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On the Observer Problem for Discrete-Time Control Systems

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Abstract— **This work studies the construction of observers for nonlinear time-varying discrete-time systems in a general context, where a certain function of the states must be estimated. Appropriate notions of robust complete observability are proposed, under which a constructive proof of existence of an observer is developed. Moreover, a "transitive observability property" is proven, with which a state observer can be generated as the series connection of two observers. The analysis and the results are developed in normed linear spaces, to cover both finite-dimensional and infinite-dimensional systems.**

I. INTRODUCTION

The observer design problem for nonlinear discrete-time systems with or without inputs has attracted the interest of many researchers and important results on the nonlinear observer design problem can be found in [1,5,6,8,13,16,17,20,24]. The observer design problem for the linear case for finite-dimensional discrete-time systems is now well understood (see the corresponding results provided in [22] and references therein), but remains still an open problem for infinite-dimensional systems. In [20] a Newton iteration approach was used for the solution of a set of nonlinear algebraic equations, which provided estimates of the states. The applications of solutions to the nonlinear observer design problem for discrete-time systems are widespread (see for instance [12,15,18] for applications to the output feedback stabilization problem). Finally, it should be emphasized the close relation of the nonlinear observer design problem for the discrete-time case with the corresponding problem in the continuous-time case when sampling is introduced (see for instance [2,9]) as well as with the observability problem for discrete-time systems (see for instance [19,22,23]).

In the present paper, the focus will be on the functional observer design problem, where the objective is to estimate a given nonlinear function of the state vector, instead of the entire state vector. Functional observers have been studied extensively for continuous-time linear systems (see [7] and references therein), and have been applied to the problem of output feedback controller synthesis, where the function to be estimated is the state feedback function.

In order to provide an informal introduction to the general ideas of the present paper, let's consider a finite-dimensional discrete-time system of the form:

$$\begin{aligned} & x(t+1) = f(t, x(t), u(t)) \\ & y(t) = h(t, x(t)) \\ & x(t) \in \Re^n \ , \ y(t) \in \Re^k \ , \ u(t) \in \Re^m \ , \ t \in Z^+ \end{aligned} \tag{1.1}$$

where $y(t)$ represents the measurement. Moreover, consider another output

$$Y(t) = \theta(t, x(t)) \tag{1.2}$$

where $\theta : Z^+ \times \Re^n \to \Re^l$ a given function of the states, and the objective is to design a functional observer to estimate the output Y, driven by the measurement of y.

In order to study the functional observer design problem, appropriate notions of (functional) observability must be introduced and, subsequently, the existence of a functional observer must be established. Motivated by problems of inferential control, where the dynamic system can be subject to unknown external disturbances, notions of robust complete (functional) observability will be introduced in the present paper, for the case where the right hand side of (1.1) also depends on unknown disturbances $d(t)$. Moreover, the analysis will be performed with the inputs, states and outputs belonging to subsets of normed linear spaces (not necessarily finite-dimensional). Infinite-dimensional discrete-time systems are currently attracting interest in the literature [3,21].

The development of the solution of the functional observer problem is constructive, leading to a set of conditions that determine the functional observer. In addition to inferential control applications, the proposed functional observer can be useful for the construction of a regular full-state observer. In particular, when a full-state observer is available for the output map $Y = \theta(t, x)$ (that has been designed using any available design method from the literature), and the proposed functional observer is

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constructed for the estimation of $Y=\theta(t,x)$ from the measured output $y=h(t,x)$, we show that their series connection generates a full-state observer from the measurement $y=h(t,x)$. We call this important property the "transitive observability property". This property can facilitate the observer design, since certain output maps may be more convenient for the design of an observer than the actual measured outputs.

