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Lyapunov Theorems for Systems Described by Retarded Functional Differential Equations

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Abstract— **Lyapunov-like characterizations for non-uniform in time and uniform robust global asymptotic stability of uncertain systems described by retarded functional differential equations are provided.**

I. INTRODUCTION

In this paper we provide Lyapunov characterizations for non-uniform in time and uniform Robust Global Asymptotic Stability (RGAS) for systems described by time-varying Retarded Functional Differential Equations (RFDEs). The notion of non-uniform in time RGAS is introduced in [7] for continuous time finite-dimensional systems and in [9] for a wide class of systems including discrete-time systems and systems described by RFDEs. This notion has been proved to be fruitful for the solution of several problems in Mathematical Control Theory (see [7,8]). The notion of uniform RGAS that we adopt in this paper is an extension of the corresponding notion for finite-dimensional continuous-time uncertain systems (see [3,14]).

The motivation for the extension of non-uniform in time and uniform RGAS to uncertain systems described by RFDEs is strong, since such models are used frequently for the description of engineering systems (see [5]). It should be emphasized that in many cases where hybrid open-loop/feedback stabilizing control laws are proposed for finite-dimensional continuous-time systems, the closed-loop system is actually a system described by time-varying RFDEs (infinite-dimensional). For example, in [19] analytic driftless control systems of the following form are considered and strategies are provided for the construction of control laws of the form $u(t)=k(t,x(t),x(lT))$ for $t\in[lT,(l+1)T)$ where l is a non-negative integer and $T>0$ denotes the updating time-period of the control. Notice that the resulting closed-loop system is actually a time-varying system described by RFDEs even if k is independent of time. The same comments apply for the synchronous controller switching strategies proposed in [21]. The possibility of switching control laws using distributed delays was exploited in [17].

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Lyapunov functions and functionals play an important role to synthesis and design in control theory and several important results have been established concerning Lyapunov-like descriptions of *uniform global asymptotic stability* (UGAS) (see [5,13] and the references therein). Our goal is to establish Lyapunov characterizations for the concepts of *non-uniform in time and uniform robust global asymptotic stability* (RGAS) analogous to the corresponding characterizations given in [3,14] for continuous-time finite-dimensional uncertain systems, which overcome the limitations imposed by previous works. Particularly, our Lyapunov characterizations apply

- to systems with disturbances that take values in a (not necessarily compact) given set
- to systems with dynamics which are not necessarily bounded with respect to time

Viability issues for systems described by functional differential inclusions (and thus uncertain systems described by RFDEs) were considered in [2]. We note that Lyapunov functionals for linear time-delay systems were constructed in [4,12,18]. Stability conditions for time-varying time-delay systems were given in [11]. Recently in many works the problem of feedback stabilization of systems described by RFDEs was studied (see for instance [6,15,16,20]). It should be emphasized that the literature concerning issues of stability and stabilization of linear time-delay systems is vast and the previous references are only given as pointers. Note also that in the present paper we are not concerned with stability conditions given by Razumikhin functions, since such conditions resemble "small-gain" conditions with Lyapunov-like characteristics (see [23]).

In the present work we provide Lyapunov-like conditions which demand the infinitesimal decrease property to hold only on subsets of the state space which contain the solutions of the system (i.e., the infinitesimal decrease property holds only after some time) along with an additional property that guarantees forward completeness on the critical time interval when the infinitesimal decrease property does not hold, namely the property $\dot{V}\le V$, where V denotes the Lyapunov functional and $\dot{V}$ the time derivative of the Lyapunov functional evaluated along the

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solutions of the system (Theorems 3.1 and 3.2). This property was shown to be necessary and sufficient for forward completeness of continuous-time finite dimensional systems in [1]. Moreover, the "weaker" property that we demand in the present paper is utilized for the construction of Lyapunov functionals for time-delay systems (see Example 3.4).

