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Robust synchronization in nonlinear network with link failure uncertainty

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Abstract— **The problem of synchronization of systems over a network, is a widely studied problem given the importance of synchronization phenomena, in various natural science and engineering applications. In this paper, we study one of the important aspect of this problem that is, robustness of synchronization to random link failure uncertainty. The link failure uncertainty is modeled as an on-off Bernoulli switch. The main results of this paper provide, for the first time, analytical conditions for the maximum tolerable link failure uncertainty to maintain mean square synchronization among the network components. The analytical conditions are expressed in terms of individual component dynamics, network properties, and link uncertainty. The main results of this paper can be used to determine, the weakest/strongest link in the network. Simulation results are provided to verify the main results of this paper.**

I. Introduction

The problem of synchronization of systems over a network, is of interest in various natural science and engineering applications such as, synchronization of generators in electrical power network, mechanical nano oscillators in sensing applications, sensor network, circadian clocks, neural networks in visual cortex in biological applications, and synchronization of fireflies [1], [2], [3]. Synchronization in coupled chaotic systems for secure communications and the problem of consensus or distributed averaging in linear time invariant (LTI) systems are also investigated [4].

One of the important problems in the study of synchronization over network is that of robustness of synchronization to the uncertainty in the network. Synchronization and robustness of small world network models to random removal of nodes has been studied in [5]. Synchronization in blinking networks with random Bernoulli switching of the interconnection links is investigated in [6], [7]. An information theoretic approach to synchronization, under constraints in information coding has been studied in [8]. Distributed averaging problems with link failures have been studied, as structured uncertainty problems and, bounds obtained on convergence rates of the system based on network topology properties [9], [10], [11].

In this paper we study synchronization of identical nonlinear systems which are connected over a network. We are interested in studying robustness properties of the synchronization over the network when some links are uncertain. The link failure uncertainty is modeled as on-off Bernoulli switch. However the proposed framework can be easily extended to more general uncertainty model with continuous probability distribution. The mean square exponential synchronization is used as a performance measure for robustness to link failure uncertainty. The main contribution of this paper is that it provides analytical conditions, expressed in terms of individual component dynamics, network property and link failure uncertainty, to maintain mean square exponential synchronization among network component dynamics. With link failure uncertainty model as on-off Bermoulli switch, we show that there exists a critical probability below which mean square exponential synchronization is not possible.

We carry out an analysis based on the linearized system obtained from the given nonlinear dynamics. Here the linearized system at any time instant is derived from the state of the nonlinear system. Hence it encodes global stability information of the nonlinear system as opposed to local information contained within the linearization around an equilibrium point. A major contribution of this work is to demonstrate how non-equilibrium global dynamics of a nonlinear system affects synchronization over a network in terms of Lyapunov exponents. Furthermore, the main results of this paper can be used to determine the weakest link in the network. The identification of such weakest critical link is important in application such as network power systems for the prevention of cascade failures [12], [13]. Furthermore the uncertainty model and the synchronization results are particularly attractive from the point of view of its application to study the synchronization in neurons [14].

The paper is organized as follows. Preliminaries and problem formulation are discussed in section II. The main results are proved in section III. Simulation results are provided in section IV followed by conclusion in section V.

II. Problem Formulation and Preliminaries

Consider M identical systems given by

$$x_i(t+1) = f(x_i(t)), \qquad \forall i \in \{1,\dots,M\} \tag{1}$$

where $x_i(t) \in X \subseteq \mathbb{R}^N$, for all $i \in \{1,\dots,M\}$. We make the following assumptions on the system (1).

Assumption 1: The system mapping f is assumed to be C^1 function of x and the Jacobian $\frac{\partial f}{\partial x}(x)$ is invertible and bounded for almost all Lebesgue measurable $x \in X$.

