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Hybrid Attitude Tracking of Output Feedback Controlled Rigid Bodies

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***Abstract*— In this paper we address the problem of output feedback attitude control of a rigid body in quaternion coordinate space through a PD+ based tracking controller using switching technique to obtain stability for all initial values. Assumptions on earlier results where either the initial state is considered bounded, or the attitude error for all time is less than 180 degrees, is removed by applying switching technique, also including hysteresis for robust stability. More precisely, we show uniform asymptotic stability in the large of a set containing the origin for the closed-loop system in the presence of unknown, bounded input disturbances. Simulation results are presented to verify our theoretical findings, showing that the system stabilizes as expected, even with high initial estimated velocity error.**

I. INTRODUCTION

Attitude control on the rotational sphere is an interesting theoretical problem since, due to the parametrization of the attitude for the unit quaternion, the model has multiple equilibrium points. From a more practical viewpoint, besides achieving stability in some sense, control of a rigid body demands fast and accurate settling using minimal control effort. Thus, a wide number of controllers have been developed during the past years, by focusing on the enhancement of performance while guaranteeing robust stability and minimizing the control effort.

Attitude tracking control naturally lies on a bulk of literature on tracking control of robot manipulators –*cf.* [1]. A classic in robot control literature is the PD+ controller of Paden and Panja –*cf.* [2] which, together with the Slotine and Li controller –*cf.* [3], was the first algorithm for which global asymptotic stability was demonstrated. A PD+ based controller for spacecraft was presented in [4], called model-dependent control.

An angular velocity observer for rigid body motion was presented in [5], using unit quaternions and a mechanical energy function approach, while a passivity approach was considered in [6] where the passivity properties were exploited in a nonlinear controller to ensure asymptotic stability without need of a model-based observer for angular velocity reconstruction. Similar results were presented in [7] where a class of Euler-Lagrange (EL) systems were determined, satisfying a dissipation propagation condition, while output feedback tracking control of a class of EL-systems subjected to monotonic loads were investigated in [8]. In [9] two different schemes were presented based on results for output control of robot manipulators (*cf.* [10]); in the first scheme, a second-order model-based observer is adopted for estimation of the angular velocity, while the second scheme is based on a lead-filter for estimation of the angular velocity error. An alternative approach using Modified Rodrigues Parameters (MRP) was presented in [11], simplifying the resulting control law. The topic of output feedback was further pursued for spacecraft control in [12], where the inertia matrix was assumed unknown, and the problem was solved using an adaptive approach. [13] presented a scheme based on a unit quaternion observer and a linear feedback control law to prove asymptotic stability of the equilibrium point, thus avoiding the use of lead filter. The problem was further pursued and almost global (in the large) asymptotic stability was obtained for a similar approach in [14], and almost global exponential stability for stabilization on SE(3) in [15].

In work such as [9], [16], [17] it is assumed that either the attitude error for the dynamics and estimator never increases beyond π rad, thus choosing one rotational direction and stick to it throughout the maneuver, or a bound on the initial values is introduced to make sure this never happens. Choosing a goal equilibrium at the start of the maneuver is mostly motivated by the theoretical analysis. From a practical viewpoint it may be more desirable to implement a decision law which determines the reference operating point online. To deal with the problem of multiple equilibria while allowing discontinuous changes of the goal equilibrium during the maneuver it is most natural to use hybrid or, more specifically, switching control (*cf.* [18], [19]). See for instance [20] for the control of an under-actuated non-symmetric rigid body and [21] where the authors use quaternion-based hybrid feedback and presents two different control laws: one derived from an energy-based Lyapunov function which only switches rotational direction when the rotational error is above π rad, and one based on backstepping design which also has the angular velocity error included in the switching threshold, thus leading to a more complex behavior.

In this paper we use a PD+ based output feedback controller, roughly speaking, include quaternion-based hybrid feedback with hysteresis for the error dynamics leading to robust stabilization with respect to measurement noise. Switching is also used to ”reset” the estimated attitude error to zero when a certain threshold is crossed to make sure that asymptotic stability with respect to Lyapunov holds for all time without restricting the set of available initial

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values. Strictly speaking, we show that a set containing the origin of the closed-loop system is uniformly practically asymptotically stable in the large with respect to unknown, bounded perturbations, which means that all trajectories converge asymptotically towards a set which size can be arbitrarily diminished by increasing controller gains. Our theoretical findings are validated in simulation for an Earth orbiting spacecraft with high initial angular velocity error, both for the dynamical and estimated part, thus switching is provoked. The result presented in this paper can be seen as an extension to the results presented in [9] - even though the observer structure differs some because we do not make use of sliding variables - where switching control is applied to remove the assumption of strict boundedness on the initial state vector.

II. PRELIMINARIES

The cross product operator $\times$ between two vectors $\mathbf{a}$ and $\mathbf{b}$ is written as $\mathbf{S}(\mathbf{a})\mathbf{b}$ where $\mathbf{S}$ is a skew-symmetric matrix. The symbol $\boldsymbol{\omega}_{b,a}^{c}$ denotes angular velocity of frame a relative to frame b, expressed in the frame c; $\mathbf{R}_a^b$ is the rotation matrix from frame a to frame b; $\|\cdot\|$ denotes the Euclidean norm or the induced $\mathcal{L}_2$-norm of matrices, and we denote $\mathbb{R}_+$ as the set of all positive real numbers. When the context is sufficiently explicit, we omit the arguments of functions.

A. Quaternions

The attitude of a rigid body is represented by a rotation matrix $\mathbf{R} \in SO(3) = \{\mathbf{R} \in \mathbb{R}^{3\times 3} \ : \ \mathbf{R}^\top \mathbf{R} = \mathbf{I},\ \det(\mathbf{R}) = 1\}$, which is the special orthogonal group of order three. Quaternions are used to parameterize members of $SO(3)$ where the unit quaternion is defined as $\mathbf{q} = [\eta,\ \boldsymbol{\epsilon}^\top]^\top \in S^3 = \{\mathbf{x} \in \mathbb{R}^4 \ : \ \mathbf{x}^\top\mathbf{x} = 1\}$, where $\eta \in \mathbb{R}$ and $\boldsymbol{\epsilon} \in \mathbb{R}^3$. The rotation matrix may be described by [22]

$$\mathbf{R} = \mathbf{I} + 2\eta\mathbf{S}(\boldsymbol{\epsilon}) + 2\mathbf{S}^2(\boldsymbol{\epsilon}). \tag{1}$$

The inverse rotation can be performed by using the inverse conjugated of $\mathbf{q}$ as $\bar{\mathbf{q}} = [\eta,\ \ -\boldsymbol{\epsilon}^\top]^\top$. The set S^3 forms a group with quaternion multiplication, which is distributive and associative, but not commutative, and the quaternion product of two arbitrary quaternions $\mathbf{q}_1$ and $\mathbf{q}_2$ is defined as

$$\mathbf{q}_1 \otimes \mathbf{q}_2 = \begin{bmatrix} \eta_1\eta_2 - \boldsymbol{\epsilon}_1^\top\boldsymbol{\epsilon}_2 \\ \eta_1\boldsymbol{\epsilon}_2 + \eta_2\boldsymbol{\epsilon}_1 + \mathbf{S}(\boldsymbol{\epsilon}_1)\boldsymbol{\epsilon}_2 \end{bmatrix}. \tag{2}$$

See [22] for further detail.