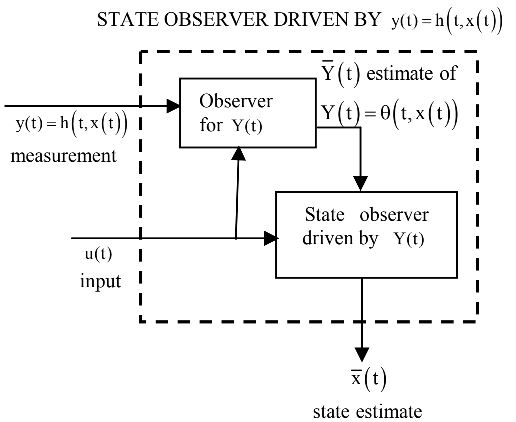

Figure 1: The Transitive Observability Property

Notations:

* By $\| \ \|_{\mathcal{X}}$, we denote the norm of the normed linear space $\mathcal{X}$. By $| \ |$ we denote the euclidean norm of $\Re^n$.
* We denote by Z^+ the set of non-negative integers and by $\Re^+$ the set of non-negative real numbers.
* Let $\mathcal{U}$ a normed linear space and let $u\in\mathcal{U}$ and $r\geq 0$. We denote by $B_{\mathcal{U}}[u,r]$ the closed ball of radius $r\geq 0$ centered at $u\in\mathcal{U}$, i.e., $B_{\mathcal{U}}[u,r]:=\left\{ v\in\mathcal{U}\,;\|v-u\|_{\mathcal{U}}\leq r \right\}$.
* We denote by $CU(Z^+\times A\,;\Theta)$, where A is a subset of the normed linear space $\mathcal{X}$ and Θ is a normed linear space, the class of continuous maps $\Psi: Z^+\times A\to\Theta$ with the properties that: (i) for every pair of bounded sets $I\subset Z^+$, $O\subset A$ the image set $\Psi(I\times O)$ is bounded and (ii) $\Psi(t,0)=0$ for all $t\in Z^+$.
* We denote by M_U the set of sequences with values in U.
* For definitions of classes K, K_∞, KL see [14]. K^+ denotes the class of positive continuous functions.

The following convention holds throughout the paper: the Cartesian product of two normed linear spaces $C:=\mathcal{X}\times\mathcal{Y}$ is a normed linear space with norm $\|(x,y)\|_C:=\sqrt{\|x\|_{\mathcal{X}}^2+\|y\|_{\mathcal{Y}}^2}$, unless stated otherwise.

II. DEFINITIONS AND PRELIMINARY RESULTS ON DISCRETE-TIME SYSTEMS WITH INPUTS

Consider the following discrete-time system:

$$x(t+1)=f(t,d(t),x(t),u(t)) \tag{2.1a}$$

$$y(t)=h(t,x(t))$$

$$x(t)\in\mathcal{X}\,,\ d(t)\in D\,,\ y(t)\in S\subseteq\mathcal{Y} \tag{2.1b}$$

$$u(t)\in U\subseteq\mathcal{U}\,,\ t\in Z^+$$

where $\mathcal{X},\mathcal{Y},\mathcal{U}$ is a triplet of normed linear spaces, D is the set of disturbances (or time-varying parameters), U is the set of inputs with $0\in U$, S is the output set with $0\in S$ and $f: Z^+\times D\times\mathcal{X}\times U\to\mathcal{X}$, $h: Z^+\times\mathcal{X}\to S\subseteq\mathcal{Y}$ with $f(t,d,0,0)=0$ and $h(t,0)=0$ for all $(t,d)\in Z^+\times D$. Our main assumption concerning system (2.1a) is:

(H1) *There exist functions* $a\in K_\infty$, $\gamma\in K^+$ *such that* $\|f(t,d,x,u)\|_{\mathcal{X}}\leq a\left(\gamma(t)\|x\|_{\mathcal{X}}\right)+a\left(\gamma(t)\|u\|_{\mathcal{U}}\right)$, *for all* $(t,x,d,u)\in Z^+\times\mathcal{X}\times D\times U$.

