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Finite-Time Global Stabilization by Means of Time-Varying Distributed Delay Feedback

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Abstract— **The paper contains certain results concerning the finite-time global stabilization for triangular control systems described by retarded functional differential equations by means of time-varying distributed delay feedback. These results enable us to present solutions to feedback stabilization problems for systems with delayed input. The results are obtained by using the backstepping technique.**

I. INTRODUCTION

It is generally known that for finite-dimensional continuous-time control systems with locally Lipschitz dynamics (e.g. $\dot{x} = f(x,u)$, where f is locally Lipschitz with respect to $(x,u) \in \Re^n \times \Re^m$), finite-time global stabilization cannot be achieved by means of a locally Lipschitz feedback law. However, it has been shown that finite-time global stabilization is possible by means of continuous (see [2,3,4,6,8] as well as the reported results in [1]) or discontinuous feedback laws (see [14]).

Recently, the option of using feedback with delays has been considered for the stabilization of continuous-time systems in various problems. Of course, the closed-loop system will be described by a system of (possibly time-varying) Retarded Functional Differential Equations (RFDEs), which is an infinite dimensional system. For example, in [20] the authors provide strategies for the construction of control laws of the form $u(t) = k(t, x(t), x(lT))$ for $t \in [lT, (l+1)T)$, where l is a non-negative integer and $T > 0$ denotes the updating time-period of the control. Notice that this type of feedback is a time-varying feedback with delays of the form $u(t) = k\left(t, x(t), x\left(\left[\frac{t}{T}\right]T\right)\right)$, where $[t/T]$ denotes the integer part of t/T, which is time-varying even if k is independent of time, i.e., $k(t,x,\xi) = k(x,\xi)$. The same comments apply for the synchronous controller switching strategies proposed in [22]. The possibility of switching control laws using distributed delays was recently exploited in [17]. Observers that make use of past values of the state estimate and guarantee convergence in finite-time were considered in [5,19]. The ability of output discrete delay feedback to stabilize minimum phase linear systems was studied in [9,21]. Recently, there has been an increasing interest for the feedback stabilization problems of systems with delayed input (see [15,18]) as well as the application of the backstepping technique for the stabilization of nonlinear time-delay systems (see [10,16]).

In the present paper it is shown that finite-time global stabilization can be achieved by time-varying locally Lispchitz distributed delay feedback. It is known that systems described by time-varying RFDEs admit solutions that converge to the equilibrium point in finite time (e.g. Property 5.1 in Chapter 3 in [7]). Using the backstepping technique (see [13,24]), the problem of finite-time global stabilization for nonlinear triangular systems is studied and solved. Moreover, the approach proposed in this paper is not limited to triangular finite-dimensional continuous-time control systems but can be directly applied to nonlinear triangular systems with delays. The case of delayed inputs is also considered. Among other cases, our results can be applied to:

- the case of triangular control systems with no delays

$$\begin{aligned} &\dot{x}_i(t) = f_i(t, x_1(t),...,x_i(t)) + x_{i+1}(t) \quad i = 1,...,n-1 \\ &\dot{x}_n(t) = f_n(t, x(t)) + u(t) \\ &x(t) := (x_1(t),...,x_n(t)) \in \Re^n \ , u(t) \in \Re \ , t \geq 0 \end{aligned} \tag{1.1}$$

- the case of a chain of delayed integrators with no limitation on the size of the delays

$$\begin{aligned} &\dot{x}_i(t) = x_{i+1}(t - \tau_i) \quad , \quad i = 1,...,n-1 \\ &\dot{x}_n(t) = u(t - \tau_n) \\ &x(t) = (x_1(t),...,x_n(t)) \in \Re^n \ , u(t) \in \Re \end{aligned} \tag{1.2}$$

where $\tau_i \geq 0$ $i = 1,...,n$ are the delays

- the case of triangular control systems with delayed drift terms:

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$$\dot{x}_i(t) = f_i(t, x_1(t-\tau_{i,1}),..., x_i(t-\tau_{i,i})) + x_{i+1}(t) \quad i = 1,...,n-1$$
$$\dot{x}_n(t) = f_n(t, x_1(t-\tau_{n,1}),..., x_n(t-\tau_{n,n})) + u(t)$$
$$x(t) := (x_1(t),..., x_n(t)) \in \Re^n \ , u(t) \in \Re \ , t \geq 0 \tag{1.3}$$

where $\min_{i=1,...,n-1} \min_{j=1,...,i} \tau_{i,j} > 0$

Notations: Throughout this paper we adopt the following notations:

* For a vector $x \in \Re^n$ we denote by $|x|$ its usual Euclidean norm. For $x \in C^0([-r,0];\Re^n)$ we define $\|x\|_r := \max_{\theta\in[-r,0]} |x(\theta)|$.
* By $C^j(A)$ ($C^j(A;\Omega)$), where $j \geq 0$ is a non-negative integer, we denote the class of functions (taking values in Ω) that have continuous derivatives of order j on A.
* We denote by K^+ the class of positive C^∞ functions defined on $\Re^+$. We say that an increasing and continuous function $\rho: \Re^+ \to \Re^+$ with $\rho(0) = 0$ is of class K_∞ if $\lim_{s\to+\infty} \rho(s) = +\infty$.
* Z^+ denotes the set of positive integers and $\Re^+$ the set of non-negative real numbers.
* A continuous mapping $f: I \times C^0([-r,0];\Re^n) \times U \to \Re^k$, where $\Re^+ \subseteq I \subseteq \Re$, $U \subseteq \Re^m$, is said to be completely locally Lipschitz with respect to $(x,u) \in C^0([-r,0];\Re^n) \times U$ if for every bounded set $S \subset I \times C^0([-r,0];\Re^n) \times U$ there exists $L \geq 0$ such that $|f(t,x,u) - f(t,y,v)| \leq L\|x-y\|_r + L|u-v|$ for all $(t,x,u) \in S$, $(t,y,v) \in S$.
* Let $x:[a-r,b) \to \Re^n$ with $b > a \geq 0$ and $r \geq 0$. By $T_r(t)x$ we denote the "history" of x from $t-r$ to t, i.e., $T_r(t)x := \{x(t+\theta) \,;\, \theta \in [-r,0]\}$, for $t \in [a,b)$.

Notice that a mapping $f: I \times C^0([-r,0];\Re^n) \times U \to \Re^k$, where $\Re^+ \subseteq I \subseteq \Re$, $U \subseteq \Re^m$, which is completely locally Lipschitz with respect to $(x,u) \in C^0([-r,0];\Re^n) \times U$, is also defined on $I \times C^0([-r-\sigma,0];\Re^n) \times U$, for every $\sigma \geq 0$, and is completely locally Lipschitz with respect to $(x,u) \in C^0([-r-\sigma,0];\Re^n) \times U$, for every $\sigma \geq 0$.

II. DEFINITIONS AND TECHNICAL RESULTS

Consider the following control system described by Retarded Functional Differential Equations (RFDEs):

$$\begin{aligned} &\dot{x}(t) = f(t, T_r(t)x, u(t-\tau(t))) \\ &x(t) \in \Re^n \ , u(t) \in U \ , t \geq 0 \end{aligned} \tag{2.1}$$

where $0 \in U \subseteq \Re^m$, $f: \Re^+ \times C^0([-r,0];\Re^n) \times U \to \Re^n$ is completely locally Lipschitz with respect to $(x,u) \in C^0([-r,0];\Re^n) \times U$ with $f(t,0,0) = 0$ for all $t \geq 0$ and $\tau: \Re^+ \to \Re^+$ is a bounded continuous function. We denote by $x(t) = x(t,t_0,x_0,u) \in \Re^n$ the solution of (2.1) initiated from $t_0 \geq 0$ with initial condition $T_r(t_0)x = x_0 \in C^0([-r,0];\Re^n)$ and corresponding to $u \in C^0(\Re;U)$. By virtue of Theorem 3.2 in [7], for every $(t_0,x_0,u) \in \Re^+ \times C^0([-r,0];\Re^n) \times C^0(\Re;U)$ there exists $t_{\max}(t_0,x_0,u) > t_0$ (called the maximal existence time) such that the solution $x(t) = x(t,t_0,x_0,u) \in \Re^n$ of (2.1) initiated from $t_0 \geq 0$ with initial condition $T_r(t_0)x = x_0 \in C^0([-r,0];\Re^n)$ and corresponding to $u \in C^0(\Re;U)$, is defined on $[t_0 - r, t_{\max})$, is continuous on $[t_0 - r, t_{\max})$ and continuously differentiable on $[t_0, t_{\max})$ and cannot be further continued, i.e., if $t_{\max} < +\infty$ then $\limsup_{t\to t_{\max}^-} |x(t)| = +\infty$. When $r = 0$ we identify the space $C^0([-r,0];\Re^n)$ with the finite-dimensional space $\Re^n$. Consequently, all the following definitions and results hold also for finite-dimensional continuous-time systems.

In order to study the properties of control system (2.1), we must clarify the differentiability properties of functionals along the solutions of (2.1).