B. Kinematics and Dynamics

The time derivative of (1) can be written as $\dot{\mathbf{R}}_b^a = \mathbf{S}\left(\boldsymbol{\omega}_{a,b}^a\right)\mathbf{R}_b^a = \mathbf{R}_b^a\mathbf{S}\left(\boldsymbol{\omega}_{a,b}^b\right)$, and the kinematic differential equations can be expressed as

$$\dot{\mathbf{q}}_{i,b} = \mathbf{T}(\mathbf{q}_{i,b})\boldsymbol{\omega}_{i,b}^b,\ \ \mathbf{T}(\mathbf{q}_{i,b}) = \frac{1}{2}\begin{bmatrix} -\boldsymbol{\epsilon}_{i,b}^\top \\ \eta_{i,b}\mathbf{I} + \mathbf{S}(\boldsymbol{\epsilon}_{i,b}) \end{bmatrix}. \tag{3}$$

The dynamical model of a rigid body can be described by a differential equation for angular velocity, and is deduced from Euler's moment equation. This equation describes the relationship between applied torque and angular momentum on a rigid body as [23]

$$\mathbf{J}\dot{\boldsymbol{\omega}}_{i,b}^b = -\mathbf{S}(\boldsymbol{\omega}_{i,b}^b)\mathbf{J}\boldsymbol{\omega}_{i,b}^b + \boldsymbol{\tau}^b, \tag{4}$$

where $\boldsymbol{\tau}^b \in \mathbb{R}^3$ is the total torque working in the body frame, and $\mathbf{J} \in \mathbb{R}^{3\times 3}$ is the rigid body inertia matrix. The torque working on the body is expressed as $\boldsymbol{\tau}^b = \boldsymbol{\tau}_a^b + \boldsymbol{\tau}_d^b$, where $\boldsymbol{\tau}_d^b$ is the disturbance torque, and $\boldsymbol{\tau}_a^b$ is the actuator torque.

C. Hybrid Control

For the purpose of analysis, we use the setting of [24], [19]. According with this framework hybrid systems are described by a continuous-time dynamics defined by a "flow map" and discrete-time dynamics, defined by a "jump map". In addition, we are equipped of a "flow set" and a "jump set". That is,

$$\mathcal{H}: \left\{ \begin{array}{lll} x \in C & \Longrightarrow & \dot{x} = F(x) \\ x \in D & \Longrightarrow & x^+ = G(x) \end{array} \right.$$

where x^+ is the state value 'immediately' after a jump.

After [24], [19] solutions to the hybrid system are defined as maps from a *hybrid time domain*, subset of $\mathbb{R}_{\geq 0} \times \mathbb{N}$, into an Euclidean space. Roughly, the hybrid time domain denoted "dom x", consists in an ordered sequence of continuous-time intervals $[t_j, t_{j+1})$ or $[t_j, t_{j+1}]$ and discrete instants $\{j\}$. During flows (if $x(t,j) \in C$) the solution is a locally absolutely continuous function that satisfies $\dot{x} = F(x)$. At jumps ($x \in D$), the state value after the jump satisfies $x^+ = G(x)$. The solution of a hybrid system is denoted $j, t \to x(t,j)$.

Then, (asymptotic) stability is defined as follows. A compact set $\mathcal{A}$ is stable for $\mathcal{H}$ if for each $\epsilon > 0$ there exists $\sigma > 0$ such that[1] $\|x(0,0)\|_{\mathcal{A}} \leq \sigma$ implies $\|x(t,j)\|_{\mathcal{A}} \leq \epsilon$ for all solutions x to $\mathcal{H}$ and all $(t,j) \in \operatorname{dom} x$. A compact set is attractive if there exists a neighborhood of $\mathcal{A}$ from which each solution is complete and converges to $\mathcal{A}$, that is $\|x(t,j)\|_{\mathcal{A}} \to 0$ as $t + j \to \infty$, where $(t,j) \in \operatorname{dom} x$.

III. CONTROL OF RIGID BODY

A. Problem Formulation

The control problem is to steer the state $\mathbf{q}_{i,b}(t)$ towards a given reference trajectory $\mathbf{q}_{i,d}(t)$ satisfying $\dot{\mathbf{q}}_{i,d} = \mathbf{T}(\mathbf{q}_{i,d})\boldsymbol{\omega}_{i,d}^b$. The tracking error in quaternion coordinates, $\tilde{\mathbf{q}} = [\tilde{\eta},\ \tilde{\boldsymbol{\epsilon}}^\top]^\top$ is given by

$$\tilde{\mathbf{q}} := \bar{\mathbf{q}}_{i,d} \otimes \mathbf{q}_{i,b} = \begin{bmatrix} \eta_{i,d}\eta_{i,b} + \boldsymbol{\epsilon}_{i,d}\boldsymbol{\epsilon}_{i,b} \\ \eta_{i,d}\boldsymbol{\epsilon}_{i,b} - \eta_{i,b}\boldsymbol{\epsilon}_{i,d} - \mathbf{S}(\boldsymbol{\epsilon}_{i,d})\boldsymbol{\epsilon}_{i,b} \end{bmatrix}, \tag{5}$$

and the quaternion velocities may be expressed as $\dot{\tilde{\mathbf{q}}} = \mathbf{T}(\tilde{\mathbf{q}})\left(\boldsymbol{\omega}_{i,b}^b - \boldsymbol{\omega}_{i,d}^b\right)$. For the purpose of establishing meaningful stability properties we define the errors

$$\mathbf{e}_q := [1 - h\tilde{\eta},\ \tilde{\boldsymbol{\epsilon}}^\top]^\top, \quad \mathbf{e}_\omega := \boldsymbol{\omega}_{i,b}^b - \boldsymbol{\omega}_{i,d}^b, \tag{6}$$

[1]As usual, $\|x\|_{\mathcal{A}} = \inf_{z\in\mathcal{A}}|z - x|$.