We note the following important fact for the time-varying case (2.1a):

Fact I: *System (2.1a) under hypothesis (H1) is Robustly Forward Complete (RFC) from the input* $u\in M_U$, *i.e., there exist functions* $\mu\in K^+$, $a\in K_\infty$ *and a constant* $R\geq 0$ *such that for every* $(t_0,x_0,d,u)\in Z^+\times\mathcal{X}\times M_D\times M_U$, *the corresponding solution* $x(t)$ *of (2.1) with* $x(t_0)=x_0$ *satisfies the following estimate:*

$$\|x(t)\|_{\mathcal{X}}\leq\mu(t)\,a\left(R+\|x_0\|_{\mathcal{X}}+\sup_{\tau\in[t_0,t]}\|u(\tau)\|_U\right),\ \forall t\geq t_0 \tag{2.2}$$

Our main assumption concerning the input set U is:

(H2): *U is a convex cone.*

The following lemma shows that under hypothesis (H2) we can obtain "sharper" estimates than estimate (2.2) for the solutions of (2.1a).

Lemma 2.1 *Suppose that hypotheses (H1-2) are fulfilled for system (2.1a). Then there exist functions* $\mu\in K^+$, $a\in K_\infty$ *such that for every* $(t_0,x_0,d,u)\in Z^+\times\mathcal{X}\times M_D\times M_U$, *the corresponding solution* $x(t)$ *of (2.1) with* $x(t_0)=x_0$ *satisfies estimate (2.2) with* $R=0$, *i.e.,*

$$\|x(t)\|_{\mathcal{X}} \le \mu(t)\, a\left(\|x_0\|_{\mathcal{X}} + \sup_{\tau\in[t_0,t]} \|u(\tau)\|_{\mathcal{U}} \right),\ \forall t \ge t_0 \quad (2.3)$$

Proof Consider the following discrete-time dynamical system:

$$\begin{aligned} & x(t+1) = \tilde{f}(t, \tilde{d}(t), x(t), z(t)) \\ & z(t+1) = z(t) \\ & (x(t), z(t)) \in C\,,\ \tilde{d}(t) \in \tilde{D}\ ,\ t \in Z^+ \end{aligned} \quad (2.4)$$

where $\tilde{d} := (d, v)$, $\tilde{D} = D \times V$, $V := U \cap B_{\mathcal{U}}[0,1]$, $\tilde{f}(t, \tilde{d}, x, z) := f(t, d, x, |z| v)$, $C := \mathcal{X} \times \Re$. Notice that by virtue of hypothesis (H1) we know that there exist functions $a \in K_\infty$, $\gamma \in K^+$ such that $\|f(t,d,x,u)\|_{\mathcal{X}} \le a(\gamma(t)\|x\|_{\mathcal{X}}) + a(\gamma(t)\|u\|_{\mathcal{U}})$, for all $(t, x, d, u) \in Z^+ \times \mathcal{X} \times D \times U$, which directly implies the following inequality:

$$\left\|\tilde{f}(t,\tilde{d},x,z)\right\|_C \le \tilde{a}\left(\tilde{\gamma}(t)\|(x,z)\|_C\right),$$
$$\text{for all } (t,x,z,\tilde{d}) \in Z^+ \times \mathcal{X} \times \Re \times \tilde{D} \quad (2.5)$$

where $\tilde{a}(s) := 2a(s) + s \in K_\infty$ and $\tilde{\gamma}(t) := \gamma(t) + 1 \in K^+$. We first show that system (2.4) is Robustly Forward Complete in the sense described in [11]. Particularly, this follows by considering arbitrary $r \ge 0$, $T \in Z^+$, then defining recursively the sequence of sets in C by $A(k) := f([0,2T] \times \tilde{D} \times A(k-1))$ for $k = 1,...,T$ with $A(0) := \left\{(x,z) \in \mathcal{X} \times \Re\,;\, \|(x,z)\|_C \le r\right\}$, which are bounded by virtue of inequality (2.5) and finally noticing that

$$\left\{ \begin{aligned} & (x(t_0+k, t_0, x_0, z_0; \tilde{d}), z_0)\,;\ \|(x_0,z_0)\|_C \le r\,, \\ & t_0 \le T\,,\, k \le T\,,\, \tilde{d} \in M_{\tilde{D}} \end{aligned} \right\} \subseteq A(k)$$

for all $k = 0,...,T$, where $x(t, t_0, x_0, z_0; \tilde{d})$ denotes the x-component of the unique solution of (2.4) initiated from $(x_0, z_0) \in C$ at time $t_0 \ge 0$ and corresponding to $\tilde{d} \in M_{\tilde{D}}$ (notice that the z-component of the unique solution of (2.4) satisfies $z(t) = z_0$ for all $t \ge t_0$).

