence estimate (12). □

Proof: We first define

$$\varphi_2(t, z_2) = \phi(R\hat{x} + Q(C\hat{x} - y)) - \phi(R\hat{x} + S(C\hat{x} - y)) \tag{15}$$

where z_2 is the difference of the arguments of $\phi(\cdot)$:

$$\begin{aligned} z_2 &= (R\hat{x} + Q(C\hat{x} - y)) - (R\hat{x} + S(C\hat{x} - y))\\ &= (S - Q)Ce, \qquad (16) \end{aligned}$$

and the time-dependence is due to variables other than z_2, which affect $\varphi_2(t, z_2)$. As in the arguments (5)-(9) for $\varphi_1(t, z_1)$, the nondecreasing nonlinearity $\phi(\cdot)$ in (15) guarantees the sector property:

$$z_2\varphi_2(t, z_2) \geq 0. \tag{17}$$

With $\varphi_2(t,z_2)$ defined as in (15), the dynamics of the observer error $e = x - \hat{x}$ are

$$\begin{aligned} \dot{e} &= (A+LC)e + G\varphi_1(t,z_1) \\ &\quad +P^{-1}C^T(Q-S)^T\varphi_2(t,z_2) \end{aligned} \tag{18}$$

$$z_1 = (H+KC)e \tag{19}$$

$$z_2 = (S-Q)Ce \tag{20}$$

and, thus, the observer Lyapunov function $V_{\text{obs}} = \frac{1}{2}e^TPe$ satisfies

$$\begin{aligned} \dot{V}_{\text{obs}} &= \frac{1}{2}e^T[(A+LC)^TP + P(A+LC)]e \\ &\quad +e^TPG\varphi_1(t,z_1) + e^TC^T(Q-S)^T\varphi_2(t,z_2). \end{aligned} \tag{21}$$

Substituting (10)-(11) in (21) we obtain

$$\begin{aligned} \dot{V}_{\text{obs}} &\leq -\frac{\epsilon}{2}|e|^2 - e^T(H+KC)^T\varphi_1(t,z_1) \\ &\quad +e^TC^T(Q-S)^T\varphi_2(t,z_2), \end{aligned} \tag{22}$$

which means, from (19)-(20) and from the sector properties (9) and (17),

$$\dot{V}_{\text{obs}} \leq -\frac{\epsilon}{2}|e|^2 - z_1\varphi_1(t,z_1) - z_2\varphi_2(t,z_2) \leq -\frac{\epsilon}{2}|e|^2. \tag{23}$$

The estimate (12) then follows from standard Lyapunov arguments. □

We now present a further property of the observer (13)-(14), which will be instrumental in proving the results of the next section:

Lemma 2: If the conditions of Lemma 1 hold for the observer (13)-(14), then $\varphi_1(t,z_1)$ and $\varphi_2(t,z_2)$ defined in (7) and (15), respectively, satisfy

$$\lim_{T\to t_f}\int_0^T |\varphi_i(t,z_i)|dt \leq h|e(0)| \quad i=1,2, \tag{24}$$

for some constant $h > 0$ which is independent of t_f. □

Inequality (24) shows that the observer renders the nonlinear terms $\varphi_1(t,z_1)$ and $\varphi_2(t,z_2)$ $\mathcal{L}_1$ on the interval $[0,t_f)$. Although $\varphi_1(t,z_1)$ and $\varphi_2(t,z_2)$ depend on the plant trajectories $x(t)$ by their definitions, the $\mathcal{L}_1$-property (24) is independent of the behavior of $x(t)$. This means that the convergence properties of $z_1(t)$ and $z_2(t)$ vary according to $x(t)$, and because z_1 and z_2 are functions of the observer error e as in (19)-(20), this argument shows that the observer tunes its convergence speed according to the growth of $x(t)$. This interplay between the observer and the plant is the key to the certainty-equivalence property achieved in the next section.