where $h \in H := \{-1, 1\}$ is considered as a switching variable determining the choice of goal equilibrium point (*cf.* [21]). Moreover, we have

$$\dot{\mathbf{e}}_q = \mathbf{T}_e(\mathbf{e}_q)\mathbf{e}_\omega, \quad \mathbf{T}_e(\mathbf{e}_q) := \frac{1}{2}\left[\begin{array}{c} h\tilde{\boldsymbol{\epsilon}}^\top \\ \tilde{\eta}\mathbf{I} + \mathbf{S}(\tilde{\boldsymbol{\epsilon}}) \end{array}\right]. \quad (7)$$

Since measurements of the angular velocity is not available we define an estimation error defined as $\mathbf{e}_{e\omega} := \boldsymbol{\omega}_{i,b}^b - \boldsymbol{\omega}_{i,e}^b$, where subscript e denotes the estimated frame, together with an attitude estimation error defined as $\mathbf{q}_{e,b} := [\eta_{e,b},\ \boldsymbol{\epsilon}_{e,b}^\top]^\top = \bar{\mathbf{q}}_{i,e} \otimes \mathbf{q}_{i,b}$, thus the error function is defined as $\mathbf{e}_{eq} := [1 - \eta_{e,b},\ \boldsymbol{\epsilon}_{e,b}^\top]^\top$ with the kinematic relation

$$\dot{\mathbf{e}}_{eq} = \mathbf{T}_{eq}(\mathbf{e}_{eq})\mathbf{e}_{e\omega}, \quad \mathbf{T}_{eq} := \frac{1}{2}\left[\begin{array}{c} -\boldsymbol{\epsilon}_{e,b}^\top \\ \eta_{e,b}\mathbf{I} + \mathbf{S}(\boldsymbol{\epsilon}_{e,b}) \end{array}\right]. \quad (8)$$

The problem is solved if the equilibrium point $(\mathbf{e}_q, \mathbf{e}_\omega, \mathbf{e}_{eq}, \mathbf{e}_{e\omega}) = (\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0})$ is practically asymptotically stabilized.

B. Controller-Observer Design

We pose the following assumptions:

***Assumption** 3.1: There exists $\beta_j, \beta_J > 0$ such that the inertia matrix $\mathbf{J}$ is symmetric and positive definite, and satisfies the inequality*

$$\beta_j \le \|\mathbf{J}\| \le \beta_J. \quad (9)$$

***Assumption** 3.2: There exists $\beta_d > 0$ such that the disturbance moments $\boldsymbol{\tau}_d^b$ are bounded as*

$$\|\boldsymbol{\tau}_d^b(t)\| \le \beta_d. \quad (10)$$

***Assumption** 3.3: The desired angular velocity and the desired angular acceleration are bounded,* i.e.$\|\boldsymbol{\omega}_{i,d}^b(t)\| \le \beta_{\omega_{i,d}^b} \in \mathbb{R}_+$ *and* $\|\dot{\boldsymbol{\omega}}_{i,d}^b(t)\| \le \beta_{\dot{\omega}_{i,d}^b} \in \mathbb{R}_+ \ \forall\ t \ge t_0 \ge 0$.

The desired angular velocity is usually given with reference to the inertial frame denoted by $\boldsymbol{\omega}_{i,d}^i$. In the body frame,

$$\boldsymbol{\omega}_{i,d}^b = \mathbf{R}_i^b \boldsymbol{\omega}_{i,d}^i \quad (11)$$

hence, the reference acceleration in the body frame is

$$\begin{aligned} \dot{\boldsymbol{\omega}}_{i,d}^b &= \dot{\mathbf{R}}_i^b \boldsymbol{\omega}_{i,d}^i + \mathbf{R}_i^b \dot{\boldsymbol{\omega}}_{i,d}^i & (12) \\ &= -\mathbf{S}(\boldsymbol{\omega}_{i,b}^b)\boldsymbol{\omega}_{i,d}^b + \mathbf{R}_i^b \dot{\boldsymbol{\omega}}_{i,d}^i. & (13) \end{aligned}$$

As is common in control of mechanical systems, the controller includes feedforward terms which depend on reference velocity and acceleration. However, notice that $\boldsymbol{\omega}_{i,d}^b$ depends on the unavailable actual velocity $\boldsymbol{\omega}_{i,b}^b$. Therefore, for control purposes we use the modified acceleration vector (*cf.* [9])

$$\begin{aligned} \mathbf{a}_d &= -\mathbf{S}(\boldsymbol{\omega}_{i,d}^b)\boldsymbol{\omega}_{i,d}^b + \mathbf{R}_i^b \dot{\boldsymbol{\omega}}_{i,d}^i & (14) \\ &= \mathbf{R}_i^b \dot{\boldsymbol{\omega}}_{i,d}^i & (15) \end{aligned}$$

that is, where the unmeasured state $\boldsymbol{\omega}_{i,b}^b$ is replaced by the reference $\boldsymbol{\omega}_{i,b}^d$.

Consider the controller

$$\boldsymbol{\tau}_a = \mathbf{J}\mathbf{a}_d - \mathbf{S}(\mathbf{J}\boldsymbol{\omega}_{i,e}^b)\boldsymbol{\omega}_{i,d}^b - k_p \mathbf{T}_e^\top \mathbf{e}_q - k_d \boldsymbol{\omega}_{d,e}^b, \quad (16)$$

with $k_p, k_d, \in \mathbb{R}_+$ considered as constant gains and $\boldsymbol{\omega}_{d,e}^b := \boldsymbol{\omega}_{i,e}^b - \boldsymbol{\omega}_{i,d}^b$ is the angular velocity of the estimated frame relative to the desired frame presented in the body frame. Roughly, this may be regarded as the difference between the desired reference velocity and the estimated velocity, generated by the observer

$$\begin{aligned} \dot{\mathbf{z}} &= \mathbf{a}_d + \mathbf{J}^{-1}\big[l_p \mathbf{T}_{eq}^\top \mathbf{e}_{eq} - k_p \mathbf{T}_e^\top \mathbf{e}_q\big], & (17) \\ \boldsymbol{\omega}_{i,e}^b &= \mathbf{z} + 2\mathbf{J}^{-1} l_d \mathbf{T}_{eq}^\top \mathbf{e}_{eq}, & (18) \end{aligned}$$

where $l_p, l_d \in \mathbb{R}_+$ are constant gains to be defined.