We are interested in the synchronization of these M systems when they are coupled together through a network where, synchronization is achieved when all the components have identical state at all times i.e. $x_i(t) = x_j(t)$, $\forall i, j$, $\forall t$. Furthermore we assume the network coupling to be undirected. Let

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the underlying network for the coupling be denoted by an undirected graph $\mathscr{G}(\mathscr{V},\mathscr{E})$. Let $\mathscr{V}$ and $\mathscr{E}$ be the set of all vertices and edges respectively, of the graph $\mathscr{G}$. Henceforth when we refer to the network we imply $\mathscr{G}$.

Assumption 2: We assume that the components are connected over the network through an identical scalar value function of states.

We can now write the equation for the networked system as

$$x_i(t+1) = f(x_i(t)) + \sum_{j=1, j\neq i}^{M} a_{ij} b\left(g(x_j(t)) - g(x_i(t))\right) \quad (2)$$

for all $i \in \{1,\ldots,M\}$, where $b \in \mathbb{R}^N$ and $g : \mathbb{R}^N \to \mathbb{R}$ a C^1 function and $a_{ij} = 1$, $\forall e_{ij} \in \mathscr{E}$. As we have assumed that the graph is undirected we have $a_{ij} = a_{ji}$, $\forall i \neq j$. We define the graph Laplacian as $L = [l_{ij}]_{n\times n}$ where

$$l_{ij} = \begin{cases} a_{ij} & \text{if } j \neq i \\ -\sum_{k,i\neq k} a_{ik} & j = i \end{cases} \quad (3)$$

Hence (2) may be written as

$$\bar{x}(t+1) = \bar{f}(\bar{x}(t)) + (L \otimes b)\, \bar{g}(\bar{x}(t)) \quad (4)$$

where $\bar{x}(t) = [x_1'(t), \cdots, x_M'(t)]'$, $x_i'(t)$ denotes the transpose of $x_i(t)$, $\bar{f}(\bar{x}(t)) = [f'(x_1(t)), \cdots, f'(x_M(t))]'$ and $\bar{g}(\bar{x}(t)) = [g(x_1(t)), \cdots, g(x_M(t))]'$. Here, $L \otimes b$ denotes the Kronecker product. We now describe the model for randomness in the network. The uncertainty in the links connecting any two components (i.e. edges of the graph $\mathscr{G}$) is modeled as a on-off Bernoulli switch $\xi_{ij} \in \{0,1\}$ with following probability distribution Prob$\left\{\xi_{ij}(t) = 1\right\} = p, \forall e_{ij} \in \mathscr{E}$. The random variable ξ_{ij} is independent for all pairs (i, j). Thus (4) with Bernoulli switching may be written as

$$\bar{x}(t+1) = \bar{f}(\bar{x}(t)) + \left(L_1 \otimes b + L_2\left(\Xi(t)\right) \otimes b\right) \bar{g}(\bar{x}(t)) \quad (5)$$

where $L_2(\Xi(t))$ is the random graph Laplacian for the random graph obtained due to the stochastic on-off coupling, where $\Xi(t) = \mathrm{diag}\{\xi_1(t), \cdots, \xi_r(t)\}$ where r is the cardinality of the total number of uncertain edges.

The synchronization manifold of the system given by $\mathscr{S} = \{s(t) \colon s(t+1) = f(s(t)),\ t \in \mathbb{Z}\}$, is the set we desire the network components to eventually converge to. We can write the error between each component state and the $\mathscr{S}$ to be $e_i(t) = x_i(t) - s(t)$. In order to achieve synchronization, it is sufficient for this error to converge to zero. Now suppose we write $\bar{e}(t) = [e_1'(t), \cdots, e_M'(t)]'$ we can write the error dynamics for each error in a compact form as

$$\begin{aligned} \bar{e}(t+1) = {} & \bar{f}(\bar{x}(t)) - \bar{f}(s(t)) + (L_1 \otimes b)\left(\bar{g}(\bar{x}(t)) - \bar{g}(s(t))\right) \\ & + \left(L_2(\Xi(t)) \otimes b\right)\left(\bar{g}(\bar{x}(t)) - \bar{g}(s(t))\right) \end{aligned} \quad (6)$$

where we use the property of every Laplacian for M node graph that, it has an eigenvector of all ones $\mathbf{1}_{M,1} := [1,\ldots,1]' \in \mathbb{R}^M$ with zero eigenvalue.

