Proceedings of the
44th IEEE Conference on Decision and Control, and
the European Control Conference 2005
Seville, Spain, December 12-15, 2005

Existence and Uniqueness of Solutions for Duncan-Mortensen-Zakai Equations

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Joint Conference on Decision and Control, and European Control Conference
December 12-15, Seville, Spain

***Abstract*— In this paper, we give a priori estimation of derivatives of solution of Duncan-Mortensen-Zakai (DMZ) equation up to 2nd order. We also prove the existence and uniqueness of weak solution for DMZ equation, which is a fundamental equation in nonlinear filtering.**

1. Introduction

In nonlinear filtering theory, the central problem is to understand the Duncan-Mortensen-Zakai (DMZ) equation which is a time dependent parabolic equation of the following form:

$$\frac{\partial u}{\partial t} = \Delta u + \sum_{i=1}^{n} F_i \frac{\partial u}{\partial x_i} - Vu, \tag{1.1}$$

where F_i and V are time dependent functions. Despite of many beautiful works by distinguished engineers and mathematicians on DMZ equation, the existence and uniqueness of the solution for DMZ equation are still not well understood. The main difficulty is that F_i and V may be unbounded on $\mathbb{R}^n$. Strook and Norris [St-No] treated the case with bounded coefficients F_i and V. Fleming and Mitter [Fl-Mi], Sussmann [Su], Baras-Blankenship-Hopkins [B-B-H] have obtained important estimates on the robust DMZ equation. However, the latter two papers are focused only on one spatial dimension while in the first paper the drift term of the nonlinear filtering is assumed to be bounded. In [Ya-Ya], we demonstrated the existence of the solution of DMZ equation if the drift term $f(x)$ and observation terms $h(x)$ in (2.1) have linear growth. This result was new even in one spatial dimension case. The purpose of this paper is to show that under very mild conditions (which essentially say that the growth of $|h|$ is greater than the growth of $|f|$), the DMZ equation admits a unique nonnegative solution $u \in W_0^{1,1}((0,T)\times\mathbb{R}^n)$.

*Research partially supported by U.S. Army Research Office

2. Preliminaries

The nonlinear filtering problem considered here is based on the following signal observation model

$$\begin{cases} dx(t) = f(x(t))dt + g(x(t)) & dv(t) \\ & x(0) = x_0 \\ dy(t) = h(x(t))dt + dw(t) & y(0) = 0 \end{cases} \tag{2.1}$$

in which x, v, y and w are, respectively, $\mathbb{R}^n, \mathbb{R}^p, \mathbb{R}^m, \mathbb{R}^m$ valued processes, and v and w have components which are independent, standard Brownian processes. We further assume that $n = p, f$, and h are smooth, and that g is an orthogonal matrix. (2.1) is a pair of the stochastic differential equations.

Let $\rho(t,x)$ denote the conditional probability density of the state given the observation $\{y(s) : 0 \le s \le t\}$. It is well known that $\rho(t,x)$ is given by normalizing a function $\sigma(t,x)$ which satisfies the following Duncan-Mortensen-Zakai equation:

$$d\sigma(t,x) = L_0\sigma(t,x)dt + \sum_{i=1}^{m} L_i\sigma(t,x)dy_i(t) \tag{2.2}$$

where $\sigma(0,x) = \sigma_0$, $L_0 = \frac{1}{2}\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2} - \sum_{i=1}^{n} f_i\frac{\partial}{\partial x_i} - \sum_{i=1}^{n}\frac{\partial f_i}{\partial x_i} - \frac{1}{2}\sum_{i=1}^{m} h_i^2$ and for $i = 1,\dots,m$, L_i is the zero degree differential operator of multiplication by h_i. We have used the Fisk-Stratonvich version of the stochastic calculus in writing (3.2).

In real applications, we are interested in constructing robust state estimators from observed sample paths with some property robustness. Davis [Da] studied this problem and he introduced a new unnormalized density

$$u(t,x) = \exp\left(-\sum_{i=1}^{n} h_i(x)y_i(t)\right)\sigma(t,x). \tag{2.3}$$

It is easy to show that $u(t,x)$ satisfies the following parabolic partial differential equation.