For the purpose of analysis, let $\mathbf{x} := [\mathbf{e}_q^\top,\ \mathbf{e}_\omega^\top,\ \mathbf{e}_{eq}^\top,\ \mathbf{e}_{e\omega}^\top, h]^\top$ and define the flow sets as

$$\begin{aligned} C_1 &= \{\mathbf{x} : h\tilde{\eta} \ge -\delta_m\} & (19) \\ C_2 &= \{\mathbf{x} : \eta_{e,b} \ge \delta_n\}, & (20) \end{aligned}$$

where $C = C_1 \cap C_2$, and δ_m and δ_n are constants to be defined. The set (19) can be seen as hysteresis similar to [21], while the second set (20) ensures that the system is "flowing" as long as the state $\eta_{e,b}$ is positive and separated from zero. Then, the jump sets are defined as

$$\begin{aligned} D_1 &= \{\mathbf{x} : h(\tilde{\eta} - \frac{1}{2k_p}\lambda\tilde{\boldsymbol{\epsilon}}^\top \mathbf{J}\mathbf{e}_\omega) \le -\delta_m\} & (21) \\ D_2 &= \{\mathbf{x} : \eta_{e,b} \le \delta_n\}, & (22) \end{aligned}$$

where $D = D_1 \cup D_2$, and the switching laws defined as

$$\dot{h} = 0 \ \forall \mathbf{x} \in C \quad (23)$$

$$\mathbf{x}^+ = G_1(\mathbf{x}) = [\mathbf{e}_q^\top, \mathbf{e}_\omega^\top, \mathbf{e}_{eq}^\top, \mathbf{e}_{e\omega}^\top, -h]^\top \ \forall \mathbf{x} \in D_1 \quad (24)$$

$$\mathbf{x}^+ = G_2(\mathbf{x}) = [\mathbf{e}_q^\top, \mathbf{e}_\omega^\top, \mathbf{0}, \mathbf{e}_{e\omega}^\top, h]^\top \ \forall \mathbf{x} \in D_2. \quad (25)$$

$G_1(\mathbf{x})$ ensures that h switches sign when the hysteresis value is passed such that the product $h\tilde{\eta}$ is positive, while $G_2(\mathbf{x})$ "resets" the estimated attitude error, ($\mathbf{e}_{eq} = \mathbf{0} \Rightarrow \eta_{e,b} = 1$) ensuring that $\eta_{e,b}$ is kept positive and separated from zero for the Lyapunov stability to hold.

***Proposition** 3.1: Let Assumptions 3.1–3.3 hold. Then, the set $\mathcal{A} = \{(\mathbf{e}_q, \mathbf{e}_\omega, \mathbf{e}_{eq}, \mathbf{e}_{e\omega}, h) : \|(\mathbf{e}_q, \mathbf{e}_\omega, \mathbf{e}_{eq}, \mathbf{e}_{e\omega})\| \le \delta\}$ where δ is to be defined, of the system* (3) *and* (4)*, in closed loop with the hybrid control law* (16) *and* (19)–(25)*, and the observer* (17)–(18) *is Uniformly Asymptotically Stable (UAS) in the large.*

The proof is given in the Appendix.

IV. SIMULATION RESULTS

We present simulation results for a spacecraft in an elliptic Low Earth Orbit (LEO). The simulations were performed in Simulink using a fixed sample-time Runge-Kutta ODE4 solver with 10^{-2} s step size. The moments of inertia were chosen as $\mathbf{J} = \mathrm{diag}\{4.35,\ 4.33,\ 3.664\}$ kgm^2, and the spacecraft orbit was chosen with perigee at 600 km, apogee at 750 km, inclination at 71°, and the argument of perigee and the right ascension of the ascending node at 0°.

We introduce measurement noise as $\sigma\mathbb{B}^n = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\| \le \sigma\}$ and add a suitable amount to the error functions according to $\tilde{\mathbf{e}}_q = (\mathbf{e}_q + 0.01\mathbb{B}^4)/\|\mathbf{e}_q + 0.01\mathbb{B}^4\|$. Since

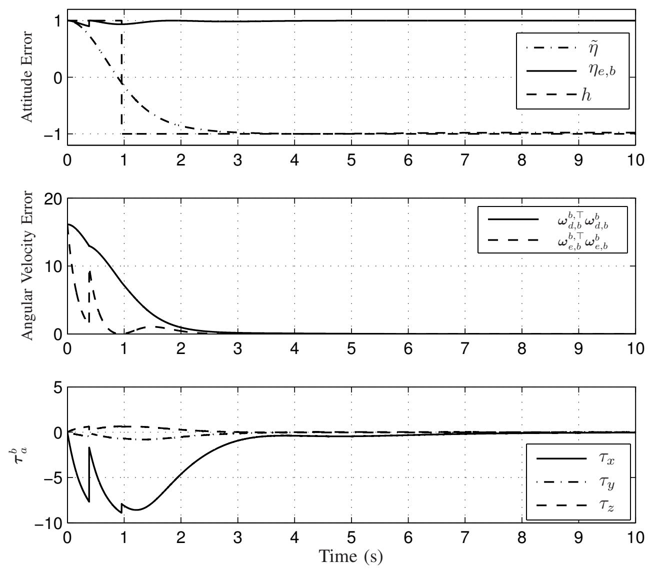

Fig. 1. Attitude error, angular velocity error and control torque during spacecraft maneuver.

we are applying a slightly elliptic LEO, we only consider the disturbance torques which are the major contributors to these kind of orbits; namely, gravity gradient torque [23], and torques generated by atmospheric drag and gravity (J_2 effect) [25], the two latter because of $\mathbf{r}_c^b = [0.1,\ 0,\ 0]^\top$ m displacement of the center of mass. All disturbances are considered continuous and bounded. For our simulations we have chosen the initial conditions as $\mathbf{x}(t_0) = [1,\ 0,\ 0,\ 0,\ 4,\ 0.2,\ -0.3,\ 1]^\top$, $\mathbf{z}(t_0) = [0,\ 0,\ 0]^\top$ and $\mathbf{q}_{i,e}^b(t_0) = \mathbf{q}_{i,b}^b(t_0)$, gains $k_p = 1$, $k_d = 3$, $l_p = 40$, $l_d = 25$, and switching variables as $\delta_m = 0.1$ and $\delta_n = 0.9$. The spacecraft were commanded to follow smooth sinusoidal trajectories around the origin with velocity profile

$$\begin{aligned}\boldsymbol{\omega}_{i,d}^i = [&3.2\cos(2\times10^{-3}t),\ 0.12\sin(1\times10^{-3}t),\\ &-3.2\sin(4\times10^{-3}t)]^\top \times 10^{-6}\ \text{rad/s}.\end{aligned} \tag{26}$$

The simulation results are depicted in Figure 1 and show that the large initial velocity error provoked switching for both the dynamical system and the observer after about 1 s and 0.4 s, as can be seen in the topmost plot, which means that h is set to -1 and $\eta_{e,b}$ is set to 1, respectively. In the second plot it is shown that the angular velocity of the body frame relative to the estimated frame converges faster than the angular velocity of the body frame relative to the desired frame, as is expected since from (35) we observe that $l_d > k_d$ and we can expect that in most cases $l_p^\star > k_d^\star$. When both angular velocity errors have converged, we conclude that also $\boldsymbol{\omega}_{d,e}^b \approx \mathbf{0}$. The actuator torque is depicted in the bottommost plot, where the required torque is consequently reduced during jumps.