Notations Throughout this paper we adopt the following notations:

* For a vector $x \in \Re^n$ we denote by $|x|$ its usual Euclidean norm and by x' its transpose. For $x \in C^0([-r,0];\Re^n)$ we define $\|x\|_r := \max_{\theta\in[-r,0]} |x(\theta)|$.
* By $C^0(A;\Omega)$), we denote the class of functions (taking values in Ω) that are continuous on A.
* $\mathcal{E}$ denotes the class of non-negative C^0 functions $\mu:\Re^+ \to \Re^+$, for which it holds: $\int_0^{+\infty} \mu(t)dt < +\infty$ and $\lim_{t\to+\infty} \mu(t) = 0$.
* We denote by K^+ the class of positive C^0 functions defined on $\Re^+$. We say that a function $\rho:\Re^+ \to \Re^+$ is positive definite if $\rho(0)=0$ and $\rho(s)>0$ for all $s>0$. For definitions of the classes K, K_∞ and KL see [10].

II. DEFINITIONS AND TECHNICAL RESULTS

Let $x:[a-r,b) \to \Re^n$ with $b > a \ge 0$ and $r \ge 0$. We define

$$T_r(t)x := \{x(t+\theta)\,;\,\theta\in[-r,0]\}\,,\text{ for } t\in[a,b) \qquad (2.1)$$

Let M_D the class of all right-continuous mappings $d:\Re^+ \to D$, with the following property:

> "there exists a countable set $A_d \subset \Re^+$ which is either finite or $A_d = \{t_k^d\,;\,k=1,...,\infty\}$ with $t_{k+1}^d > t_k^d > 0$ for all $k=1,2,...$ and $\lim t_k^d = +\infty$, such that the mapping $t\in\Re^+ \setminus A_d \to d(t) \in D$ is continuous"

We denote by $x(t)$ with $t \ge t_0$ the unique solution of the initial-value problem:

$$\begin{aligned} &\dot{x}(t) = f(t, T_r(t)x, d(t))\,,\ t \ge t_0 \\ &x(t)\in\Re^n\,,\ d(\cdot)\in M_D \end{aligned} \qquad (2.2)$$

with initial condition $T_r(t_0)x = x_0 \in C^0([-r,0];\Re^n)$, where $D \subseteq \Re^l$, $r \ge 0$ is a constant and the mapping $f:\Re^+ \times C^0([-r,0];\Re^n)\times D \to \Re^n$ with $f(t,0,d)=0$ for all $(t,d)\in\Re^+\times D$ satisfies the following hypotheses:

(H1) The mapping $(x,d)\to f(t,x,d)$ is continuous for each fixed $t\ge 0$ and such that for every bounded $I\subseteq\Re^+$ and for every bounded $S\subset C^0([-r,0];\Re^n)$, there exists a constant $L\ge 0$ such that:

$$\begin{aligned} &\big(x(0)-y(0)\big)'\big(f(t,x,d)-f(t,y,d)\big) \le \\ &L \max_{\tau\in[-r,0]} |x(\tau)-y(\tau)|^2 = L\|x-y\|_r^2 \\ &\forall t\in I\,,\ \forall (x,y)\in S\times S\,,\ \forall d\in D \end{aligned}$$

This assumption is equivalent to the existence of a continuous function $L:\Re^+\times\Re^+\to\Re^+$ such that for each fixed $t\ge 0$ the mappings $L(t,\cdot)$ and $L(\cdot,t)$ are non-decreasing, with the following property:

$$\begin{aligned} &\big(x(0)-y(0)\big)'\big(f(t,x,d)-f(t,y,d)\big) \le \\ &L(t,\|x\|_r+\|y\|_r)\,\|x-y\|_r^2 \\ &\forall (t,x,y,d)\in\Re^+\times C^0([-r,0];\Re^n)\times C^0([-r,0];\Re^n)\times D \end{aligned} \qquad (2.3)$$

(H2) For every bounded $\Omega\subset\Re^+\times C^0([-r,0];\Re^n)$ the image set $f(\Omega\times D)\subset\Re^n$ is bounded.

(H3) There exists a countable set $A\subset\Re^+$, which is either finite or $A=\{t_k\,;\,k=1,...,\infty\}$ with $t_{k+1}>t_k>0$ for all $k=1,2,...$ and $\lim t_k = +\infty$, such that mapping $(t,x,d)\in(\Re^+\setminus A)\times C^0([-r,0];\Re^n)\times D \to f(t,x,d)$ is continuous. Moreover, for each fixed $(t_0,x,d)\in\Re^+\times C^0([-r,0];\Re^n)\times D$, we have $\lim_{t\to t_0^+} f(t,x,d) = f(t_0,x,d)$.