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = L_0 u(t,x) + \sum_{i=1}^{n} y_i(t)[L_0, L_i]u(t,x) \\ +\frac{1}{2}\sum_{i,j=1}^{m} y_i(t)y_j(t)[[L_0, L_i], L_j]u(t,x) \\ u(0,x) = \sigma_0 \end{cases} \tag{2.4}$$

where $[\cdot\ ,\ \cdot]$ denotes the Lie bracket. It is shown in [Ya-Ya, p.236] that the robust DMZ equation (2.4) is of the form

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = \frac{1}{2}\Delta u(t,x) + (-f(x) \\ \qquad +\nabla K(t,x)) \cdot \nabla u(t,x) \\ \qquad +\big(-\mathrm{div} f(x) - \frac{1}{2}|h(x)|^2 \\ \qquad +\frac{1}{2}\Delta K(t,x) - f(x)\cdot\nabla K(t,x) \\ \qquad +\frac{1}{2}|\nabla K(t,x)|^2\big)u(t,x) \\ u(0,x) = \sigma_0(x), \end{cases} \tag{2.5}$$

where $K = \sum_{j=1}^{m} y_j(t)h_j(x)$, $f = (f_1, \ldots, f_n)$ and $h = (h_1, \ldots, h_m)$.

3. A PRIORI ESTIMATION OF DERIVATIVES UP TO SECOND ORDERS

In this section, we shall give a priori estimation of zero, first and second derivatives of the solution of robust DMZ equation on $[0,T]\times B_R$, where B_R is a ball of radius R centered at origin.

Theorem 3.1: Consider the robust DMZ equation on $[0,T]\times B_R$, where $B_R = \{x\in\mathbb{R}^n : |x| \le R\}$ is a ball of radius R,

$$\begin{cases} \frac{\partial u_R}{\partial t}(t,x) = \frac{1}{2}\Delta u_R(t,x) - (f(x) \\ \qquad -\nabla K(t,x))\cdot\nabla u_R(t,x) \\ \qquad -\big[\frac{1}{2}|h(x)|^2 + \mathrm{div}\ f(x) - \frac{1}{2}\Delta K(t,x) \\ \qquad +f(x)\cdot\nabla K(t,x) \\ \qquad -\frac{1}{2}|\nabla K(t,x)|^2\big]u_R(t,x) \\ u_R(t,x)\Big|_{\partial B_R} = 0 \\ u_R(0,x) = \sigma_0(x), \end{cases} \tag{3.1}$$

where $K(t,x) = \sum_{i=1}^{m} y_i(t)h_i(x)$, $\nabla K(t,x) = \big(\frac{\partial K}{\partial x_1}(t,x), \ldots, \frac{\partial K}{\partial x_n}(t,x)\big)$ and $\Delta K(t,x) = \sum_{i=1}^{n}\frac{\partial^2 K}{\partial x_i^2}(t,x)$. Let $C_1 = \max_{0\le t\le T}\big[\sum_{i=1}^{m}|y_i(t)|^2\big]^{\frac{1}{2}}$ be the smallest constant such that

$$|\nabla K(t,x)| \le C_1|\nabla h(x)| \text{ for } (t,x)\in[0,T]\times B_R \tag{3.2}$$

where $|\nabla h|^2 = \sum_{i=1}^{m}|\nabla h_i(x)|^2$.

Suppose that there exists a constant $C > 0$ such that for any $r \ge 0$

$$\min_{|x|=r}\frac{|h|^2 + \mathrm{div} f + C}{\sqrt{|f|^2 + |h|^2 + \mathrm{div} f + C} + |f|} - C_1\max_{|x|=r}|\nabla h| \ge 0. \tag{3.3}$$

Let $g(x)$ be a positive radial symmetric function on $\mathbb{R}^n$ (i.e. $g = g(r)$ where $r = |x| = \big(\sum_{i=1}^{n} x_i^2\big)^{\frac{1}{2}}$ such that

$$|g'(r)| \le \min_{|x|=r}\frac{|h|^2 + \mathrm{div} f + C}{\sqrt{|f|^2 + |h|^2 + \mathrm{div} f + C} + |f|} - C_1\max_{|x|=r}|\nabla h|. \tag{3.4}$$

Then, for $0 \le t \le T$,

$$\int_{B_R} e^{2g}u_R^2(t,x) \le e^{Ct}\int_{B_R} e^{2g}\sigma_0^2(x). \tag{3.5}$$

Remark 3.2: Theorem 3.1 gives a priori estimation of the solution of the robust DMZ equation on $[0,T]\times B_R$. Notice that from (3.4), if h grows fast, then we can allow g to grow fast.