V. CONCLUSION

In this paper we have designed a PD+ based output feedback control law utilizing switching technique to remove assumptions on previous work regarding a bound on the initial state vector. The a set containing the origin of the closed-loop system is proven uniformly asymptotically stable, and simulation results of an Earth orbiting spacecraft are presented to to verify the theoretical findings.

APPENDIX

The error dynamics can be written on state space form

$$\dot{\mathbf{x}} = f(t,\mathbf{x}) = [(\mathbf{T}_e\mathbf{e}_\omega)^\top, (\mathbf{J}_l^{-1}\xi_1)^\top, (\mathbf{T}_{eq}\mathbf{e}_{e\omega})^\top, (\mathbf{J}_l^{-1}\xi_2)^\top, 0]^\top,$$

where

$$\begin{aligned}\xi_1 =& \mathbf{S}(\mathbf{J}\boldsymbol{\omega}_{i,b}^b)\mathbf{e}_\omega + \mathbf{S}(\mathbf{J}\mathbf{e}_{e\omega})\boldsymbol{\omega}_{i,d}^b - k_p\mathbf{T}_e^\top\mathbf{e}_q \\ &- k_d(\mathbf{e}_\omega - \mathbf{e}_{e\omega}) - \mathbf{J}\mathbf{S}(\boldsymbol{\omega}_{i,d}^b)\mathbf{e}_\omega + \boldsymbol{\tau}_d,\end{aligned} \tag{27}$$

$$\begin{aligned}\xi_2 =& \mathbf{S}(\mathbf{J}\boldsymbol{\omega}_{i,b}^b)\mathbf{e}_\omega - k_d\mathbf{e}_\omega + \mathbf{S}(\mathbf{J}\mathbf{e}_{e\omega})\boldsymbol{\omega}_{i,d}^b + k_d\mathbf{e}_{e\omega} \\ &- \frac{l_d}{2}[\eta_{e,b}\mathbf{I} + \mathbf{S}(\boldsymbol{\epsilon}_{e,b})]\mathbf{e}_{e\omega} - l_p\mathbf{T}_{eq}^\top\mathbf{e}_{eq} + \boldsymbol{\tau}_d.\end{aligned} \tag{28}$$

Consider the Lyapunov function candidate

$$V(\mathbf{x}) := \frac{1}{2}\Big[\mathbf{e}_q^\top k_p\mathbf{e}_q + \mathbf{e}_\omega^\top\mathbf{J}\mathbf{e}_\omega + \mathbf{e}_{eq}^\top l_p\mathbf{e}_{eq} + \mathbf{e}_{e\omega}^\top\mathbf{J}\mathbf{e}_{e\omega}\Big] \tag{29}$$

which is quadratic and thus positive definite because $k_p, l_p, \beta_j > 0$. The total time derivative of V along the closed-loop trajectories yields

$$\begin{aligned}\dot{V} =& -k_d\mathbf{e}_\omega^\top\mathbf{e}_\omega + \mathbf{e}_\omega^\top\mathbf{S}(\mathbf{J}\mathbf{e}_{e\omega})\boldsymbol{\omega}_{i,d}^b - \mathbf{e}_\omega^\top\mathbf{J}\mathbf{S}(\boldsymbol{\omega}_{i,d}^b)\mathbf{e}_\omega \\ &+ \mathbf{e}_{e\omega}^\top\mathbf{S}(\mathbf{J}\boldsymbol{\omega}_{i,b}^b)\mathbf{e}_\omega + \mathbf{e}_{e\omega}^\top\mathbf{S}(\mathbf{J}\mathbf{e}_\omega)\boldsymbol{\omega}_{i,d}^b \\ &- \left(\frac{l_d}{2}\eta_{e,b} - k_d\right)\mathbf{e}_{e\omega}^\top\mathbf{e}_{e\omega} + \left(\mathbf{e}_\omega^\top + \mathbf{e}_{e\omega}^\top\right)\boldsymbol{\tau}_d.\end{aligned} \tag{30}$$

Since the matrix $\mathbf{S}(\cdot)$ is linear in its arguments, we have [9]

$$\|\mathbf{S}(\mathbf{J}\mathbf{a})\mathbf{b}\| \le \beta_J\|\mathbf{a}\|\|\mathbf{b}\|. \tag{31}$$

By applying (31), Young's inequality and Assumptions 3.1–3.3 we have

$$\mathbf{e}_\omega^\top\mathbf{S}(\mathbf{J}\mathbf{e}_{e\omega})\boldsymbol{\omega}_{i,d}^b \le \frac{1}{2}\beta_J\beta_{\omega_{i,d}^b}(\|\mathbf{e}_\omega\|^2 + \|\mathbf{e}_{e\omega}\|^2) \tag{32}$$

$$\begin{aligned}\mathbf{e}_\omega^\top\mathbf{J}\mathbf{S}(\boldsymbol{\omega}_{i,d}^b)\mathbf{e}_\omega &\le \beta_J\beta_{\omega_{i,d}^b}\|\mathbf{e}_\omega\|^2 \\ \mathbf{e}_{e\omega}^\top\mathbf{S}(\mathbf{J}\boldsymbol{\omega}_{i,b}^b)\mathbf{e}_\omega &\le \frac{1}{2}\beta_J(\|\mathbf{e}_\omega\|^2 + \|\mathbf{e}_{e\omega}\|^2)(\|\mathbf{e}_\omega\| + \beta_{\omega_{i,d}^b}) \\ \mathbf{e}_{e\omega}^\top\mathbf{S}(\mathbf{J}\mathbf{e}_\omega)\boldsymbol{\omega}_{i,d}^b &\le \beta_J\beta_{\omega_{i,d}^b}\|\mathbf{e}_{e\omega}\|^2.\end{aligned} \tag{33}$$

Inserting the bounds (32)–(33) into (30), we obtain

$$\begin{aligned}\dot{V} \le& -\phi(k_d, \|\mathbf{e}_\omega\|)\|\mathbf{e}_\omega\|^2 - \psi(k_d, l_d, \|\mathbf{e}_\omega\|)\|\mathbf{e}_{e\omega}\|^2 \\ &+ \left(\mathbf{e}_\omega^\top + \mathbf{e}_{e\omega}^\top\right)\boldsymbol{\tau}_d,\end{aligned} \tag{34}$$

where

$$\phi(k_d, \|\mathbf{e}_\omega\|) = k_d - \frac{1}{2}\beta_J\left(4\beta_{\omega_{i,d}^b} + \|\mathbf{e}_\omega\|\right) \tag{35a}$$

$$\psi(k_d, l_d, \|\mathbf{e}_\omega\|) = \frac{l_d}{2}\eta_{e,b} - k_d - \frac{1}{2}\beta_J(4\beta_{\omega_{i,d}^b} + \|\mathbf{e}_\omega\|). \tag{35b}$$