Definition 2.1 *Let a functional* $\varphi: I \times C^0([-r,0];\Re^n) \to \Re$ *where* $\Re^+ \subseteq I \subseteq \Re$, $\varphi(t,x)$ *being completely locally Lipschitz with respect to* $x \in C^0([-r,0];\Re^n)$, *with* $\varphi(t,0) = 0$ *for all* $t \in I$. *We say that:*

* φ *is* ***ultimately differentiable*** *along the solutions of (2.1) with time constant* $T \geq 0$, *if there exists a constant* $T \geq 0$ *and a functional* $D\varphi: I \times C^0([-r,0];\Re^n) \times U \to \Re$ *with* $D\varphi(t,x,u)$ *being completely locally Lipschitz with respect to* $(x,u) \in C^0([-r,0];\Re^n) \times U$ *and* $D\varphi(t,0,0) = 0$ *for all* $t \in I$, *such that for every*

$(t_0, x_0, u) \in \Re^+ \times C^0([-r,0];\Re^n) \times C^0(\Re;U)$ *for which* $t_0 + T < t_{\max}(t_0, x_0, u)$*, the mapping* $t \to \varphi(t, T_r(t)x)$ *is of class* C^1 *on* $[t_0 + T, t_{\max})$ *and it holds that:*

$$\frac{d}{dt}\varphi(t, T_r(t)x) = D\varphi(t, T_r(t)x, u(t-\tau(t)))\,,\ \forall t \in [t_0 + T, t_{\max})$$

The functional $D\varphi : I \times C^0\left([-r,0];\Re^n\right) \times U \to \Re$ *is called the derivative of* φ *along the solutions of (2.1).*

* φ *is* ***differentiable*** *along the solutions of (2.1), if* φ *is ultimately differentiable along the solutions of (2.1) with time constant* $T = 0$.

Example 2.2 Consider the following planar system:

$$\begin{aligned} &\dot{x}_1(t) = x_2(t) \quad , \quad \dot{x}_2(t) = u(t) \\ &(x_1(t), x_2(t)) \in \Re^2 \ , u(t) \in \Re \end{aligned} \tag{2.2}$$

Let $r > 0$ and consider the functional $\varphi(t,x) := x_1(-r)$ defined on $\Re^+ \times C^0\left([-r,0];\Re^2\right)$. It can be easily shown that φ is ultimately differentiable along the solutions of (2.2) with time constant $T = r > 0$ but φ is **not** differentiable along the solutions of (2.2). Moreover, we have $D\varphi(t,x,u) := x_2(-r)$, since $\dot{x}_1(t-r) = x_2(t-r)$ for all $t \ge t_0 + r$.

Definition 2.3 *Let* $0 \le \mu \le r$*. A functional* p *defined on* $I \times C^0\left([-(r-\mu),0];\Re^n\right)$ *where* $\Re^+ \subseteq I \subseteq \Re$, $p(t,x)$ *being completely locally Lipschitz with respect to* $x \in C^0\left([-(r-\mu),0];\Re^n\right)$*, with* $p(t,0) = 0$ *for all* $t \in I$*, is called* ***l-differentiable along the solutions of (2.1) with delay*** μ*, if there exist functionals* $D^i p$, $i = 1,2,...,l$*, called the i-th derivatives of* p*, defined on* $I \times C^0\left([-(r-\mu),0];\Re^n\right)$*, each* $D^i p(t,x)$ *being completely locally Lipschitz with respect to* $x \in C^0\left([-(r-\mu),0];\Re^n\right)$ *with* $D^i p(t,0) = 0$ *for all* $t \in I$*, such that for every* $(t_0, x_0, u) \in \Re^+ \times C^0([-r,0];\Re^n) \times C^0(\Re;U)$*, the mapping* $t \to p(t, T_{r-\mu}(t-\mu)x)$ *is of class* C^l *on* $[t_0, t_{\max} + \mu)$ *and the following holds for all* $t \in [t_0, t_{\max} + \mu)$ *and* $i = 1,2,...,l$*:*

$$\frac{d^i}{dt^i} p(t, T_{r-\mu}(t-\mu)x) = D^i p(t, T_{r-\mu}(t-\mu)x)$$

Remark 2.4: (i) If $p : I \times C^0\left([-(r-\mu),0];\Re^n\right) \to \Re$, where $\Re^+ \subseteq I \subseteq \Re$, $0 \le \mu \le r$, is l-differentiable along the solutions of (2.1) with delay μ, then its derivatives $D^i p : I \times C^0\left([-(r-\mu),0];\Re^n\right) \to \Re$, $i = 1,2,...,l-1$, are functionals which are $(l-i)$-differentiable along the solutions of (2.1) with delay μ.