We must next give a priori estimation of the first derivatives and second derivatives of the solution of the robust DMZ equation on $[0,T]\times B_R$. We first observe that for estimation of second derivatives it is sufficient to estimate the Laplacian of the solution.

Lemma 3.3: Let ρ be a smooth function with compact support in B_R. Let u_R be the solution of (3.1). Then

$$\int_{B_R}\sum_{i,j=1}^{n}\rho^2(u_R)_{ji}^2 \le 4\int_{B_R}\rho^2(\Delta u_R)^2 + 6\sup|\nabla\rho|^2\int_{B_R}|\nabla u_R|^2. \tag{3.6}$$

Now we are ready to give a priori estimation of the first and second derivatives of the solution of the robust DMZ equation on $[0,T]\times B_R$.

Theorem 3.4: Consider the robust DMZ equation (3.1) on $[0,T]\times B_R$, where $B_R = \{x\in\mathbb{R}^n : |x| < R\}$ is a ball of radius R. Assume that

$$\sqrt{\frac{1}{2}|h|^2 + \mathrm{div} f - \frac{1}{2}\Delta K + f\cdot\nabla K - \frac{1}{2}|\nabla K|^2 + \frac{C}{2}} - |f| - |\nabla K| \ge 0, \tag{3.7}$$

where C is the constant in Theorem 3.1. Choose a nonnegative function $\widetilde{g}$ so that

$$|\nabla\widetilde{g}| \le \sqrt{\frac{1}{2}|h|^2 + \mathrm{div} f - \frac{1}{2}\Delta K + f\cdot\nabla K - \frac{1}{2}|\nabla K|^2 + \frac{C}{2}} - |f| - |\nabla K| \tag{3.8}$$

and

$$e^{2\widetilde{g}}\left|\nabla\left(\frac{1}{2}|h|^2 + \mathrm{div} f - \frac{1}{2}\Delta K + f\cdot\nabla K - \frac{1}{2}|\nabla K|^2\right)\right|^2 \le e^{2g} \tag{3.9}$$

where g is chosen as in Theorem 3.1. Then

$$\int_{B_R} e^{2\widetilde{g}}|\nabla u_R|^2(T,x) + \frac{1}{2}\int_0^T\int_{B_R} e^{2\widetilde{g}}(\Delta u_R)^2(t,x) \le \int_{B_R} e^{2\widetilde{g}}|\nabla u_R|^2(0,x) + T\int_{B_R} e^{2g}\sigma_0^2(x). \tag{3.10}$$

4. Existence of weak solution for DMZ equation

Let $Q=(0,T)\times\mathbb{R}^n$ and $L^2(Q)$ be the space of functions that are square integrable over Q. The scalar product of two elements v_1, v_2 of $L^2(Q)$ is defined by the equation

$$(v_1,v_2)=\iint\limits_Q v_1 v_2\,dx\,dt.$$

The class of C^∞ functions in $\overline{Q}$ with compact supports in Q will be denoted by $C_0^\infty(Q)$.

Definition 4.1: A locally L^2-integrable function w is called a generalized derivative of a locally L^2-integrable function $v(t,x)$ in Q with respect to x if for each $\Phi(t,x)\in C_0^\infty(Q)$ the equation

$$\iint\limits_Q \left(v\frac{\partial\Phi}{\partial x_k}+w\Phi\right)dx\,dt=0 \qquad (4.1)$$

holds. In this case, we write $w=\frac{\partial v}{\partial x_k}$. Generalized derivative with respect to t and generalized derivatives of higher order are defined similarly (see [So]).

Remark 4.2: If the sequence of functions $v_m(t,x)$ tends weakly to $v(t,x)$ in the space $L^2(Q)$ as $m\to\infty$ and the norms of $\frac{\partial v_m}{\partial x_k}$ in $L^2(Q)$ are uniformly bounded with respect to m, then $v(t,x)$ has a generalized derivative $\frac{\partial v}{\partial x_k}\in L^2(Q)$ and $\frac{\partial v_m}{\partial x_k}$ tends weakly to $\frac{\partial v}{\partial x_k}$ (see [So]).