Proof of Lemma 2: Using (19)-(20), and substituting $H + KC = -G^TP$ from (11), the dynamic equations for z_1 and z_2 are

$$\begin{aligned} \dot{z}_1 &= -G^TPG\varphi_1(t,z_1) \\ &\quad -G^TC^T(Q-S)^T\varphi_2(t,z_2) + d_1 \end{aligned} \tag{25}$$

$$\begin{aligned} \dot{z}_2 &= (S-Q)CG\varphi_1(t,z_1) \\ &\quad -(Q-S)CP^{-1}C^T(Q-S)^T\varphi_2(t,z_2) + d_2 \end{aligned} \tag{26}$$

where

$$d_1 := -G^TP(A+LC)e \tag{27}$$

$$d_2 := (S-Q)C(A+LC)e. \tag{28}$$

Using

$$\text{sgn}(z_i)\phi_i(t,z_i) = |\phi_i(t,z_i)| \quad i=1,2, \tag{29}$$

which follows from the sector properties (9) and (17), we conclude from (25)-(26) that, for almost all $t \geq 0$,

$$\begin{aligned} \frac{d}{dt}|z_1| &\leq -G^TPG|\varphi_1(t,z_1)| \\ &\quad +|(S-Q)CG|\,|\varphi_2(t,z_1)| + |d_1| \end{aligned} \tag{30}$$

$$\begin{aligned} \frac{d}{dt}|z_2| &\leq |(S-Q)CG|\,|\varphi_1(t,z_1)| \\ &\quad -(Q-S)CP^{-1}C^T(Q-S)^T|\varphi_2(t,z_2)| + |d_2|. \end{aligned} \tag{31}$$

We next claim we can find a constant $\lambda > 0$ such that

$$k_1 := \lambda G^TPG - |(S-Q)CG| > 0 \tag{32}$$

$$\begin{aligned} k_2 &:= (Q-S)CP^{-1}C^T(Q-S)^T \\ &\quad -\lambda|(S-Q)CG| > 0. \end{aligned} \tag{33}$$

For now we take this to be true, and note from (30)-(31) that

$$\begin{aligned} \frac{d}{dt}(\lambda|z_1| + |z_2|) &\leq -k_1|\varphi_1(t,z_1)| - k_2|\varphi_2(t,z_2)| \\ &\quad +(\lambda|d_1| + |d_2|). \end{aligned} \tag{34}$$

Integrating both sides of (34), we obtain

$$\begin{aligned} &k_1\int_0^T |\varphi_1(t,z_1)|dt + k_2\int_0^T |\varphi_2(t,z_2)|dt \\ &\qquad \leq \lambda|z_1(0)| + |z_2(0)| + \int_0^T (\lambda|d_1| + |d_2|)dt \\ &\qquad \leq \bar{h}|e(0)|, \end{aligned} \tag{35}$$

in which the second inequality holds for some constant $\bar{h} > 0$, because z_1, z_2, d_1 and d_2 all depend linearly on e, and because $e(t)$ satisfies the exponential convergence estimate (12) from Lemma 1. We thus conclude that (24) holds with $h = \bar{h}/\min\{k_1,k_2\}$.

To complete the proof we now show that a constant $\lambda > 0$ as in (32)-(33) indeed exists. To this end we rewrite (32)-(33) as

$$\lambda|w_1|^2 - |w_1^T w_2| \;>\; 0 \tag{36}$$
$$|w_2|^2 - \lambda|w_1^T w_2| \;>\; 0, \tag{37}$$

where

$$w_1 := P^{1/2}G \quad \text{and} \quad w_2 := P^{-1/2}C^T(S-Q)^T. \tag{38}$$

Because w_1 and w_2 are non-zero (we only consider the case $(S-Q)C \neq 0$ because, otherwise, $\varphi_2(t, z_2) \equiv 0$, and (24) already holds), and are not co-linear with each other (this follows from $PG = -(H + KC)^T$ and the assumption that H is linearly independent from the rows of C), we conclude from the Schwartz Inequality that

$$|w_1^T w_2| < |w_1|\,|w_2|, \tag{39}$$

which means that we can choose λ such that

$$\frac{|w_1^T w_2|}{|w_1|^2} < \lambda < \frac{|w_2|^2}{|w_1^T w_2|} \tag{40}$$

and satisfy (36)-(37). □

IV. Recovery of Stability in Certainty-Equivalence Implementation

In this section we present a design for $\vartheta(\hat{x}, y)$ in the augmented observer (13) that guarantees GAS for certainty-equivalence controllers. For the underlying state-feedback controller we employ an assumption adapted from [11]:

Assumption 1: There exists a locally Lipschitz state-feedback control law $u = \alpha(x)$, a continuously differentiable, radially unbounded, and positive definite Lyapunov function $V(x)$, and a continuously differentiable and class-$\mathcal{K}_\infty$ function $\mathcal{L}(\cdot)$, such that the origin $x = 0$ of the closed-loop system

$$\dot{x} = Ax + G\gamma(Hx) + \beta(Cx, \alpha(x)) =: f(x) \tag{41}$$

is globally asymptotically stable, and

$$\frac{\partial V}{\partial x} f(x) \leq 0 \tag{42}$$

$$|\mathcal{L}'(V(x))| \left|\frac{\partial V}{\partial x}\right| \leq 1 \quad \forall x \in I\!R^n. \tag{43}$$

□

Inequality (43) means that, upon scaling with $\mathcal{L}(\cdot)$, the new Lyapunov function $\mathcal{L}(V(x))$ has a globally bounded gradient. Although this assumption is not restrictive, there exist systems for which such a Lyapunov function does not exist (see the examples presented in [13], [14]). Before presenting our main result, we further restrict the function $\beta(y, u)$ in (1) as in Assumption 2 below. We will, however, eliminate this assumption with a relaxed certainty-equivalence design in the next section.

Assumption 2: The function $\beta(y, u)$ in (1) is such that, for all $y, \hat{y} \in I\!R^p$, and $u \in I\!R^m$,

$$|\beta(y, u) - \beta(\hat{y}, u)| \leq \omega(|y - \hat{y}|) \tag{44}$$

for some class-$\mathcal{K}$ function $\omega(\cdot)$ which is differentiable at zero. □

We now present our observer which ensures global asymptotic stability with the certainty-equivalence controller $u = \alpha(\hat{x})$:

Theorem 1: Suppose that system (1) satisfies Assumption 2, there exists a state-feedback controller $u = \alpha(x)$ as in Assumption 1, and there exist matrices P, K and L satisfying Proposition 1. Then, the certainty-equivalence controller $u = \alpha(\hat{x})$, with $\hat{x}$ obtained from the observer (13) with

$$\vartheta(\hat{x}, y) = \theta P^{-1}C^T K^T[\gamma(H\hat{x} + K(C\hat{x} - y)) - \gamma(H\hat{x})] \tag{45}$$

$\theta > 0$, renders the origin $(x, \hat{x}) = 0$ of the closed-loop system (1),(13) GAS. □

Proof: We first note that (45) conforms to the structure (14), with $\phi(\cdot) = \theta\gamma(\cdot)$, $R = H$, $Q = K$, and $S = 0$, which, from Lemma 2, means that

$$\lim_{T \to t_f} \int_0^T \Big|\gamma(H\hat{x} + K(C\hat{x} - y)) - \gamma(H\hat{x})\Big|\, dt \leq \frac{h}{\theta}|e(0)|. \tag{46}$$

Next, adding and subtracting $G\gamma(H\hat{x})$ and $\beta(\hat{y}, u)$, we rewrite the observer equation (13),(45) as

$$\dot{\hat{x}} = A\hat{x} + G\gamma(H\hat{x}) + \beta(C\hat{x}, \alpha(\hat{x})) + d_3 \tag{47}$$

$$d_3 := -LCe + [\beta(y, \alpha(\hat{x})) - \beta(\hat{y}, \alpha(\hat{x}))] + \tag{48}$$
$$(G - \theta P^{-1}C^T K^T)[\gamma(H\hat{x} + K(C\hat{x} - y)) - \gamma(H\hat{x})].$$

Using inequality (46), Assumption 2, and the exponential decay property of the observer error e as in Proposition 1, it is not difficult to show that d_3 satisfies

$$\lim_{T \to t_f} \int_0^T |d_3(t)|\, dt \leq \Omega(|e(0)|) \tag{49}$$

for some class-$\mathcal{K}$ function $\Omega(\cdot)$ which is independent of t_f. Then, from Assumption 1, the derivative of the Lyapunov function $\mathcal{L}(V(\hat{x}))$ along the trajectories of (47) is

$$\frac{d}{dt}\mathcal{L}(V(\hat{x})) \leq |d_3| \tag{50}$$

and, integration from 0 to T and substitution of (49) yield

$$\lim_{T\to t_f} \mathcal{L}(V(\hat{x}(T))) \leq \mathcal{L}(V(\hat{x}(0))) + \Omega(|e(0)|). \quad (51)$$