(ii) If $p : I \times C^0\left([-(r-\mu),0];\Re^n\right) \to \Re$, where $\Re^+ \subseteq I \subseteq \Re$, $0 \le \mu \le r$, is *l*-differentiable along the solutions of (2.1) with delay μ, then for every $\tau \in [0,\mu]$ the functional $k(t,x) := p(t+\tau, x)$ is *l*-differentiable along the solutions of (2.1) with delay $\mu - \tau$ with derivatives $D^i k(t,x) = D^i p(t+\tau, x)$. Moreover, the functional $k(t,x) := p(t+\tau, T_{r'}(\tau-\mu)x)$, where $r' \ge r + \tau - \mu$ is differentiable along the solutions of (2.1) with derivative $Dk(t,x) = Dp(t+\tau, T_{r'}(\tau-\mu)x)$.

Definition 2.5 *Let a functional* $\varphi : I \times C^0\left([-r,0];\Re^n\right) \to \Re$*, where* $\Re^+ \subseteq I \subseteq \Re$*. We say that* φ *is* $T-$*periodic if there exists* $T > 0$ *such that* $\varphi(t+T, x) = \varphi(t,x)$ *for all* $(t,x) \in I \times C^0\left([-r,0];\Re^n\right)$.

The following lemma provides classes of functionals, which are infinitely differentiable with delay μ along the solutions of (2.1).

Lemma 2.6 *The following statements hold:*

(i) *The functional*

$$k(t,x) := \int_{-2\mu}^{-\mu} \mu^{-1} h\left(\frac{s}{\mu}\right) \varphi\left(t+s, T_R(s+\mu)x\right) dw \tag{2.3}$$

defined on $\Re \times C^0\left([-R-\mu,0];\Re^n\right)$*, where* $\mu > 0$, $R \ge 0$, $h \in C^l\left(\Re;\Re^+\right)$ *with* $h(s) = 0$ *for all* $s \notin (-2,-1)$ *and* $\varphi : \Re \times C^0\left([-R,0];\Re^n\right) \to \Re$, $\varphi(t,x)$ *being completely locally Lipschitz with respect to* $x \in C^0\left([-R,0];\Re^n\right)$*, with* $\varphi(t,0) = 0$ *for all* $t \in \Re$*, is l-differentiable along the solutions of (2.1) with delay* μ*, with derivatives for* $i = 1,2,...,l$*:*

$$D^i k(t,x) := \frac{(-1)^i}{\mu^{i+1}} \int_{-2\mu}^{-\mu} \frac{d^i h}{dt^i}\left(\frac{s}{\mu}\right) \varphi\left(t+s, T_R(s+\mu)x\right) dw \tag{2.4}$$

Moreover, if $\varphi : \Re \times C^0\left([-R,0];\Re^n\right) \to \Re$ *is* $T-$*periodic (or linear) then* k *and its derivatives* $D^i k$ *are* $T-$*periodic (or linear).*

(ii) *Let* $a \in C^l\left(\Re;\Re\right)$ *and* $p : \Re \times C^0\left([-R-\mu,0];\Re^n\right) \to \Re$ *a* $(l+i)-$*differentiable functional along the solutions of (2.1) with delay* μ*. The functional*

$$k(t,x) := D^i p(t,x) + a(t) \int_{-2\mu}^{-\mu} \mu^{-1} h\left(\frac{s}{\mu}\right) \varphi(t+s, T_R(s+\mu)x) dw \quad (2.5)$$

defined on $\Re \times C^0([-R-\mu,0];\Re^n)$, *where* $\mu > 0$, $R \ge 0$, $h \in C^l(\Re;\Re^+)$ *with* $h(s)=0$ *for all* $s \notin (-2,-1)$, $D^i p$ *is the i-th derivative of* p *and* $\varphi : \Re \times C^0([-R,0];\Re^n) \to \Re$, $\varphi(t,x)$ *being completely locally Lipschitz with respect to* $x \in C^0([-R,0];\Re^n)$, *with* $\varphi(t,0)=0$ *for all* $t \in \Re$, *is l-differentiable along the solutions of (2.1) with delay* μ. *Moreover, if* $a \in C^l(\Re;\Re^+)$, φ, p *and its derivatives* $D^j p$ ($j=1,...,l+i$) *are* $T-$*periodic, then it follows that* k *and its derivatives* $D^j k$ ($j=1,...,l$) *are* $T-$*periodic. Finally, if* φ, p *and its derivatives* $D^j p$ ($j=1,...,l+i$) *are linear, then it follows that* k *and its derivatives* $D^j k$ ($j=1,...,l$) *are linear.*

Having clarified the notions of differentiability of functionals along the solutions of a control system, we next proceed to the notion of finite-time stabilizability of a control system.