Definition 4.3: We denote $W^1(\mathbb{R}^n)$ the space of functions $\phi(x)$ such that $\phi(x)\in L^2(\mathbb{R}^n)$ and $\frac{\partial\phi}{\partial x_i}\in L^2(\mathbb{R}^n)$ for $i=1,\dots,n$ with the scalar product

$$(\phi_1,\phi_2)_1:=\int_{\mathbb{R}^n}\phi_1(x)\phi_2(x)dx+\int_{\mathbb{R}^n}\sum_{i=1}^n\frac{\partial\phi_1}{\partial x_i}\frac{\partial\phi_2}{\partial x_i}dx. \qquad (4.2)$$

We shall denote by $W^{1,1}(Q)$ the space of functions $v(t,x)$ for which $v(t,x)\in L^2(Q)$, $\frac{\partial v(t,x)}{\partial x_i}\in L^2(Q)(i=1,\dots,n)$ and $\frac{\partial v(t,x)}{\partial t}\in L^2(Q)$, with the scalar product

$$(v_1,v_2)_{1,1}:=\iint\limits_Q v_1(t,x)v_2(t,x)dt\,dt + \iint\limits_Q\left(\sum_{i=1}^n\frac{\partial v_1}{\partial x_i}\frac{\partial v_2}{\partial x_i}+\frac{\partial v_1}{\partial t}\frac{\partial v_2}{\partial t}\right)dx\,dt. \qquad (4.3)$$

It is known [So] that $W^1(\mathbb{R}^n)$ and $W^{1,1}(\mathbb{R})$ are complete. The norms in $L^2(Q)$, $W^1(\mathbb{R}^n)$, and $W^{1,1}(Q)$ will be written $\|v\|_0$, $\|v\|_1$ and $\|v\|_{1,1}$ respectively.

Remark 4.4: It follows from the embedding theorems of S.L. Sobolev that a function of $W^{1,1}(Q)$ can be modified on a set of measure zero in such a way that it is L^2-integrable on the section of the cylinder Q by any n-dimensional plane or n-dimensional C^1 surface. In particular, such a function is L^2-integrable on the section of Q by any plane t=constant. Moreover, the values of $v(t,x)\in W^{1,1}(Q)$ on sufficiently close n-dimensional planes will differ in mean by as little as we please [So]. In particular, if $v(t,x)\in W^{1,1}(Q)$ and $v(x,0)=\phi(x)$, then $\int_Q[v(t,x)-\phi(x)]^2dx\longrightarrow 0$ as $t\longrightarrow 0$.

Definition 4.5: The subspace of $W^1(\mathbb{R}^n)$ consisting of functions that have compact supports in $\mathbb{R}^n$ is written $W_0^1(\mathbb{R}^n)$, and the subspace of $W^{1,1}(Q)$ consisting of functions $v(t,x)$ which have compact supports in $\mathbb{R}^n$ for any t is written $W_0^{1,1}(Q)$.

Definition 4.6: The function $u(t,x)$ in $W_0^{1,1}(Q)$ is called a weak solution of the initial value problem

$$\begin{cases} \sum\limits_{i,j=1}^n\frac{\partial}{\partial x_i}\big(A_{ij}(t,x)\frac{\partial u}{\partial x_j}\big)+\sum\limits_{i=1}^n B_i(t,x)\frac{\partial u}{\partial x_i} \\ \qquad +C(t,x)u=\frac{\partial u}{\partial t} \\ u(0,x)=\phi(x) \end{cases} \qquad (4.4)$$

if for any function $\Phi(t,x)\in W_0^{1,1}(Q)$ the following relation is valid:

$$\iint\limits_Q\big[\sum_{i,j=1}^n A_{ij}\frac{\partial u}{\partial x_j}\frac{\partial\Phi}{\partial x_i}-\big(\sum_{i=1}^n B_i\frac{\partial u}{\partial x_i}+Cu-\frac{\partial u}{\partial t}\big)\Phi\big]dx,dt=0 \qquad (4.5)$$

and $u(0,x)=\phi(x)$.

We now recall some facts concerning convergence in Hilbert spaces.

Remark 4.7: A sequence $\{u_m\}$, in a Hilbert space H with scalar product (,), is said to be *weakly convergent* (to u) if the sequence $\{(u_m,f)\}$ is convergent (to (u,f)) for any $f\in H$. A weakly convergent sequence is bounded. From any bounded sequence $\{u_m\}$ in H one can extract a weakly convergence subsequence. If $\{u_m\}$ is weakly convergent to u, then there exists a subsequence $\{u_{m'}\}$ whose arithmetic means converge to u in the norm H (see p. 273 of [Fr]).