This, combined with (12), implies that $t_f = \infty$ (that is, $x = \hat{x} + e$ does not exhibit finite escape time) and that the origin $(\hat{x}, e) = 0$ is stable. Finally, because $e \to 0$ as $t \to \infty$, the dynamics on the ω-limit set of every trajectory coincides with (41) which, by Assumption 1, has the GAS equilibrium $x = 0$. In view of the Invariance Principle [5, Lemma 4.1] we conclude that the equilibrium $(x, \hat{x}) = 0$ of the observer-based control system is globally attractive which, taken together with stability, proves GAS. □

Example 1: For the system

$$\begin{aligned} \dot{x}_1 &= x_2 & (52)\\ \dot{x}_2 &= x_2 - x_2^5 + u & (53)\\ y &= x_1, & (54) \end{aligned}$$

a solution to the conditions in Proposition 1 is

$$P = \begin{bmatrix} 5 & -1.382 \\ -1.382 & 1 \end{bmatrix}, \quad L = \begin{bmatrix} -4 \\ -5 \end{bmatrix},$$

K=-1.382, and results in the observer

$$\begin{aligned} \dot{\hat{x}}_1 &= -4(\hat{x}_1 - y) + \hat{x}_2 \\ \dot{\hat{x}}_2 &= -5(\hat{x}_1 - y) + \hat{x}_2 + u \\ &\quad -(\hat{x}_2 - 1.382(\hat{x}_1 - y))^5. \quad (55) \end{aligned}$$

However, when employed in a certainty-equivalence implementation in [11], this observer was shown to result in finite escape time. We now apply the redesign (45), which yields

$$\vartheta(\hat{x}, y) = \theta \underbrace{\begin{bmatrix} -0.4472 \\ -0.6181 \end{bmatrix}}_{P^{-1}C^T K} [(\hat{x}_2 - 1.382(\hat{x}_1 - y))^5 - \hat{x}_2^5], \quad (56)$$

and note from (53) that $\beta(y, u) = u$, which satisfies Assumption 2. This means from Theorem 1 that, with the augmented observer

$$\begin{aligned} \dot{\hat{x}}_1 &= -4(\hat{x}_1 - y) + \hat{x}_2 & (57)\\ &\quad +0.4472\,\theta\,[(\hat{x}_2 - 1.382(\hat{x}_1 - y))^5 - \hat{x}_2^5] \\ \dot{\hat{x}}_2 &= -5(\hat{x}_1 - y) + \hat{x}_2 - 0.6181\theta\,\hat{x}_2^5 + u & (58)\\ &\quad -(1 - 0.6181\,\theta)\,(\hat{x}_2 - 1.382(\hat{x}_1 - y))^5, \end{aligned}$$

where $\theta > 0$ is a design parameter, the certainty-equivalence implementation $u = \alpha(\hat{x})$ of any controller satisfying Assumption 1 achieves GAS of the equilibrium $(x, \hat{x}) = 0$. □

V. A Relaxed Certainty-Equivalence Design

Assumption 2 restricts the growth of output-dependent nonlinearities in the system (1). In this section we remove this assumption with an alternative certainty-equivalence implementation in which the measured components of the state vector are directly incorporated in the control law, and not replaced with their observer estimates. For this implementation we assume, without loss of generality, that the matrices C and H in (1) have the form

$$C = [\, I_{p\times p} \;\; 0_{p\times(n-p)} \,], \qquad H = [\, 0_{1\times p} \;\; H_2 \,]; \quad (59)$$

that is, the state vector is

$$x = \begin{bmatrix} y \\ x_2 \end{bmatrix}, \quad (60)$$

where the output y comprises the first p entries, and the nonlinearity $\gamma(Hx)$ depends only on x_2. Because H is linearly independent from the rows of C, a linear state transformation always exists to bring (1) to this form. We then implement the "relaxed" certainty-equivalence controller $u = \alpha(y, \hat{x}_2)$ which employs the true value of the output y rather than its estimate $\hat{y}$:

Theorem 2: Consider the system (1),(59), and suppose there exists a state-feedback controller $u = \alpha(y, x_2)$ satisfying Assumption 1, and there exist matrices P, K and L satisfying Proposition 1. Then, the certainty-equivalence controller $u = \alpha(y, \hat{x}_2)$ implemented with the redesigned observer (13),(45) renders the origin of the closed-loop system (1),(13) GAS. □