(H4) For every $\varepsilon>0$, $t\in\Re^+$, there exists $\delta:=\delta(\varepsilon,t)>0$ such that $\sup\{|f(\tau,x,d)|\,;\,\tau\in\Re^+,\, d\in D,\, |\tau-t|+\|x\|_r<\delta\}<\varepsilon$.

We denote by $\phi(t,t_0,x_0;d):=T_r(t)x$ the solution of (2.2) with initial condition $T_r(t_0)x = x_0 \in C^0([-r,0];\Re^n)$ corresponding to input $d\in M_D$. It can be shown for the initial-value problem (2.2) under hypotheses (H1-4) that:

1) for every $(t_0,x_0,d)\in\Re^+\times C^0([-r,0];\Re^n)\times M_D$ there exists $t_{\max}\in(t_0,+\infty]$ such that the initial-value problem admits a unique solution $x(t)$, which is continuous on $[t_0-r,t_{\max})$ and cannot be further continued. Moreover, if

$t_{\max} < +\infty$ then we must necessarily have $\limsup_{t \to t_{\max}^-} |x(t)| = +\infty$.

2) the solution $x(t)$ is an absolutely continuous function on $[t_0, t_{\max})$, which satisfies $\dot{x}(t) = f(t, T_r(t)x, d(t))$ for all $t \in [t_0, t_{\max}) \setminus (A \cup A_d)$ and $\lim_{h \to 0^+} \frac{x(t+h) - x(t)}{h} = f(t, T_r(t)x, d(t))$ for all $t \in [t_0, t_{\max})$.

3) Finally, the unique solution of (2.2) satisfies for all $(t_0, x_0, y_0, d) \in \Re^+ \times C^0([-r,0];\Re^n) \times C^0([-r,0];\Re^n) \times M_D$ and for all $t \ge t_0$ so that $\phi(t, t_0, x_0; d)$ and $\phi(t, t_0, y_0; d)$ are both defined:

$$\|\phi(t,t_0,x_0;d) - \phi(t,t_0,y_0;d)\|_r \le \|x_0 - y_0\|_r \exp\left(\widetilde{L}(t - t_0)\right) \quad (2.4)$$

where $\widetilde{L} := L\left(t, \sup_{t \in [t_0,t]} \|\phi(t,t_0,x_0;d)\|_r + \sup_{t \in [t_0,t]} \|\phi(t,t_0,y_0;d)\|_r\right)$ and L is the function involved in (2.3).

Remark 2.1:

(a) When $r = 0$ we identify the space $C^0([-r,0];\Re^n)$ with the finite-dimensional space $\Re^n$ and we obtain the familiar finite-dimensional continuous-time case. Consequently, all the following results hold also for finite-dimensional continuous-time systems.

(b) A major difference between the case of uncertain finite-dimensional continuous-time systems considered in [14] and the case of uncertain systems described by RFDEs is the nature of the class of allowed inputs M_D. This happens because there is a fundamental difference between the two cases: in the finite-dimensional case the map describing the evolution of the state is absolutely continuous with respect to time while in the infinite-dimensional case the map describing the evolution of the state is (simply) continuous with respect to time (see Lemma 2.1 in [5] and notice that the state for the infinite-dimensional case is $T_r(t)x \in C^0([-r,0];\Re^n)$). This fact has an important consequence: Lyapunov functionals evaluated on the solutions of system (2.2) will be (simply) continuous and not absolutely continuous maps with respect to time and in order to guarantee their monotonicity, we must require that an appropriate decrease condition holds for all times (and not almost everywhere, see the discussion in [3], Chapter 6). Thus we cannot allow M_D contain arbitrary measurable mappings.

(c) In all the following results we assume that the inputs belong to the class M_D. It is clear that the same conclusions hold for inputs $d : \Re^+ \to D$, for which there exists $d' \in M_D$ such that $d(t) = d'(t)$ almost everywhere.