In view of the definition of the flow and sets, (20), (22) and the jump map (25), we have $\eta_{e,b} \ge \delta_n\ \forall t \ge t_0$ hence, $\psi(\cdot)$ may be made positive for sufficiently large gains. It follows that $\dot{V}$ is negative semi-definite for bounded values of $\mathbf{e}_\omega$, thus for any $r > 0$ there exists $\Delta^\dagger(r) > 0$ such that $\sup_{t\geq t_0}\|[\mathbf{e}_\omega^\top(t),\ \mathbf{e}_{e\omega}^\top(t)]^\top\| \leq \Delta^\dagger$ for all initial conditions $\|\boldsymbol{\chi}(t_0)\| < r,\ t_0 \geq 0$ where $\boldsymbol{\chi} = [\mathbf{e}_q^\top, \mathbf{e}_\omega^\top, \mathbf{e}_{eq}^\top, \mathbf{e}_{e\omega}^\top]^\top$. For any $\Delta^\dagger$, let $\lambda(\Delta^\dagger) > 0$ be a real valued constant to be determined. Now, consider the Lyapunov function candidate

$$\mathcal{V}(\mathbf{x}) := V(\mathbf{x}) + \lambda W(\mathbf{x}), \tag{36}$$

where $W(\mathbf{x}) := \mathbf{e}_q^\top \mathbf{T}_e \mathbf{J}\mathbf{e}_\omega + \mathbf{e}_{eq}^\top \mathbf{T}_{eq}\mathbf{J}\mathbf{e}_{e\omega}$, and observe that $\mathcal{V}$ is positive positive definite and proper for $\lambda \leq 1$, that is, $\mathcal{V}$ is lower and upper bounded as $\underline{\alpha}(\mathbf{x}) \leq \mathcal{V}(\mathbf{x}) \leq \overline{\alpha}(\mathbf{x})$ where

$$\underline{\alpha}(\mathbf{x}) := \boldsymbol{\chi}^\top p_m \boldsymbol{\chi}, \quad \overline{\alpha}(\mathbf{x}) := \boldsymbol{\chi}^\top p_M \boldsymbol{\chi} \tag{37}$$

and p_m, p_M are such that $p_m \leq \|\mathbf{P}\| \leq p_M$ where

$$\mathbf{P} := \frac{1}{2}\begin{bmatrix} k_p\mathbf{I} & \lambda\mathbf{T}_e\mathbf{J} & \mathbf{0} & \mathbf{0} \\ \lambda\mathbf{J}\mathbf{T}_e^\top & \mathbf{J} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & l_p\mathbf{I} & \lambda\mathbf{T}_{eq}\mathbf{J} \\ \mathbf{0} & \mathbf{0} & \lambda\mathbf{J}\mathbf{T}_{eq}^\top & \mathbf{J} \end{bmatrix}. \tag{38}$$

The total time derivative of W along the closed-loop trajectories yields

$$\begin{aligned} \dot{W} =& \mathbf{e}_\omega^\top h/4[\tilde{\eta}\mathbf{I} + \mathbf{S}(\tilde{\boldsymbol{\epsilon}})]\mathbf{e}_\omega + \mathbf{e}_q^\top \mathbf{T}_e \mathbf{S}(\mathbf{J}\boldsymbol{\omega}_{i,b}^b)\mathbf{e}_\omega \\ &+ \mathbf{e}_q^\top \mathbf{T}_e \mathbf{S}(\mathbf{J}\mathbf{e}_\omega)\boldsymbol{\omega}_{i,d}^b - \mathbf{e}_q^\top \mathbf{T}_e k_p \mathbf{T}_e^\top \mathbf{e}_q \\ &- \mathbf{e}_q^\top \mathbf{T}_e k_d(\mathbf{e}_\omega - \mathbf{e}_{e\omega}) - \mathbf{e}_q^\top \mathbf{T}_e \mathbf{J}\mathbf{S}(\boldsymbol{\omega}_{i,d}^b)\mathbf{e}_\omega + \mathbf{e}_q^\top \mathbf{T}_e \boldsymbol{\tau}_d \\ &+ \mathbf{e}_{e\omega}^\top/4[\eta_{e,b} + \mathbf{S}(\boldsymbol{\omega}_{e,b})]\mathbf{J}\mathbf{e}_{e\omega} + \mathbf{e}_{eq}^\top \mathbf{T}_{eq}\mathbf{S}(\mathbf{J}\boldsymbol{\omega}_{i,b}^b)\mathbf{e}_\omega \\ &- \mathbf{e}_{eq}^\top \mathbf{T}_{eq} k_d \mathbf{e}_\omega + \mathbf{e}_{eq}^\top \mathbf{T}_{eq}\mathbf{S}(\mathbf{J}\mathbf{e}_{e\omega})\boldsymbol{\omega}_{i,d}^b \\ &+ \mathbf{e}_{eq}^\top \mathbf{T}_{eq} k_d \mathbf{e}_{e\omega} - \mathbf{e}_{eq}^\top \mathbf{T}_{eq} l_d/2[\eta_{e,b}\mathbf{I} + \mathbf{S}(\epsilon_{e,b})]\mathbf{e}_{e\omega} \\ &- \mathbf{e}_{eq}^\top \mathbf{T}_{eq} l_p \mathbf{T}_{eq}^\top \mathbf{e}_{eq} + \mathbf{e}_{eq}^\top \mathbf{T}_{eq}\boldsymbol{\tau}_d \end{aligned} \tag{39}$$

and by applying (32)–(33) on (39) and add the result with (34) according to (36) we obtain

$$\dot{\mathcal{V}}(\boldsymbol{\chi}) \leq -\boldsymbol{\chi}^\top \mathbf{Q}(\boldsymbol{\omega}_{i,b}^b)\boldsymbol{\chi} + 2\beta_d\|\boldsymbol{\chi}\| \tag{40}$$

where $\mathbf{Q}(\boldsymbol{\omega}_{i,b}^b) = [\mathbf{q}_{ij}], \quad i,j = 1,2,3,4$ with