Proof: Using (59) we rewrite (1) as

$$\begin{aligned} \dot{y} &= A_1 x_2 + G_1\gamma(H_2 x_2) + \beta_1(y, u) & (61)\\ \dot{x}_2 &= A_2 x_2 + G_2\gamma(H_2 x_2) + \beta_2(y, u), & (62) \end{aligned}$$

where all linear terms in y are now incorporated in $\beta_1(y, u)$ and $\beta_2(y, u)$. Likewise, the $\hat{x}_2$-component of the redesigned observer is

$$\begin{aligned} \dot{\hat{x}}_2 = A_2\hat{x}_2 + G_2\gamma(H_2\hat{x}_2 + K(\hat{y} - y)) + \beta_2(y, u) \\ +L_2(\hat{y} - y) - \Gamma[\gamma(H_2\hat{x}_2 + K(\hat{y} - y)) - \gamma(H_2\hat{x}_2)], \end{aligned} \quad (63)$$

where

$$\Gamma := [0_{(n-p)\times p} \; I_{(n-p)\times(n-p)}]\theta P^{-1} K^T. \quad (64)$$

Adding and subtracting $A_1\hat{x}_2$ and $G_1\gamma(H_2\hat{x}_2)$ in (61), $G_2\gamma(H_2\hat{x}_2)$ in (63), and substituting $u = \alpha(y, \hat{x}_2)$, we get

$$\dot{y} = A_1\hat{x}_2 + G_1\gamma(H_2\hat{x}_2) + \beta_1(y, \alpha(y, \hat{x}_2)) + d_4 \quad (65)$$

$$\dot{\hat{x}}_2 = A_2\hat{x}_2 + G_2\gamma(H_2\hat{x}_2) + \beta_2(y, \alpha(y, \hat{x}_2)) + d_5 \quad (66)$$

where

$$d_4 := A_1(x_2 - \hat{x}_2) + G_1[\gamma(H_2x_2) - \gamma(H_2\hat{x}_2)] \quad (67)$$

$$d_5 := (G_2 - \Gamma)[\gamma(H_2\hat{x}_2 + K(\hat{y} - y)) - \gamma(H_2\hat{x}_2)] + L_2(\hat{y} - y). \quad (68)$$

Because, by Lemma 2,

$$\varphi_1 = \gamma(H_2x_2) - \gamma(H_2\hat{x}_2 + K(\hat{y} - y)) \quad (69)$$

$$\varphi_2 = \gamma(H_2\hat{x}_2 + K(\hat{y} - y)) - \gamma(H_2\hat{x}_2) \quad (70)$$

are both $\mathcal{L}_1$ and, because d_4 above contains $[\gamma(H_2x_2) - \gamma(H_2\hat{x}_2)] = \varphi_1 + \varphi_2$, and d_5 contains φ_2, we conclude that d_4 and d_5 are $\mathcal{L}_1$ as in (49). The proof of GAS then follows from the arguments in Theorem 1. ☐

We finally note that, if $K = 0$ in the original observer (3), then $\vartheta(\hat{x}, y) \equiv 0$ in (45) and, hence, Theorems 1 and 2 imply GAS without a redesign of this observer. We wish to emphasize, hovewer, that feasibility conditions for Proposition 1 with $K = 0$ are restrictive (see [2]).

Corollary 1: If the conditions of Proposition 1 hold with $K = 0$, and if $u = \alpha(y, x_2)$ satisfies Assumption 1, then the controller $u = \alpha(y, \hat{x}_2)$ with the observer (3) achieves GAS. If, further, (44) holds, then the controller $u = \alpha(\hat{y}, \hat{x}_2)$ also achieves GAS.

VI. Conclusions

Output-feedback control of nonlinear systems continues to be a challenging research area due to the shortage of constructive procedures for designing observers, and for incorporating them in controllers. This paper first broadened the class of observers in [1] by augmenting them with an additional nonlinear term and, next, exploited the new design flexibility to achieve stability of certainty-equivalence controllers. It would be of interest to further study the augmented class of observers in Section III to achieve enhanced performance and robustness properties. Another research direction would be to reveal state-dependent convergence properties in other nonlinear observer design methods, and to pursue output-feedback designs within the framework of [11].

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