Definition 2.7 *Let* $b := \sup_{t \ge 0} \tau(t)$. *We say that system (2.1) is finite-time stabilizable if there exists constant* $T \ge 0$ *and a functional* $k : \Re^+ \times C^0([-R,0];\Re^n) \to U$, $k(t,x)$ *being completely locally Lipschitz with respect to* $x \in C^0([-R,0];\Re^n)$ *with* $k(t,0)=0$ *for all* $t \ge 0$ *such that:*

(P1) *for every* $(t_0,x_0) \in \Re^+ \times C^0([-\tilde{r},0];\Re^n)$, *where* $\tilde{r} := \max(r, R+b)$, *the solution* $x(t)=x(t,t_0,x_0) \in \Re^n$ *of the closed-loop system (2.1) with* $u(t)=k(t,T_R(t)x)$ *initiated from* $t_0 \ge 0$ *with initial condition* $T_{\tilde{r}}(t_0)x = x_0 \in C^0([-\tilde{r},0];\Re^n)$ *exists for all* $t \ge t_0$ *and satisfies* $x(t)=0$, *for all* $t \ge t_0+T$.

(P2) *for every* $s \ge 0$ *it holds that:*

$$\sup\left\{ |x(t_0+h,t_0,x_0)| \,;\; h \in [-\tilde{r},s],\, \|x_0\|_{\tilde{r}} \le s,\, t_0 \in [0,s] \right\} < +\infty$$

where $x(t)=x(t,t_0,x_0) \in \Re^n$ *of the closed-loop system (2.1) with* $u(t)=k(t,T_R(t)x)$ *initiated from* $t_0 \ge 0$ *with initial condition* $T_{\tilde{r}}(t_0)x = x_0 \in C^0([-\tilde{r},0];\Re^n)$.

Particularly, if system (2.1) is finite-time stabilizable then we say that the closed-loop system (2.1) with $u(t)=k(t,T_R(t)x)$ *satisfies the dead-beat property of order* T.

Remark 2.8 Using Lemma 3.3 in [12] it can be shown that if the closed-loop system (2.1) with $u(t)=k(t,T_R(t)x)$ satisfies the dead-beat property of order T then the equilibrium point $0 \in C^0([-r,0];\Re^n)$ is non-uniformly in time Globally Asymptotically Stable for system (2.1) with $u(t)=k(t,T_R(t)x)$.

The following lemma shows that finite-time global stabilization is possible if time-varying distributed delay feedback is used.

Lemma 2.9 *Consider the one-dimensional control system:*

$$\dot{x}(t) = u(t-\tau) + v(t)\,,\; x(t) \in \Re\,,\; u(t) \in \Re\,,\; v(t) \in \Re \quad (2.6)$$

where $\tau \ge 0$ *is a constant. Then for every* $\mu > \tau$, *the solution of the closed-loop system (2.6) with*

$$u(t) = -p_\mu\left(t+\tau, T_\mu(t+\tau-\mu)x\right) \quad (2.7)$$

where $p_\mu : \Re \times C^0([-\mu,0];\Re) \to \Re$ *is the linear* 3μ-*periodic functional defined by:*

$$p_\mu(t,x) := \mu^{-1} a\left(\frac{t}{\mu}\right) \int_{-2\mu}^{-\mu} \mu^{-1} h\left(\frac{s}{\mu}\right) x(s+\mu) ds \quad (2.8)$$

for certain $a \in C^0(\Re;\Re^+)$ *being periodic function with period* 3 *with* $a(t)=0$ *for* $t \in [0,2]$ *and* $\int_2^3 a(t)dt = 1$ *and* $h \in C^0(\Re;\Re^+)$ *with* $\int_{-2}^{-1} h(s)ds = \int_{-2\mu}^{-\mu} \mu^{-1} h\left(\frac{s}{\mu}\right) ds = 1$, *initiated from arbitrary* $t_0 \ge 0$ *with arbitrary initial condition* $T_{2\mu}(t_0)x = x_0 \in C^0([-2\mu,0];\Re)$ *and corresponding to arbitrary input* $v \in C^0(\Re^+;\Re)$ *satisfies for all* $t \ge t_0$:

$$|x(t)| \le \exp(3L)\sigma\left(\frac{t-t_0}{\mu}\right)\|x_0\|_{2\mu} + 10\mu\exp(3L) \sup_{\max(t_0,t-6\mu)\le s\le t} |v(s)| \quad (2.9)$$

where $\sigma(t) := \begin{cases} 1 & if \quad t < 6 \\ 0 & if \quad t \ge 6 \end{cases}$ *and* $L := \max_{t\in[0,3]} a(t)$.