Since $f(t,0,d) = 0$ for all $(t,d) \in \Re^+ \times D$, it follows that $\phi(t,t_0,0;d) = 0 \in C^0([-r,0];\Re^n)$ for all $(t_0,d) \in \Re^+ \times M_D$ and $t \ge t_0$. Furthermore, for every $\varepsilon > 0$, $T, h \ge 0$ there exists $\delta := \delta(\varepsilon, T, h) > 0$ such that:

$$\|x\|_r < \delta \quad \Rightarrow$$
$$\sup\left\{\|\phi(\tau,t_0,x;d)\|_r \; ; \; d \in M_D \, , \, \tau \in [t_0, t_0 + h] \, , \, t_0 \in [0,T]\right\} < \varepsilon$$

Thus $0 \in C^0([-r,0];\Re^n)$ is a robust equilibrium point for (2.2) in the sense described in [9]. The following definition of non-uniform in time RGAS coincides with the definition of non-uniform in time RGAS given in [9], for a wide class of systems.

Definition 2.2: *We say that* $0 \in C^0([-r,0];\Re^n)$ *is* ***non-uniformly in time Robustly Globally Asymptotically Stable (RGAS)*** *for system (2.2) if the following properties hold:*

P1 $0 \in C^0([-r,0];\Re^n)$ *is* ***Robustly Lagrange Stable****, i.e., for every* $s \ge 0$, $T \ge 0$, *it holds that*

$$\sup\left\{\|\phi(t,t_0,x_0;d)\|_r \; ; \; t \in [t_0,+\infty) \, , \, \|x_0\|_r \le s \, , t_0 \in [0,T] \, , d \in M_D\right\} < +\infty$$

(Robust Lagrange Stability)

P2 $0 \in C^0([-r,0];\Re^n)$ *is* ***Robustly Lyapunov Stable****, i.e., for every* $\varepsilon > 0$ *and* $T \ge 0$ *there exists a* $\delta := \delta(\varepsilon, T) > 0$ *such that:*

$$\|x_0\|_r \le \delta \, , \, t_0 \in [0,T] \Rightarrow \|\phi(t,t_0,x_0;,d)\|_r \le \varepsilon \, ,$$
$$\forall t \in [t_0,+\infty) \, , \, \forall d \in M_D$$

(Robust Lyapunov Stability)

P3 $0 \in C^0([-r,0];\Re^n)$ satisfies the **Robust Attractivity Property**, i.e. *for every* $\varepsilon > 0$, $T \ge 0$ *and* $R \ge 0$, *there exists a* $\tau := \tau(\varepsilon, T, R) \ge 0$, *such that:*

$$\|x_0\|_r \le R \, , \, t_0 \in [0,T] \Rightarrow \|\phi(t,t_0,x_0;d)\|_r \le \varepsilon \, ,$$
$$\forall t \in [t_0 + \tau,+\infty) \, , \, \forall d \in M_D$$

The two following lemmas are given in [9] (as Lemma 3.3 and Lemma 3.4, respectively) for a wide class of systems that include systems of RFDEs under hypotheses (H1-4). They provide essential characterizations of the notion of non-uniform in time RGAS.

Lemma 2.3: *Suppose that (2.2) is Robustly Forward Complete, i.e., for every* $s \ge 0$, $T \ge 0$, *it holds that*

$$\sup\left\{\|\phi(t_0+h,t_0,x_0;d)\|_r \ ;\ h\in[0,T],\ \|x_0\|_r \le s,\ t_0\in[0,T],\ d\in M_D\right\}<+\infty$$

and that $0 \in C^0([-r,0];\Re^n)$ *satisfies the Robust Attractivity Property (property P3 of Definition 2.2) for system (2.2). Then* $0 \in C^0([-r,0];\Re^n)$ *is non-uniformly in time RGAS for system (2.2).*

Lemma 2.4: $0 \in C^0([-r,0];\Re^n)$ *is non-uniformly in time RGAS for system (2.2) if and only if there exist functions* $\sigma \in KL$, $\beta \in K^+$ *such that the following estimate holds for all* $(t_0, x_0, d) \in \Re^+ \times C^0([-r,0];\Re^n) \times M_D$ *and* $t \in [t_0, +\infty)$ *:*

$$\|\phi(t,t_0,x_0;d)\|_r \ \le\ \sigma\left(\beta(t_0)\|x_0\|_r\ ,\, t-t_0\right) \quad (2.5)$$

Finally, we also provide the definition of uniform RGAS, in terms of KL functions, which is completely analogous to the finite-dimensional case (see [3,14]). It is clear that such a definition is equivalent to a $\delta-\varepsilon$ definition (analogous to Definition 2.2).