We next show that $0 \in \mathcal{X} \times \Re$ is a robust equilibrium point for (2.4) in the sense described in [11]. It suffices to show that for every $\varepsilon > 0$, $N \in Z^+$ and $T \in Z^+$ there exists $\delta := \delta(\varepsilon, N, T) \in (0, \varepsilon]$ such that:

$$\begin{aligned} & \|(x_0,z_0)\|_C \le \delta\,,\, t_0 \le T\ \Rightarrow \\ & \sup\left\{ \left\|(x(t_0+k,t_0,x_0,z_0;\tilde{d}),z_0)\right\|_C ;\, 0 \le k \le N\,,\, \tilde{d} \in M_{\tilde{D}} \right\} \le \varepsilon \end{aligned}$$

where $x(t, t_0, x_0, z_0; \tilde{d})$ denotes the x-component of the unique solution of (2.4) initiated from $(x_0, z_0) \in C$ at time $t_0 \ge 0$ and corresponding to $\tilde{d} \in M_{\tilde{D}}$ (notice that the z-component of the unique solution of (2.4) satisfies $z(t) = z_0$ for all $t \ge t_0$). We prove this fact by induction on $N \in Z^+$. First notice that the fact holds for $N = 0$ (by selecting $\delta(\varepsilon, 0, T) = \varepsilon$). We next assume that the fact holds for some $N \in Z^+$ and we prove it for the next integer $N+1$. In order to have $\left\|(x(t_0+N+1,t_0,x_0,z_0;\tilde{d}),z_0)\right\|_C \le \varepsilon$, by virtue of inequality (2.5) it suffices to have $\left\|(x(t_0+N,t_0,x_0,z_0;\tilde{d}),z_0)\right\|_C \le \rho(\varepsilon,N,T) := \dfrac{\tilde{a}^{-1}(\varepsilon)}{\max\{\tilde{\gamma}(t)\,;\, 0 \le t \le T+N\}}$.

It follows that the selection $\delta(\varepsilon,N+1,T) := \min\{\delta(\varepsilon,N,T), \delta(\rho(\varepsilon,N,T),N,T)\} > 0$ guarantees that $\sup\left\{\left\|(x(t_0+k,t_0,x_0,z_0;\tilde{d}),z_0)\right\|_C ;\, 0 \le k \le N\,,\, \tilde{d} \in M_{\tilde{D}}\right\} \le \varepsilon$ and $\left\|(x(t_0+N+1,t_0,x_0,z_0;\tilde{d}),z_0)\right\|_C \le \varepsilon$, for all $\|(x_0,z_0)\|_C \le \delta$, $t_0 \le T$.

It follows from Lemma 3.5 in [11] that there exist functions $\mu \in K^+$, $a \in K_\infty$ such that for every $(x_0, z_0) \in \mathcal{X} \times \Re$, $\tilde{d} \in M_{\tilde{D}}$, $t_0 \in Z^+$, the solution $(x(t), z(t))$ of (2.4) with initial condition $(x(t_0), z(t_0)) = (x_0, z_0)$ and corresponding to input $\tilde{d} \in M_{\tilde{D}}$ satisfies:

$$\|(x(t), z(t))\|_C \le \mu(t)\, a\left(\|x_0\|_{\mathcal{X}} + |z_0|\right),\ \forall t \ge t_0 \quad (2.6)$$

Finally, notice that that for every $x_0 \in \mathcal{X}$, $t_0 \in Z^+$, $t \in Z^+$ with $t \ge t_0$, $(u, d) \in M_U \times M_D$, the unique solution $x(t)$ of (2.1a), with initial condition $x(t_0) = x_0$ and corresponding to $(u, d) \in M_U \times M_D$ coincides on the interval $[t_0, t]$ with the component $x(t)$ of the solution of (2.4) with initial condition $(x(t_0), z(t_0)) = (x_0, \sup_{\tau\in[t_0,t]} \|u(\tau)\|_{\mathcal{U}})$ and corresponding to $(d, v) \in M_{\tilde{D}}$, where