III. Main Results

In the present paper we consider triangular time-varying systems described by RFDEs:

$$\begin{aligned}
&\dot{x}_i(t) = f_i(t, T_r(t)x_1, \ldots, T_r(t)x_i) + x_{i+1}(t-\tau_i) \quad i = 1, \ldots, n-1 \\
&\dot{x}_n(t) = f_n(t, T_r(t)x) + u(t-\tau_n) \qquad (3.1) \\
&x(t) := (x_1(t), \ldots, x_n(t)) \in \Re^n \ , u(t) \in \Re \, , t \geq 0
\end{aligned}$$

where $r \geq \tau_i \geq 0$, $i = 1, \ldots, n$, the mappings $f_i : \Re^+ \times C^0([-r,0]; \Re^i) \to \Re$ $i = 1, \ldots, n$ are completely locally Lipschitz with respect to $x \in C^0([-r,0]; \Re^n)$ with $f_i(t,0) = 0$ for all $t \geq 0$ and satisfy one of the following assumptions:

(A1) *There exist mappings* $\varphi_i : \Re \times C^0([-r+\tau_i, 0]; \Re^i) \to \Re$, $i = 1, \ldots, n$, *which are differentiable along the solutions of (3.1) and satisfy the following identities for all* $t \geq 0$ *and* $x \in C^0([-r,0]; \Re^n)$:

$$\varphi_1(t-\tau_1, T_{r-\tau_1}(-\tau_1)x_1) := f_1(t, x_1) \qquad (3.2a)$$

$$\begin{aligned}
&\varphi_{i+1}(t-\tau_{i+1}, T_{r-\tau_{i+1}}(-\tau_{i+1})x_1, \ldots, T_{r-\tau_{i+1}}(-\tau_{i+1})x_i, T_{r-\tau_{i+1}}(-\tau_{i+1})x_{i+1}) := \\
&D\varphi_i(t, x_1, \ldots, x_i, x_{i+1}(-\tau_i)) + f_{i+1}(t, x_1, \ldots, x_i, x_{i+1}) \quad i = 1, \ldots, n-1
\end{aligned} \qquad (3.2b)$$

(A2) *There exist mappings* $\varphi_i : \Re \times C^0([-r+\tau_i, 0]; \Re^i) \to \Re$, $i = 1, \ldots, n$, *which are ultimately differentiable along the solutions of (3.1) with time constant* $T > 0$ *and satisfy identities (3.2). Moreover, there exists a constant* $R \in (0, r]$ *and a continuous function* $L : \Re^+ \times \Re^+ \to \Re^+$ *such that for all* $(t,x) \in \Re^+ \times C^0([-r,0]; \Re^n)$ *we have:*

$$\begin{aligned}
&\sum_{i=1}^{n-1} x_i(0) f_i(t, x_1, \ldots, x_i) - x_n(0) D\varphi_{n-1}(t,x) \leq \\
&L\left(t, \sup_{-r \leq \theta \leq -R} |x(\theta)|\right)\left(|x(0)|^2 + 1\right)
\end{aligned} \qquad (3.3)$$

Our first main result states that system (3.1) is finite-time stabilizable under assumption (A1).

Theorem 3.1 *Consider system (3.1) and suppose that assumption (A1) holds. Let* $b_{i,m} := \sum_{k=i}^{m} \tau_k$. *Then for every* $\mu > b_{1,n}$ *and* $l \in Z^+$, *there exist functions* $\gamma \in K^+$, $\rho \in K_\infty$, *functionals* $p_i : \Re \times C^0([-r_n + \mu, 0]; \Re^i) \to \Re$, $i = 1, \ldots, n$ *where* $r_n := r + 2n\mu$, *which are l-differentiable along the solutions of (3.1) with delay* $\mu > 0$ *and a constant* $T > 0$, *such that:*

(i) *the closed-loop system (3.1) with* $u(t) = k(t, T_{r_n - \mu}(t)x)$ *satisfies the dead-beat property of order* T, *where* $k : \Re \times C^0([-r_n + \mu, 0]; \Re^n) \to \Re$ *is completely locally Lipschitz with respect to* $x \in C^0([-r_n + \mu, 0]; \Re^n)$ *with* $k(\cdot, 0) = 0$ *and is defined by:*

$$\begin{aligned}
&k(t,x) := -\varphi_n(t,x) \\
&-\sum_{i=1}^{n} p_i\left(t + b_{i,n}, T_{r_n-\mu}(b_{i,n} - \mu)x_1, \ldots, T_{r_n-\mu}(b_{i,n} - \mu)x_i\right)
\end{aligned} \qquad (3.4)$$