Definition 2.5: *We say that* $0 \in C^0([-r,0];\Re^n)$ *is Uniformly Robustly Globally Asymptotically Stable (URGAS) for system (2.2) if and only if there exist a function* $\sigma \in KL$ *such that the following estimate holds for all* $(t_0, x_0, d) \in \Re^+ \times C^0([-r,0];\Re^n) \times M_D$ *and* $t \in [t_0, +\infty)$ *:*

$$\|\phi(t,t_0,x_0;d)\|_r \ \le\ \sigma\left(\|x_0\|_r\ ,\, t-t_0\right) \quad (2.6)$$

The following corollary must be compared to Lemma 1.1, page 131 in [5]. It shows that for periodic systems of RFDEs non-uniform in time RGAS is equivalent to URGAS. We say that (2.2) is $T-periodic$ if there exists $T>0$ such that $f(t+T,x,d) = f(t,x,d)$ for all $(t,x,d) \in \Re^+ \times C^0([-r,0];\Re^n) \times D$.

Corollary 2.6: *Suppose that* $0 \in C^0([-r,0];\Re^n)$ *is non-uniformly in time RGAS for system (2.2) and that (2.2) is* $T-periodic$. *Then* $0 \in C^0([-r,0];\Re^n)$ *is URGAS for system (2.2).*

Let $x \in C^0\left([-r,0];\Re^n\right)$. By $E_h(x;v)$, where $0 \le h < r$ and $v \in \Re^n$ we denote the following operator:

$$E_h(x;v) := \begin{cases} x(0)+(\theta+h)v & \text{for } -h<\theta\le 0 \\ x(\theta+h) & \text{for } -r\le\theta\le -h \end{cases} \quad (2.7)$$

Let $V : \Re^+ \times C^0\left([-r,0];\Re^n\right) \to \Re$. We define

$$V^0(t,x;v) := \limsup_{\substack{h\to 0^+ \\ y\to 0,\, y\in C^0([-r,0];\Re^n)}} \frac{V(t+h, E_h(x;v)+hy) - V(t,x)}{h} \quad (2.8)$$

The following lemma presents an important property of this generalized derivative.

Lemma 2.7 *Let* $V : \Re^+ \times C^0\left([-r,0];\Re^n\right) \to \Re$ *and let* $x \in C^0([t_0-r, t_{\max});\Re^n)$ *a solution of (2.2) under hypotheses (H1-4) corresponding to certain* $d \in M_D$. *Then for all* $t \in [t_0, t_{\max})$ *it holds that*

$$\limsup_{h\to 0^+} \frac{V(t+h, T_r(t+h)x) - V(t, T_r(t)x)}{h} \le V^0(t, T_r(t)x; f(t, T_r(t)x, d(t))) \quad (2.9)$$

The following comparison principle is an extension of the comparison principle in [10] and will be used frequently in this paper in conjunction with Lemma 2.7 for the derivation of estimates of values of Lyapunov functionals.