$v(\tau) := \dfrac{u(\tau)}{\sup_{s\in[t_0,t]} \|u(s)\|_{\mathcal{U}}} \in V$, if $\sup_{\tau\in[t_0,t]} \|u(\tau)\|_{\mathcal{U}} > 0$ and $\tau \in [t_0, t]$ and $v(\tau) = 0 \in V$ if otherwise. This observation, in conjunction with inequality (2.6) gives the desired estimate (2.3). The proof is complete. ◁

We suppose that the output map of system (2.1) satisfies:

(H3): *The output map* $h : Z^+ \times \mathcal{X} \to S \subseteq \mathcal{Y}$ *is of class* $CU(Z^+ \times \mathcal{X}\,; \mathcal{Y})$ *(see notations).*

The following lemma is an important tool for the derivation of estimates of the solutions of discrete-time systems and will be used extensively in the next section of the present paper. Its proof is omitted for space reasons.

Lemma 2.2 *Suppose that hypotheses (H1-3) are fulfilled for system (2.1). Then the following statements are equivalent:*

(i) *System (2.1) satisfies the* ***Robust Output Attractivity Property****, i.e. for every* $\varepsilon > 0$, $T \in Z^+$ *and* $R \geq 0$, *there exists* $\tau := \tau(\varepsilon, T, R) \in Z^+$, *such that for every* $x_0 \in \mathcal{X}$, $(u,d) \in M_U \times M_D$, *with* $\|x_0\|_{\mathcal{X}} + \sup_{\tau \in Z^+} \|u(\tau)\|_{\mathcal{U}} \leq R$, $t_0 \in [0, T]$, *the solution* $x(t)$ *of (2.1) with initial condition* $x(t_0) = x_0$ *and corresponding to input* $(u,d) \in M_U \times M_D$ *satisfies:*

$$\|h(t, x(t))\|_{\mathcal{Y}} \leq \varepsilon, \ \forall t \in [t_0 + \tau, +\infty)$$

(ii) *There exist functions* $\sigma \in KL$ *and* $\beta \in K^+$, *such that for every* $x_0 \in \mathcal{X}$, $t_0 \in Z^+$, $(u,d) \in M_U \times M_D$, *the unique solution* $x(t)$ *of (2.1), with initial condition* $x(t_0) = x_0$ *and corresponding to* $(u,d) \in M_U \times M_D$, *satisfies for all* $t \in [t_0, +\infty)$:

$$\|h(t, x(t))\|_{\mathcal{Y}} \leq \sigma\left(\beta(t_0)\left(\|x_0\|_{\mathcal{X}} + \sup_{\tau \in [t_0, t]} \|u(\tau)\|_{\mathcal{U}} \right), t - t_0 \right) \quad (2.7)$$

Finally, the following definitions introduce the notions of estimators and observers for discrete-time systems.

Definition 2.3 *Let* $\theta : Z^+ \times \mathcal{X} \times U \to \Theta$, *where* Θ *is a normed linear space with* $\theta(\cdot, 0, 0) = 0$ *and* $q \in K^+$ *with* $\inf_{t \geq 0} q(t) > 0$. *Let* $\mathcal{Z}$ *a normed linear space and consider the system:*

$$\begin{aligned} z(t+1) &= g(t, z(t), h(t, x(t)), u(t)) \\ \bar{\theta}(t) &= \Psi(t, z(t), h(t, x(t)), u(t)) \end{aligned} \quad (2.8)$$