Theorem 4.8: Under the hypothesis of Theorem 3.4 the robust DMZ equation on $[0,T]\times\mathbb{R}^n$

$$\begin{cases} \frac{\partial u}{\partial t}(t,x)=\frac{1}{2}\Delta u(t,x)-\Big(f(x)-\nabla K(t,x)\Big) \\ \qquad \cdot\nabla u(t,x)-\Big[\frac{1}{2}|h(x)|^2 \\ \qquad +\mathrm{div}f(x)-\frac{1}{2}\Delta K(t,x)+f(x) \\ \qquad \cdot\nabla K(t,x)-\frac{1}{2}|\nabla K(t,x)|^2\Big]u(t,x) \\ u(0,x)=\sigma_0(x) \end{cases} \qquad (4.6)$$

where $\sigma_0(x)\in W_0^1(\mathbb{R}^n)$, has a weak solution.

Proof: Let $\{R_k\}$ be a sequence of positive number such that $\lim\limits_{k\to\infty}R_k=\infty$. Let $u_k(x)$ be the solution of the robust DMZ equation on $[0,T]\times B_{R_k}$, where $B_{R_k}=\{x\in\mathbb{R}^n:$

$|x| \le R_k\}$ is a ball of radius R_k,

$$\begin{cases} \dfrac{\partial u_{R_k}}{\partial t}(t,x) = \dfrac{1}{2}\Delta u_{R_k}(t,x) - \big(f(x) - \nabla K(t,x)\big) \\ \qquad \cdot \nabla u_{R_k}(t,x) - \big[\dfrac{1}{2}|h(x)|^2 \\ \qquad + \mathrm{div} f(x) - \dfrac{1}{2}\Delta K(t,x) + f(x) \\ \qquad \cdot \nabla K(t,x) - \dfrac{1}{2}|\nabla K(t,x)|^2\big] u_{R_k}(t,x) \\ u_{R_k}(t,x)\Big|_{\partial B_{R_k}} = 0 \\ u_{R_k}(0,x) = \sigma_0(x)\Big|_{B_{R_k}}. \end{cases}$$

Let

$$u_k(t,x) = \begin{cases} u_{R_k}(t,x) & \text{if } x \in B_{R_k} \\ 0 & \text{if } x \notin B_{R_k} \end{cases},$$

$$\sigma_k(x) = \begin{cases} \sigma_0(x) & \text{if } x \in B_{R_k} \\ 0 & \text{if } x \notin B_{R_k} \end{cases}.$$

In view of Theorem 3.1 and Theorem 3.4, the sequence $\{u_k\}$ is a bounded set in $W_0^{1,1}(Q)$. By Remark 4.7, there exists a subsequence $\{u_{k'}\}$ which is weakly convergent to u. Moreover, $u(t,x)$ has generalized derivative $\dfrac{\partial u}{\partial x_i}$, $\dfrac{\partial^2 u}{\partial x_i^2} \in L^2(Q)$ and $\dfrac{\partial u_{k'}}{\partial x_i}, \dfrac{\partial^2 u_{k'}}{\partial x_i^2}$ tend weakly to $\dfrac{\partial u}{\partial x_i}, \dfrac{\partial^2 u}{\partial x_i^2}$ respectively. Now we claim that the weak derivative $\dfrac{\partial u}{\partial t}$ exists and is equal to right hand side of (4.3). To see this, let $\Phi(t,x) \in W_0^{1,1}(Q)$. Then

$$\begin{aligned} &\iint \Big[\frac{1}{2}\Delta u - (f(x) - \nabla K)\cdot \nabla u \\ &\qquad - \Big(\frac{1}{2}|h|^2 + \mathrm{div} f - \frac{1}{2}\Delta K \\ &\qquad + f\cdot\nabla K - \frac{1}{2}|\nabla K|^2\Big)u\Big]\Phi(t,x)dx\,dt \\ = &\lim_{k'\to\infty}\iint \Big[\frac{1}{2}\Delta u_{k'} - (f(x) - \nabla K)\cdot u_{k'} \\ &\qquad - \Big(\frac{1}{2}|h|^2 + \mathrm{div} f - \frac{1}{2}\Delta K \\ &\qquad + f\cdot\nabla K - \frac{1}{2}|\nabla K|^2\Big)u_{k'}\Big]\Phi(t,x)dx\,dt \\ = &\lim_{k'\to\infty}\iint \frac{\partial u_{k'}}{\partial t}\Phi(t,x)dx\,dt \\ = &-\lim_{k'\to\infty}\iint u_{k'}\frac{\partial \Phi}{\partial t}(t,x)dx\,dt \\ = &-\iint u\frac{\partial \Phi}{\partial t}(t,x)dx\,dt. \end{aligned}$$