(ii) *for all* $(t_0, x_0, v) \in \Re^+ \times C^0([-r_n, 0]; \Re^n) \times C^0(\Re; \Re)$ *the solution of the closed-loop system (3.1) with* $u(t) = k(t, T_{r_n-\mu}(t)x) + v(t)$ *satisfies the estimate:*

$$|x(t)| \leq \gamma(t) \rho\left(\|x_0\|_{r_n} + \sup_{t_0 - \tau_n \leq s \leq t} |v(s)| \right), \ \forall t \geq t_0 \qquad (3.5)$$

Moreover, if the mappings $\varphi_i : \Re \times C^0([-r+\tau_i, 0]; \Re^i) \to \Re$, $i = 1, \ldots, n$, *are independent of* t, *then the functionals* p_i $i = 1, \ldots, n$ *and* k *as defined by (3.4) can be chosen to be* $3\mu -$*periodic. Finally, if the mappings* $\varphi_i : \Re \times C^0([-r+\tau_i, 0]; \Re^i) \to \Re$, $i = 1, \ldots, n$, *are linear then the functionals* p_i $i = 1, \ldots, n$ *and* k *as defined by (3.4) can be chosen to be linear.*

Our second main result states that system (3.1) is finite-time stabilizable under assumption (A2).

Theorem 3.2 *Consider system (3.1) and suppose that assumption (A2) holds. Then for every* $\mu \geq R + \sum_{i=1}^{n} \tau_i$ *where* $R > 0$ *is the constant involved in assumption (A2), there exists a functional* $k : \Re \times C^0([-\tilde{r}, 0]; \Re^n) \to \Re$ *where* $\tilde{r} := 2r + 2n\mu$, *which is completely locally Lipschitz with respect to* $x \in C^0([-\tilde{r}, 0]; \Re^n)$ *and a constant* $T' > T$ *(where* $T \geq 0$ *is the time constant involved in assumption (A2)), such that the closed-loop system (3.1) with* $u(t) = k(t, T_{\tilde{r}}(t)x)$ *satisfies the dead-beat property of order* T'. *Moreover, if the mappings* $\varphi_i : \Re \times C^0([-r+\tau_i, 0]; \Re^i) \to \Re$, $i = 1, \ldots, n$ *are independent of* t *then the functional* k *can be chosen to be* $3\mu -$*periodic. Finally, if the mappings* $\varphi_i : \Re \times C^0([-r+\tau_i, 0]; \Re^i) \to \Re$, $i = 1, \ldots, n$ *are linear then the functional* k *can be chosen to be linear.*

Example 3.3: The chain of delayed integrators (1.2), where $\tau_i \geq 0$, has been considered in the literature for the particular case of $\tau_1 = ... = \tau_{n-1} = 0$ in [15] with the additional requirement that the stabilizing feedback must be bounded. Here we consider the general case and we demand finite-time stabilization of the corresponding closed-loop system. Since assumption (A1) holds with $\varphi_i \equiv 0$ for $i = 1,...,n$, by virtue of Theorem 3.1 we conclude that for every $\mu > \sum_{i=1}^{n} \tau_i$ there exists a linear $3\mu-$periodic time-varying distributed delay feedback such that the closed-loop system satisfies the dead-beat property. Notice that there is no limitation on the size of the delays. Specifically, for the case $n = 2$ we obtain the following feedback, which guarantees the dead-beat property of order $T = 14$:

$$u(t) = -\dot{a}(t+\tau_1+\tau_2)\int_{-2}^{-1} h(s) x_1(t+\tau_1+\tau_2+s)ds$$

$$+ a(t+\tau_1+\tau_2)\int_{-2}^{-1} \dot{h}(s) x_1(t+\tau_1+\tau_2+s)ds$$

$$- a(t+\tau_2)\int_{-2}^{-1} h(s) x_2(t+\tau_2+s)ds$$

$$- a(t+\tau_2)\int_{-2}^{-1} h(s) a(t+\tau_1+\tau_2+s)\int_{-2}^{-1} h(w) x_1(t+\tau_1+\tau_2+s+w)dwds$$

where $a \in C^1(\Re;\Re^+)$ is a periodic function with period 3, with $a(t) = 0$ for $t \in [0,2]$ and $\int_{2}^{3} a(t)dt = 1$, $h \in C^1(\Re;\Re^+)$ with $h(s) = 0$ for all $s \notin (-2,-1)$, $\int_{-2}^{-1} h(s)ds = 1$. $\lhd$

REFERENCES

[1] A. Bacciotti and L. Rosier, "Liapunov Functions and Stability in Control Theory", Lecture Notes in Control and Information Sciences, 267, Springer-Verlag, London, 2001.