Lemma 2.8 (Comparison Principle) *Consider the scalar differential equation:*

$$\dot w = f(t,w) \quad ; \quad w(t_0) = w_0 \quad (2.10)$$

where $f(t,w)$ *is continuous in* $t \ge 0$ *and locally Lipschitz in* $w \in J$, *where* $J \subseteq \Re$ *is an open interval. Let* $T > t_0$ *such that the solution* $w(t)$ *of the initial value problem (2.10) exists and satisfies* $w(t) \in J$ *for all* $t \in [t_0, T)$. *Let* $v(t)$ *a lower semi-continuous function that satisfies the differential inequality for all* $t \in [t_0, T)$ *:*

$$D^+ v(t) := \limsup_{h\to 0^+} \frac{v(t+h)-v(t)}{h} \le f(t, v(t)) \quad (2.11)$$

Suppose furthermore:

$$v(t_0) \le w_0 \quad (2.12a)$$

$$v(t) \in J,\ \forall t \in [t_0, T) \quad (2.12b)$$

If one of the following holds:

(i) *the mapping* $f(t,\cdot)$ *is non-decreasing on* $J \subseteq \Re$, *for each fixed* $t \in [t_0, T)$.

(ii) *there exists* $\phi \in C^0(\Re^+)$ *such that* $f(t,w) \le \phi(t)$, *for all* $(t,w) \in [t_0, T) \times J$.

then $v(t) \le w(t)$, *for all* $t \in [t_0, T)$.

III. MAIN RESULTS

We are now in a position to state our main results for non-uniform in time RGAS and URGAS.

Theorem 3.1 *Consider system (2.2) under hypotheses (H1-4). Then the following statements are equivalent:*

(a) $0 \in C^0([-r,0];\Re^n)$ *is non-uniformly in time RGAS for (2.2).*

(b) *There exists a continuous mapping* $(t,x) \in \Re^+ \times C^0([-r,0];\Re^n) \to V(t,x) \in \Re^+$, *with the following properties:*

(i) *There exist functions* $a_1, a_2 \in K_\infty$, $\beta \in K^+$ *such that:*

$$a_1(\|x\|_r) \le V(t,x) \le a_2\left(\beta(t)\|x\|_r\right), \quad \forall (t,x) \in \Re^+ \times C^0([-r,0];\Re^n) \tag{3.1}$$

(ii) *It holds that:*

$$V^0(t,x;f(t,x,d)) \le -V(t,x), \quad \forall (t,x,d) \in \Re^+ \times C^0([-r,0];\Re^n) \times D \tag{3.2}$$

(infinitesimal decrease property)

(iii) *There exists a non-decreasing function* $M : \Re^+ \to \Re^+$ *such that for every* $R \ge 0$, *it holds:*

$$\left|V(t,y) - V(t,x)\right| \le M(R)\left\|y - x\right\|_r, \forall t \in [0,R], \quad \forall x, y \in \left\{x \in C^0([-r,0];\Re^n) \,;\, \|x\|_r \le R\right\} \tag{3.3}$$

(c) *There exist* $\tau \ge 0$, *a lower semi-continuous mapping* $V : \Re^+ \times C^0([-r-\tau,0];\Re^n) \to \Re^+$, *constants* $R \ge 0$, $c > 0$, *functions* $a_1, a_2 \in K_\infty$, $\beta_i \in K^+ (i = 1,...,4)$ *with* $\int_0^{+\infty} \beta_4(t)dt = +\infty$, $\mu \in \mathcal{E}$ *(see Notations) and* $\rho \in C^0(\Re^+;\Re^+)$ *being positive definite and locally Lipschitz, such that the following inequalities hold:*

$$a_1\left(|x(0)|\right) \le V(t,x) \le a_2\left(\beta_1(t)\|x\|_{r+\tau}\right), \quad \forall (t,x) \in \Re^+ \times C^0([-r-\tau,0];\Re^n) \tag{3.4}$$

$$V^0(t,x;f(t,T_r(0)x,d)) \le \beta_2(t)V(t,x) + R\,\beta_3(t), \quad \forall (t,x,d) \in \Re^+ \times C^0([-r-\tau,0];\Re^n) \times D \tag{3.5a}$$

$$V^0(t,x;f(t,T_r(0)x,d)) \le -\beta_4(t)\rho\left(V(t,x)\right) + \beta_4(t)\mu\left(\int_0^t \beta_4(s)ds\right) \quad \forall (t,d) \in [\tau,+\infty) \times D, \forall x \in S(t) \tag{3.5b}$$