where $g : Z^+ \times \mathcal{Z} \times S \times U \to \mathcal{Z}$, $\Psi : Z^+ \times \mathcal{Z} \times S \times U \to \Theta$ *with* $g(t, 0, 0, 0) = 0$ *and* $\Psi(t, 0, 0, 0) = 0$ *for all* $t \in Z^+$. *Suppose that there exist functions* $\sigma \in KL$ *and* $\beta \in K^+$, *such that for every* $(x_0, z_0) \in \mathcal{X} \times \mathcal{Z}$, $t_0 \in Z^+$, $(u,d) \in M_U \times M_D$, *the unique solution of (2.1) and (2.8) with initial condition* $(x(t_0), z(t_0)) = (x_0, z_0)$ *corresponding to inputs* $(u,d) \in M_U \times M_D$, *satisfies for all* $t \in [t_0, +\infty)$:

$$q(t)\|\theta(t, x(t), u(t)) - \Psi(t, z(t), y(t), u(t))\|_{\Theta} \leq \sigma\left(\beta(t_0)\left(\|x_0\|_{\mathcal{X}} + \|z_0\|_{\mathcal{Z}} + \sup_{\tau \in [t_0, t]} \|u(\tau)\|_{\mathcal{U}} \right), t - t_0 \right) \quad (2.9)$$

Then system (2.8) is called a Robust $q-$***Estimator*** *for* θ *with respect to (2.1). System (2.8) is called a Robust* $q-$***Estimator*** *for system (2.1) if* θ *is the identity map* $\theta(t, x, u) := x$. *If* $q(t) \equiv 1$, *then (2.8) is simply called a Robust* ***Estimator*** *for* θ *with respect to (2.1). In any case, the map* $\Psi : Z^+ \times \mathcal{Z} \times S \times U \to \Theta$ *is called the reconstruction map of the Robust* ($q-$)*Estimator for* θ *with respect to (2.1)*.

Definition 2.4 *Let* $\theta : Z^+ \times \mathcal{X} \times U \to \Theta$, *where* Θ *is a normed linear space with* $\theta(\cdot, 0, 0) = 0$ *and* $q \in K^+$ *with* $\inf_{t \geq 0} q(t) > 0$. *Suppose that (2.8) is a* $q-$*estimator for* θ *with respect to (2.1). Moreover, suppose that (2.8) satisfies the* ***Consistent Initialization Property****, i.e., for every* $(t_0, x_0) \in Z^+ \times \mathcal{X}$ *there exists* $z_0 \in \mathcal{Z}$ *such that the unique solution of (2.1) and (2.8) with initial condition* $(x(t_0), z(t_0)) = (x_0, z_0)$ *and corresponding to arbitrary* $(d,u) \in M_D \times M_U$, *satisfies for all* $(d,u) \in M_D \times M_U$

$$\theta(t, x(t), u(t)) = \Psi(t, z(t), y(t), u(t)), \ \forall t \geq t_0 \quad (2.10)$$

Then we say that system (2.8) is a robust global $q-$*observer for* θ *with respect to (2.1). If* $q(t) \equiv 1$ *then we say that system (2.8) is a robust global observer for* θ *with respect to (2.1). If* θ *is the identity map* $\theta(t, x, u) := x$, *then we say that system (2.8) is a robust global* $q-$*observer for (2.1).*

Remark 2.5 We next discuss the consequences of Definition 2.4.

i) Notice that according to Definition 2.4, an observer for (2.1) guarantees convergence of the estimates of the states only for bounded inputs $u \in M_U$. The phenomenon that the state estimates of an observer converge to the states for bounded inputs and not necessarily for all possible inputs is a purely nonlinear one. For linear finite-dimensional systems with linear identity observers this phenomenon cannot happen. Moreover, it should be emphasized that there exist more demanding versions of the notion of an observer (see for example [22]), where convergence for all inputs is required.

ii) The notion of state observer given by Definition 2.4 is weaker than the one of Definition 7.1.3 in [22], even when we consider autonomous systems without inputs and full order autonomous observers because Definition 2.4 does not guarantee the "Lyapunov stability" property for the error $e=\Psi(t,z,y,u)-x$: i.e. if the initial value for the error $e_0=\bar{x}_0-x_0$ is "sufficiently small", we cannot guarantee that all future values of the error will be "small". This stronger property would be satisfied if instead of (2.9) with $\theta(t,x,u)\equiv x$ and $q(t)\equiv 1$ the following estimate were satisfied for the solution $(x(\cdot),z(\cdot))$ of system (2.1) with (2.8) for all $t\ge t_0$:

$$\left|\Psi(t,z(t),y(t),u(t))-x(t)\right|=\left|e(t)\right|\le\sigma\left(\beta(t_0)\left|e(t_0)\right|,t-t_0\right) \quad (2.9)'$$