Clearly $u(0,x) = \lim\limits_{k'\to\infty} u_{k'}(0,x) = \lim\limits_{k'\to\infty}\sigma_{k'}(x) = \sigma_0(x)$. ■

5. Uniqueness of weak solution for DMZ equation

We are now ready to establish the uniqueness of weak solution for DMZ equation. We shall follow the notations in previous sections.

Theorem 5.1: Let $Q = (0,T)\times\mathbb{R}^n$. Assume that for some $c > 0$,

$$\sup_{0\le t\le T}\int_{\mathbb{R}^n} e^{cr}u^2(t,x)dx < \infty \tag{5.1}$$

$$\int_0^T\int_{\mathbb{R}^n} e^{cr}|\nabla u(t,x)|^2 dx\,dt < \infty \tag{5.2}$$

where $r = \sqrt{x_1^2 + \cdots + x_n^2}$. Suppose that there exists a finite number α such that

$$\begin{aligned} &\Big|\frac{c}{2}\nabla r + f - \nabla K\Big|^2 - 2\Big(\frac{1}{2}|h|^2 + \mathrm{div}\ f \\ &\quad -\frac{1}{2}\Delta K + f\cdot\nabla K - \frac{1}{2}|\nabla K|^2\Big) \le \alpha. \end{aligned} \tag{5.3}$$

Then the weak solution $u(t,x)$ of the robust DMZ equation on Q

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = \dfrac{1}{2}\Delta u(t,x) - (f(x) \\ \qquad -\nabla K(t,x)) \\ \qquad \cdot\nabla u(t,x) - \Big[\dfrac{1}{2}|h(x)|^2 + \mathrm{div}\ f(x) \\ \qquad -\dfrac{1}{2}\Delta K(t,x) + f(x) \\ \qquad \cdot\nabla K(t,x) - \dfrac{1}{2}|\nabla K(t,x)|^2\Big]u(t,x) \\ u(0,x) = \sigma_0(x) \end{cases} \tag{5.4}$$

is unique.

Proof: We only need to prove that $u(t,x) = 0$ on Q if $u(0,x) = 0$. By iteration, we may assume that $\alpha T < 1$. Let $\Phi \in C_0^\infty(\mathbb{R}^n)$. According to the definition of weak solution (4.5), we have

$$\begin{aligned} &\iint\limits_Q \frac{\partial u}{\partial t}\Phi dt\,dx = \\ &-\frac{1}{2}\int_0^T\int_{\mathbb{R}^n}\nabla u\cdot\nabla\Phi dx\,dt \\ &-\int_0^T\int_{\mathbb{R}^n}(f - \nabla K)\cdot\nabla u\Phi dx\,dt \\ &-\int_0^T\int_{\mathbb{R}^n}\Big[\frac{1}{2}|h|^2 + \mathrm{div}\ f \\ &-\frac{1}{2}\Delta K + f\cdot\nabla K \\ &-\frac{1}{2}|\nabla K|^2\Big]u\Phi dx\,dt. \end{aligned} \tag{5.5}$$

Replacing Φ by Φe^{cr}, we have

$$\int_{\mathbb{R}^n} u(T,x)\Phi e^{cr} = -\frac{1}{2}\int_0^T\int_{\mathbb{R}^n} \nabla u\cdot(e^{cr}\nabla\Phi) - \frac{c}{2}\int_0^T\int_{\mathbb{R}^n}\Phi e^{cr}\nabla r\cdot\nabla u + \int_0^T\int_{\mathbb{R}^n} e^{cr}\Phi(-f+\nabla K)\cdot\nabla u - \int_0^T\int_{\mathbb{R}^n}\Big[\frac{1}{2}|h|^2 + \text{div } f - \frac{1}{2}\Delta K + f\cdot\nabla K - \frac{1}{2}|\nabla K|^2\Big]\Phi u e^{cr} + \int_0^T\int_{\mathbb{R}^n} u\frac{\partial\Phi}{\partial t}e^{cr}. \tag{5.6}$$