[2] Bhat, S.P. and D.S. Bernstein, "Continuous Finite-Time Stabilization of the Translational and Rotational Double Integrators", *IEEE Transactions on Automatic Control*, 43, 1998, 678-682.

[3] Bhat, S.P. and D.S. Bernstein, "Finite Time Stability of Continuous Autonomous Systems", *SIAM Journal on Control and Optimization*, 38, 2000, 751-766.

[4] Coron, J.M., "Stabilization in Finite-Time of Locally Controllable Systems by Means of Continuous Time-Varying Feedback Law", *SIAM Journal on Control and Optimization*, 33, 1995, 804-833.

[5] Engel, R. and G. Kreisselmeier, "A Continuous Time Observer which Converges in Finite Time", *IEEE Transactions on Automatic Control*, 47(7), 2002, 1202-1204.

[6] Haimo, V.T., "Finite Time Controllers", *SIAM Journal on Control and Optimization*, 24, 1986, 760-770.

[7] J.K. Hale and S.M.V. Lunel, "Introduction to Functional Differential Equations", Springer-Verlag, New York, 1993.

[8] Hong, Y., J. Huang and Y. Xu, "On an Output Feedback Finite-Time Stabilization Problem", *Proceedings of the IEEE Conference on Decision and Control*, Phoenix, 1999, 1302-1307.

[9] Ilchmann, A. and C.J. Sangwin, "Output Feedback Stabilization of Minimum Phase Systems by Delays", *Systems and Control Letters*, 52, 2004, 233-245.

[10] Jankovic, M., "Control Lyapunov-Razumikhin Functions and Robust Stabilization of Time Delay Systems", *IEEE Transactions on Automatic Control*, 46(7), 2001, 1048-1060.

[11] Karafyllis, I. and J. Tsinias, "A Converse Lyapunov Theorem for Non-Uniform in Time Global Asymptotic Stability and its Application to Feedback Stabilization", *SIAM Journal on Control and Optimization*, 42(3), 2003, 936-965.

[12] Karafyllis, I., "The Non-Uniform in Time Small-Gain Theorem for a Wide Class of Control Systems with Outputs", *European Journal of Control*, 10(4), 2004, 307-323.

[13] M. Krstic, I. Kanellacopoulos and P. Kokotovic, "Nonlinear and Adaptive Control Design", New York, Wiley, 1995.

[14] Levant, A., "Higher-Order Sliding Modes, Differentiation and Output Feedback Control", *International Journal of Control*, 76, 2003, 924-941.

[15] Mazenc, F. S. Mondie and S.I. Niculescu, "Global Asymptotic Stabilization for Chains of Integrators with a Delay in the Input", *IEEE Transactions on Automatic Control*, 48(1), 2003, 57-63.

[16] Mazenc, F. and P.A. Bliman, "Backstepping Design for Time-Delay Nonlinear Systems", *Proceedings of the 42nd IEEE Conference on Decision and Control,* Maui, Hawaii, U.S.A., December 2003.

[17] Mazenc, F. and C. Prieur, "Switching Using Distributed Delays", *Proceedings of the 4th IFAC Workshop on Time Delay Systems*, Rocquencourt, France, 2003.

[18] Mazenc, F. S. Mondie and S.I. Niculescu, "Global Stabilization of Oscillators with Bounded Delayed Input", to appear in *Systems and Control Letters*.

[19] Menold, P.H., R. Findeisen and F. Allgower, "Finite Time Convergent Observers for Nonlinear Systems", *Proceedings of the 42nd IEEE Conference on Decision and Control*, Hawaii, U.S.A., 2003, 5673-5678.

[20] Morin, P. and C. Samson, "Robust Point-Stabilization of Nonlinear Affine Control Systems", in "Stability and Stabilization of Nonlinear Systems", D. Aeyels, F. Lamnabhi-Lagarrigue and A. van der Schaft (Eds), Springer-Verlag, London, 1999, 215-237.

[21] S.I. Niculescu, "Delay Effects on Stability, A Robust Control Approach", Heidelberg, Germany, Springer-Verlag, 2001.

[22] A.V. Savkin and R.J. Evans, "Hybrid Dynamical Systems", Birkhauser, Boston, 2002.

[23] Sontag, E.D., "Comments on Integral Variants of ISS", *Systems and Control Letters*, 34, 1998, 93-100.

[24] Tsinias, J., "Sufficient Lyapunov like Conditions for Stabilization", *Mathematics of Control, Signals and Systems*, 2, 1989, 343-357.