Instead, our asymptotic property (2.9) guarantees that if the initial condition (z_0,x_0) and the input u are "sufficiently small", then all future values of the error will be "small". For example, if zero is a non-uniformly in time globally asymptotically stable equilibrium point for the system without inputs $x(t+1)=f(t,x(t))$, then according to Definition 2.4 the system $z(t+1)=f(t,z(t))$, $\bar{x}=z$ is a Global Identity Observer. On the other hand, if the definition of the observer were based on (2.9)' instead of (2.9), then the system $z(t+1)=f(t,z(t))$, $\bar{x}=z$ would not be an observer for the system $x(t+1)=f(t,x(t))$, unless the system $x(t+1)=f(t,x(t))$ had special structure (e.g. linear systems).

III. Main Results

In order to be able to state our main results, it is necessary to introduce the notion of Robust (Strong) Complete Observability. The definition of the notion of robust (strong) complete observability for discrete-time systems directly extends the corresponding notions given in [10] for continuous-time autonomous systems.

Definition 3.1 *Consider system (2.1) and let $(d_i,u_i)\in D\times U$, $i=0,1,...$ and define recursively the following family of mappings:*

$$F_0(t,x)=x\,,\; F_1(t,x,d^{(1)},u^{(1)})=f(t,d_0,x,u_0)$$
$$\left.\begin{array}{l}F_i(t,x,d^{(i)},u^{(i)}):=\\ f(t+i-1,d_{i-1},F_{i-1}(t,x,d^{(i-1)},u^{(i-1)}),u_{i-1})\end{array}\right.,\; i\ge 2$$

$$y_0(t,x)=h(t,x)\,,$$
$$y_i(t,x,d^{(i)},u^{(i)}):=h(t+i,F_i(t,x,d^{(i)},u^{(i)}))\,,\; i\ge 1$$

where $d^{(i)}:=(d_0,...,\,d_{i-1})$, $u^{(i)}:=(u_0,...,\,u_{i-1})$ for $i\ge 1$. Let an integer $p\ge 1$ and define the following mapping for all $(t,x,d^{(p)},u^{(p)})\in Z^+\times X\times D^p\times U^p$:

$$y^{(p+1)}(t,x,d^{(p)},u^{(p)}):=$$
$$(\,y_0(t,x),...,\,y_{p-1}(t,x,d^{(p-1)},u^{(p-1)}),y_p(t,x,d^{(p)},u^{(p)}))$$

We say that a continuous function $\theta\in CU(Z^+\times X;\Theta)$ where Θ is a normed linear space, is ***robustly completely observable from the output*** *$y=h(t,x)$* ***with respect to (2.1)*** *if there exists an integer $p\in Z^+$ and a continuous function (called the reconstruction map) $\Psi\in CU(Z^+\times U^p\times S^p\times S;\Theta)$, such that for all $(t,x,d^{(p)},u^{(p)})\in Z^+\times X\times D^p\times U^p$ it holds that*

$$\begin{aligned}&\theta(t+p,F_p(t,x,d^{(p)},u^{(p)}))\\&=\Psi(t+p,u^{(p)},y^{(p+1)}(t,x,d^{(p)},u^{(p)}))\end{aligned} \quad (3.1)$$