Approximately u by Φ in the $W^{1,1}(Q)$ norm, we get

$$\begin{aligned}
\int_{\mathbb{R}^n} u^2(T,x)e^{cr} &= -\frac{1}{2}\int_0^T\int_{\mathbb{R}^n} e^{cr}|\nabla u|^2 - \frac{c}{2}\int_0^T\int_{\mathbb{R}^n} ue^{cr}\nabla r\cdot\nabla u + \int_0^T\int_{\mathbb{R}^n} e^{cr}u(-f+\nabla K)\cdot\nabla u \\
&\quad - \int_0^T\int_{\mathbb{R}^n}\Big[\frac{1}{2}|h|^2 + \text{div } f - \frac{1}{2}\Delta K + f\cdot\nabla K - \frac{1}{2}|\nabla K|^2\Big]u^2 e^{cr} + \int_0^T\int_{\mathbb{R}^n} u\frac{\partial u}{\partial t}e^{cr} \\
&= -\frac{1}{2}\int_0^T\int_{\mathbb{R}^n} e^{cr}\Big\{|\nabla u|^2 + \Big[cu\nabla r - 2u(-f+\nabla K)\Big]\cdot\nabla u\Big\} \\
&\quad - \int_0^T\int_{\mathbb{R}^n}\Big[\frac{1}{2}|h|^2 + \text{div } f - \frac{1}{2}\Delta K + f\cdot\nabla K - \frac{1}{2}|\nabla K|^2\Big]u^2 e^{cr} \\
&\quad + \int_0^T\int_{\mathbb{R}^n}\frac{1}{2}e^{cr}u\Delta u - \int_0^T\int_{\mathbb{R}^n} e^{cr}u(f-\nabla K)\cdot\nabla u \\
&\quad - \int_0^T\int_{\mathbb{R}^n} e^{cr}u^2\Big[\frac{1}{2}|h|^2 + \text{div } f - \frac{1}{2}\Delta K + f\cdot\nabla K - \frac{1}{2}|\nabla K|^2\Big] \\
&= -\int_0^T\int_{\mathbb{R}^n} e^{cr}\Big\{|\nabla u|^2 + \Big[cu\nabla r - 2u(-f+\nabla K)\Big]\cdot\nabla u\Big\} \\
&\quad -2\int_0^T\int_{\mathbb{R}^n}\Big[\frac{1}{2}|h|^2 + \text{div } f - \frac{1}{2}\Delta K + f\cdot\nabla K - \frac{1}{2}|\nabla K|^2\Big]u^2 e^{cr} \\
&= -\int_0^T\int_{\mathbb{R}^n} e^{cr}\Big|\nabla u - \frac{cu}{2}\nabla r - uf + u\nabla K\Big|^2 \\
&\quad + \int_0^T\int_{\mathbb{R}^n} e^{cr}u^2\Big\{\Big|\frac{c}{2}\nabla r + f - \nabla K\Big|^2 - \Big(|h|^2 + 2\text{ div } f - \Delta K + 2f\cdot\nabla K - |\nabla K|^2\Big)\Big\} \\
&\leq \alpha\int_0^T\int_{\mathbb{R}^n} e^{cr}u^2(t,x).
\end{aligned} \tag{5.7}$$

By mean value theorem, there exists $T_1\in(0,T)$ such that

$$\int_0^T\int_{\mathbb{R}^n} e^{cr}u^2(t,x) = T\int_{\mathbb{R}^n} e^{cr}u^2(T_1,x).$$

In view of (5.7), we have

$$\int_{\mathbb{R}^n} u^2(T,x)e^{cr} \leq \alpha T\int_{\mathbb{R}^n} u^2(T_1,x)e^{cr}. \tag{5.8}$$

By applying (5.7) successfully, there exists $T_m\in(0,T)$ such that

$$\int_{\mathbb{R}^n} u^2(T,x)e^{cr} \leq (\alpha T)^m\int_{\mathbb{R}^n} u^2(T_m,x)e^{cr}. \tag{5.9}$$

As $\alpha T<1$, we conclude that $u=0$. ■

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