$$\mathbf{q}_{11} = \lambda\mathbf{T}_e k_p \mathbf{T}_e^\top \tag{41a}$$
$$\mathbf{q}_{12} = \mathbf{q}_{21}^\top = \frac{\lambda}{2}\mathbf{T}_e\left[k_d\mathbf{I} - \mathbf{S}(\mathbf{J}\boldsymbol{\omega}_{i,b}^b) + \mathbf{J}\mathbf{S}(\boldsymbol{\omega}_{i,d}^b)\right] \tag{41b}$$
$$\mathbf{q}_{13} = \mathbf{q}_{31}^\top = \mathbf{0} \tag{41c}$$
$$\mathbf{q}_{14} = \mathbf{q}_{41}^\top = \frac{\lambda}{2}\mathbf{T}_e\left[\mathbf{S}(\boldsymbol{\omega}_{i,d}^b)\mathbf{J} - k_d\mathbf{I}\right] \tag{41d}$$
$$\mathbf{q}_{22} = \phi(k_d, \|\mathbf{e}_\omega\|)\mathbf{I} - \lambda\frac{1}{4}\mathbf{J} \tag{41e}$$
$$\mathbf{q}_{23} = \mathbf{q}_{32}^\top = \frac{\lambda}{2}\left[k_d\mathbf{I} - \mathbf{S}(\mathbf{J}\boldsymbol{\omega}_{i,b}^b)\right]\mathbf{T}_{eq}^\top \tag{41f}$$
$$\mathbf{q}_{24} = \mathbf{q}_{42}^\top = \mathbf{0} \tag{41g}$$
$$\mathbf{q}_{33} = \lambda\mathbf{T}_{eq} l_p \mathbf{T}_{eq}^\top \tag{41h}$$
$$\mathbf{q}_{34} = \mathbf{q}_{43}^\top = \frac{\lambda}{2}\mathbf{T}_{eq}\Big[\mathbf{S}(\boldsymbol{\omega}_{i,d}^b)\mathbf{J} + \frac{l_d}{2}[\eta_{e,b}\mathbf{I} + \mathbf{S}(\boldsymbol{\epsilon}_{e,b})] - k_d\mathbf{I}\Big] \tag{41i}$$
$$\mathbf{q}_{44} = \psi(k_d, l_d, \|\mathbf{e}_\omega\|)\mathbf{I} - \lambda\frac{1}{4}\mathbf{J}. \tag{41j}$$

Next, we observe that in view of the quaternion constraint,

$$\mathbf{e}_q^\top \mathbf{T}_e \mathbf{T}_e^\top \mathbf{e}_q \geq \frac{1}{8}\mathbf{e}_q^\top \mathbf{e}_q. \tag{42}$$

idem for $\mathbf{e}_{eq}$. Therefore, for each Δ' and for all $\mathbf{e}_\omega$ such that $\|\mathbf{e}_\omega\| \leq \Delta'$ there exist lower and upper bounds $q_{ij,m}$ and $q_{ij,M}$ on the norms of the sub-blocks $\mathbf{q}_{ij}$ of $\mathbf{Q}$ such that, after applying the triangle inequality repeteadly, we obtain

$$\begin{aligned} \boldsymbol{\chi}^\top \mathbf{Q}\boldsymbol{\chi} \geq& \frac{1}{2}(q_{11,m}\|\mathbf{e}_q\|^2 + q_{22,m}\|\mathbf{e}_\omega\|^2 \\ &+ q_{33,m}\|\mathbf{e}_{eq}\|^2 + q_{44,m}\|\mathbf{e}_{e\omega}\|^2). \end{aligned} \tag{43}$$

It follows that (43) holds, that is $\mathbf{Q}$ is positive definite, if defining (from (41))

$$k_d^\star := \frac{1}{2}\beta_J(4\beta_{\omega_{i,d}^b} + \Delta') \tag{44}$$
$$k_p^\star := j_M(\Delta' + 2\beta_{\omega_{i,d}^b}) \tag{45}$$
$$l_d^\star := [2k_d - \beta_J(4\beta_{\omega_{i,d}^b} + \Delta')]/\delta_n \tag{46}$$
$$l_p^{\star'} := j_M(\Delta' + \beta_{\omega_{i,d}^b}) + \frac{l_d}{2} \tag{47}$$
$$\geq j_M(\Delta' + \beta_{\omega_{i,d}^b}) + \frac{l_d}{2}\eta_{e,b}, \tag{48}$$

we choose gains $k_d > k_d^\star$, $k_p > k_p^\star$, $l_d > l_d^\star$, $l_p > l_p^{\star'}$ and

$$\lambda \leq \min\left\{\frac{\phi(k_d,\Delta)}{\frac{1}{4}j_M + 2k_d + j_M(2\Delta' + \beta_{\omega_{i,d}^b})},\ \frac{\psi(k_d,l_d,\Delta)}{\frac{1}{4}j_M + 2\beta_{\omega_{i,d}^b}j_M + \frac{l_d}{2}2k_d}, 1\right\}.$$

Thus,

$$\dot{\mathcal{V}} \leq -q_m\|\boldsymbol{\chi}\|^2 + 2\beta_d\|\boldsymbol{\chi}\|, \tag{49}$$

where $q_m(\Delta') > 0$ is a lower bound on the smallest eigenvalue of $\mathbf{Q}(\Delta')$. The derivative $\dot{\mathcal{V}} < 0$ for all $\mathbf{x} \in \mathcal{H} := \{\mathbf{x} \in \mathcal{S}^3 \times \mathbb{R}^3 \times \mathcal{S}^3 \times \mathbb{R}^3 \times H : \delta \leq \|\boldsymbol{\chi}\| \leq \Delta\}$, where $\delta := 2\beta_d/q_m$. Given any positive constants $\delta^\star$, $\Delta^\star$ such that $\delta^\star < \Delta^\star$, we have that there exists $\Delta > \delta > 0$ such that

$$\underline{\alpha}^{-1} \circ \overline{\alpha}(\delta) = \sqrt{\frac{p_M\delta^2}{p_m}} \leq \delta^\star \tag{50}$$

$$\overline{\alpha}^{-1} \circ \underline{\alpha}(\Delta) = \sqrt{\frac{p_m\Delta^2}{p_M}} \geq \Delta^\star. \tag{51}$$

In accordance with [26, Theorem 10], all the conditions are satisfied.