Furthermore, we say that $\theta\in CU(Z^+\times X;\Theta)$ is ***robustly strongly completely observable from the output*** *$y=h(t,x)$* ***with respect to (2.1)*** *if θ is robustly completely observable from the output $y=h(t,x)$ with respect to (2.1) and for every $t\in Z^+$, $x\in X$ there exists $w=(w_1,...,w_p,w_{p+1},...,w_{2p})\in U^p\times S^p$ such that for all $(d^{(p-1)},u^{(p-1)})\in D^{p-1}\times U^{p-1}$ and $i=1,...,p-1$ it holds that*

$$\begin{aligned}&\theta(t+i,F_i(t,x,d^{(i)},u^{(i)}))=\\&\Psi(t+i,w_1,...,w_{p-i},u^{(i)},w_p,...,w_{2p-i},y^{(i+1)}(t,x,d^{(i)},u^{(i)}))\end{aligned} \quad (3.2a)$$

$$\theta(t,x)=\Psi(t,w_1,...,w_p,w_{p+1},...,w_{2p},h(t,x)) \quad (3.2b)$$

We say that system (2.1) is ***robustly (strongly) completely observable from the output*** *$y=h(t,x)$ if the identity function $\theta(t,x)=x$ is robustly (strongly) completely observable from the output $y=h(t,x)$ with respect to (2.1).*

Remark 3.2: Notice that for every input $(d,u)\in M_D\times M_U$ and for every $(t_0,x_0)\in Z^+\times X$, the unique solution $x(t)$ of (2.1) corresponding to (d,u) and initiated from x_0 at time t_0, satisfies the following relation for all $t\ge t_0+p$:

$$\begin{aligned}&\theta(t,x(t))=\\&\Psi(t,u(t-p),...,u(t-1),y(t-p),y(t-p+1),...,y(t-1),y(t))\end{aligned}$$

The following proposition provides sufficient conditions for the solvability of the robust $q-$Estimator problem for (2.1).

Proposition 3.3 *Let $\theta\in CU(Z^+\times X;\Theta)$ where Θ is a normed linear space. Suppose that $\theta\in CU(Z^+\times X;\Theta)$ is robustly completely observable from the output $y=h(t,x)$ with respect to (2.1) and that in addition to (H1-3) the following hypothesis also holds:*

(H4) *There exist functions* $a \in CU(\mathcal{Y};S)$, $b \in CU(\mathcal{U};U)$, *such that for every* $y \in S$, $u \in U$, *it holds that* $a(y) = y$ *and* $b(u) = u$.

Then for every $q \in K^+$ *with* $\inf_{t \geq 0} q(t) > 0$ *the robust global* $q-$*Estimator problem for* θ *with respect to (2.1) is solvable.*

We are now in a position to state our main result.

Theorem 3.4 (Transitive Observability Property) *Let* $\theta \in CU(Z^+ \times \mathcal{X}; \Theta)$ *and* $q \in K^+$ *with* $\inf_{t \geq 0} q(t) > 0$. *Suppose that:*

(A1) θ *is robustly strongly completely observable from the output* $y = h(t,x)$ *with respect to (2.1)*

(A2) *The robust global* $q-$*observer problem for (2.1a) with output* $y = \theta(t,x)$ *is solvable. Particularly, there exists* $\varphi : Z^+ \times \mathcal{Z} \times \Theta \times U \to \mathcal{Z}$, $\widetilde{\Psi} \in CU(Z^+ \times \mathcal{Z} \times \Theta; \mathcal{X})$, *functions* $p \in K_\infty$, $q \in K^+$ *such that* $\|\varphi(t,z,\theta,u)\|_{\mathcal{Z}} \leq p\left(q(t)\|z\|_{\mathcal{Z}}\right) + p\left(q(t)\|\theta\|_{\Theta}\right) + p\left(q(t)\|u\|_{\mathcal{U}}\right)$ *for all* $(t,z,\theta,u) \in Z^+ \times \mathcal{Z} \times \Theta \times U$ *in such a way that the following system*

$$z(t+1) = \varphi(t, z(t), \theta(t, x(t)), u(t))$$
$$\bar{x}(t) = \widetilde{\Psi}(t, z(t), \theta(t, x(t)))$$
$$z(t) \in \mathcal{Z},\ \bar{x}(t) \in \mathcal{X},\ t \in Z^+$$

is a robust global $q-$*observer for (2.1a) with output* $y = \theta(t,x)$.

Then under hypotheses (H1-4), the robust global $q-$*Observer problem for (2.1) with output* $y = h(t,x)$ *is solvable.*

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