(infinitesimal decrease property)

where the set-valued map $S(t) \subseteq C^0([-r-\tau,0];\Re^n)$ *is defined for* $t \ge \tau$ *by:*

$$S(t) := \left\{ x \in \bar{S}(t) \,;\, x(\theta) = x(-\tau) + \int_{-\tau}^{\theta} f(t+s, T_r(s)x, d(\tau+s))ds, \ \forall \theta \in [-\tau,0], d \in M_D \right\} \tag{3.6a}$$

and $\bar{S}(t) \subseteq C^0([-r-\tau,0];\Re^n)$ *is any set-valued map satisfying for all* $t \ge 0$

$$\left\{ x \in C^0([-r-\tau,0];\Re^n) \,;\, a_2\left(\beta_1(t)\|x\|_{r+\tau}\right) \ge \eta\left(\int_0^t \beta_4(s)ds\,,0\,,c\right) \right\} \subseteq \bar{S}(t) \tag{3.6b}$$

where $\eta(t,t_0,\eta_0)$ *denotes the unique solution of the initial value problem:*

$$\dot{\eta} = -\rho(\eta) + \mu(t) \quad ; \quad \eta(t_0) = \eta_0 \ge 0 \tag{3.6c}$$

Theorem 3.2 *Consider system (2.2) under hypotheses (H1-4). Then the following statements are equivalent:*

(a) $0 \in C^0([-r,0];\Re^n)$ *is URGAS for (2.2).*

(b) *There exists a continuous mapping* $(t,x) \in \Re^+ \times C^0([-r,0];\Re^n) \to V(t,x) \in \Re^+$, *satisfying properties (i), (ii) and (iii) of statement (b) of Theorem 3.1 with* $\beta(t) \equiv 1$. *Moreover, if system (2.2) is* $T-$*periodic*, *then* V *is* $T-$*periodic (i.e.* $V(t+T,x) = V(t,x)$ *for all* $(t,x) \in \Re^+ \times C^0([-r,0];\Re^n)$*) and if (2.2) is autonomous then* V *is independent of* t.

(c) *There exist constants* $\beta, \tau \ge 0$, *a lower semi-continuous mapping* $V : \Re^+ \times C^0([-r-\tau,0];\Re^n) \to \Re^+$, *functions* $a_1, a_2 \in K_\infty$ *and* $\rho \in C^0(\Re^+;\Re^+)$ *being positive definite and locally Lipschitz such that the following inequalities hold:*

$$a_1\left(|x(0)|\right) \le V(t,x) \le a_2\left(\|x\|_{r+\tau}\right), \quad \forall (t,x) \in \Re^+ \times C^0([-r-\tau,0];\Re^n) \tag{3.7}$$

$$V^0(t,x;f(t,T_r(0)x,d)) \le \beta\, V(t,x)\,,$$
$$\forall (t,x,d) \in \Re^+ \times C^0([-r-\tau,0];\Re^n) \times D \quad (3.8a)$$

$$V^0(t,x;f(t,T_r(0)x,d)) \le -\rho\big(V(t,x)\big)\,,$$
$$\forall (t,d) \in [\tau,+\infty) \times D\,,\ \forall x \in S(t) \quad (3.8b)$$

(infinitesimal decrease property)

where the set $S(t) \subseteq C^0([-r-\tau,0];\Re^n)$ *is defined for* $t \ge \tau$ *by (3.6a) with* $\bar{S}(t) := C^0([-r-\tau,0];\Re^n)$.

Remark 3.3: a) Although the conditions for non-uniform in time RGAS seem more complicated than the corresponding conditions for URGAS, it should be emphasized that the conditions for non-uniform in time RGAS are "weaker" than the corresponding conditions for URGAS. Particularly, the main difference lies in that the infinitesimal decrease condition does not have to be satisfied for states sufficiently close to the equilibrium point in the non-uniform in time case.

b) Notice that we demand the infinitesimal decrease property to hold only on a subset of the state space ($S(t)$) which contains the solutions of the system. However, an additional property that guarantees forward completeness on the critical time interval $[t_0, t_0+\tau]$ has to be satisfied, namely (3.5a) in the non-uniform in time case and (3.8a) in the uniform case. Notice that for finite-dimensional continuous-time systems, it was shown in [1] that this additional property is necessary and sufficient for forward completeness.

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