We have to ensure that the Lyapunov function decreases over jumps (*cf.* [19]), such that

$$\mathcal{V}(G_1(\mathbf{x})) - \mathcal{V}(\mathbf{x}) < 0 \tag{52}$$
$$\mathcal{V}(G_2(\mathbf{x})) - \mathcal{V}(\mathbf{x}) < 0 \tag{53}$$

is fulfilled. We see that

$$\mathcal{V}(G_1(\mathbf{x})) - \mathcal{V}(\mathbf{x}) = 2hk_p(\tilde{\eta} - \frac{1}{2k_p}\lambda\tilde{\boldsymbol{\epsilon}}^\top \mathbf{J}\mathbf{e}_\omega), \tag{54}$$

and due to (21), (52) is fulfilled. We have that

$$\mathcal{V}(G_2(\mathbf{x}))-\mathcal{V}(\mathbf{x})=\frac{1}{2}\Big[\left(\boldsymbol{\omega}_{i,b}-\mathbf{z}\right)^\top \mathbf{J}\left(\boldsymbol{\omega}_{i,b}^b-\mathbf{z}\right)-l_p\mathbf{e}_{eq}^\top\mathbf{e}_{eq} - \left(\boldsymbol{\omega}_{i,b}^b-\boldsymbol{\omega}_{i,e}^b\right)^\top \mathbf{J}\left(\boldsymbol{\omega}_{i,b}^b-\boldsymbol{\omega}_{i,e}^b\right)\Big] - \lambda\mathbf{e}_{eq}^\top\mathbf{T}_{eq}\mathbf{J}\left(\boldsymbol{\omega}_{i,b}^b-\boldsymbol{\omega}_{i,e}^b\right) \quad (55)$$

$$=\frac{1}{2}\Big[\left(\boldsymbol{\omega}_{i,b}^b-\mathbf{z}\right)^\top \mathbf{J}\left(\boldsymbol{\omega}_{i,b}^b-\mathbf{z}\right)-l_p\mathbf{e}_{eq}^\top\mathbf{e}_{eq} - \left(\boldsymbol{\omega}_{i,b}^b-\mathbf{z}-2\mathbf{J}^{-1}l_d\mathbf{T}_{eq}^\top\mathbf{e}_{eq}\right)^\top \quad (56)$$
$$\times\mathbf{J}\left(\boldsymbol{\omega}_{i,b}^b-\mathbf{z}-2\mathbf{J}^{-1}l_d\mathbf{T}_{eq}^\top\mathbf{e}_{eq}\right)\Big] -\lambda\mathbf{e}_{eq}^\top\mathbf{T}_{eq}\mathbf{J}\big(\boldsymbol{\omega}_{i,b}^b-\mathbf{z}-2\mathbf{J}^{-1}l_d\mathbf{T}_{eq}^\top\mathbf{e}_{eq}\big),$$

where $\mathbf{e}_{e\omega}=\boldsymbol{\omega}_{i,b}^b-\boldsymbol{\omega}_{i,e}^b$ and (18) was inserted, and the fact that $\mathbf{T}_{eq}^\top\mathbf{e}_{eq}=\mathbf{0}$ for $\mathbf{x}^+\in D_2$ which implies that $\mathbf{e}_{e\omega}=\boldsymbol{\omega}_{i,b}^b-\mathbf{z}$. For (53) to be fulfilled according to (22) we require that

$$l_p\mathbf{e}_{eq}^\top\mathbf{e}_{eq}+4l_d^2\mathbf{e}_{eq}^\top\mathbf{T}_{eq}\mathbf{J}^{-1}\mathbf{T}_{eq}^\top\mathbf{e}_{eq} \quad (57)$$
$$\geq 4l_d\boldsymbol{\omega}_{i,b}^{b,\top}\mathbf{T}_{eq}^\top\mathbf{e}_{eq}-4l_d\mathbf{z}^\top\mathbf{T}_{eq}^\top\mathbf{e}_{eq}-2\lambda\boldsymbol{\omega}_{i,b}^{b,\top}\mathbf{J}\mathbf{T}_{eq}^\top\mathbf{e}_{eq} + 2\lambda\mathbf{z}^\top\mathbf{J}\mathbf{T}_{eq}^\top\mathbf{e}_{eq}+4\lambda l_d\mathbf{e}_{eq}^\top\mathbf{T}_{eq}\mathbf{T}_{eq}^\top\mathbf{e}_{eq}. \quad (58)$$

For $\eta_{e,b}=\delta_n$ we have that $\boldsymbol{\epsilon}_{eq}^\top\boldsymbol{\epsilon}_{eq}=1-\delta_n^2$, $\|\mathbf{T}_{eq}^\top\mathbf{e}_{eq}\|=1/2\|\boldsymbol{\epsilon}_{eq}\|=1/2\sqrt{1-\delta_n^2}$ and $\mathbf{e}_{eq}^\top\mathbf{e}_{eq}=(1-\delta_n)^2+\boldsymbol{\epsilon}_{eq}^\top\boldsymbol{\epsilon}_{eq}=2(1-\delta_n)$, such that

$$l_p^{\star''}:=\frac{(2l_d+\lambda j_M)(\Delta'+\Delta_z)\sqrt{1-\delta_n^2}-l_d\left(\frac{l_d}{j_M}-\lambda\right)(1-\delta_n^2)}{2(1-\delta_n)},$$

where $\|\mathbf{z}\|\leq\Delta_z$ for a constant upper bound $\Delta_z>0$. This can be argumented by looking at (17), where $\mathbf{T}_{eq}^\top\mathbf{e}_{eq}$ and $\mathbf{T}_e^\top\mathbf{e}_q$ will, according to the previous part of the proof, converge towards a subset of $\mathcal{A}$ for $\mathbf{x}\in C$, and, if $\mathbf{x}$ is entering a jump set (D_1 or D_2), the value of $\mathbf{z}$ will not abruptly change during a jump, thus $\dot{\mathbf{z}}$ will converge towards $\mathbf{a}_d$ as given in (15), which can not be constant and sign-definite because that would violate Assumption 3.3, thus $\lim_{t\to\infty}\int_{t_0}^t\mathbf{a}_d d\tau<\infty$. Last, we need to be sure that l_p is chosen such that there exists an $\delta_n\in(0,1)$. According to (57), we have that

$$2l_p^{\star''}(1-\delta_n)=c_1\sqrt{1-\delta_n^2}-c_2(1-\delta_n^2), \quad (59)$$

where $c_1>0$, and in most cases $c_2>0$ since $\lambda\ll 1$, and it can be seen that there exist a $l_p^{\star''}$ large enough such that there exists a solution for $\delta_n>0$, and as $l_p^{\star''}\to\infty$, $\delta_n\to 1$. On the other hand, one can choose a $\delta_n\in(0,1)$ and solve (57) for $l_p^{\star''}$. Thus, by defining $l_p^\star:=\max\{l_p^{\star'},l_p^{\star''}\}$ and choosing $l_p>l_p^\star$, we ensure that all conditions are fulfilled according to [24, Corollary 7.7], and it can be concluded that the set $\mathcal{A}$ of the closed loop system is UAS in the large. □

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