We are interested in deriving conditions for the stability of the synchronization error dynamics (6). The stochastic nature of error dynamics due to on-off switching in the network coupling, requires an appropriate stochastic notion of stability, for the synchronization of the dynamics. Consider a general random dynamical system (RDS) of the form:

$$x(t+1) = S(x(t), \zeta(t)) \quad (7)$$

where $x(t) \in X \subset \mathbb{R}^N$, a compact set, $\zeta(t) \in W \equiv \{0,1\}$ compact for $t \geq 0$, is a sequence of i.i.d random variables, $S : X \times W \to X$ is the nonlinear map assumed to be atleast C^1 with respect to $x(t) \in X$ for any given fixed $\zeta(t) \in W$. We assume that $x = 0$ is an equilibrium point for (7) i.e., $S(0, \zeta(t)) = 0$. The following notion of stability can be defined for (7) [15], [16].

Definition 3 (Exponential Mean Square (EMS) Stable): The solution $x = 0$ is said to be EMS stable for (7) if there exists a positive constants $K < \infty$ and $\beta < 1$ such that

$$E_{\zeta_0^t}\left[\| x(t+1) \|^2\right] \leq K\beta^t \|x(0)\|^2,\ \forall t \geq 0$$

for almost all w.r.t. Lebesgue measure initial condition $x(0) \in X$ where $E_{\zeta_0^t}[\cdot]$ is the expectation taken over $\{\zeta(0), \ldots, \zeta(t)\}$. For further information regarding the definition we direct the reader to [17], [15], [16].

Remark 4: We will say the coupled system (5) is synchronized in exponential mean square sense if for the error dynamics (6), origin is exponentially mean square stable.

Next we define Lyapunov exponents from ergodic theory of dynamical systems. For more details refer to [18], [19].

Definition 5 (Lyapunov exponents): For deterministic system $x(t+1) = f(x(t))$, let $x \mapsto \mathscr{M}(x)$ where $\mathscr{M}(x)$ is an $N \times N$ invertible matrix. Furthermore, suppose $M(x(t))$ satisfies the condition $\lim_{t\to\infty} \frac{1}{t} \log(||\mathscr{M}(x(t))||)$ and $\lim_{t\to\infty} \frac{1}{t} \log(||\mathscr{M}(x(t))^{-1}||)$, then we can define

$$\Lambda(x(0)) = \lim_{t\to\infty} \left(\mathscr{M}(x(0),t)'\mathscr{M}(x(0),t)\right)^{\frac{1}{2t}} \quad (8)$$

where $\mathscr{M}(x(0),t) := M(x(t))\ldots M(x(0))$. Let λ_{exp}^i for $i = 1,\ldots,N$ be the eigenvalues of $\Lambda(x(0))$ such that $\lambda_{exp}^1 \geq \lambda_{exp}^2 \geq \cdots \geq \lambda_{exp}^N$. Then the Lyapunov exponents Λ_{exp}^i are defined as $\Lambda_{exp}^i = \log \lambda_{exp}^i$ for $i = 1,\ldots,N$. Also Λ_{exp}^1 is known as the maximum Lyapunov exponent. Furthermore, if $\Lambda(x(0)) \neq 0$ then

$$\lim_{t\to\infty} \frac{1}{t} \log\left|\det\left(\mathscr{M}(x(0),t)\right)\right| = \log \prod_{k=1}^{N} \lambda_{exp}^k(x(0)) \quad (9)$$

The limit (8) is known to exists by Oseledet's Multiplicative Ergodic theorem [20]. For the proof of (9), we refer the readers to [21] Proposition 1.3 and Theorem 1.6 and [22].

III. Main results

There are two main results of this paper. Our first main result provides necessary condition for the mean square exponential synchronization where only one link in the network is uncertain. The second main result provides necessary condition for mean square exponential synchronization when all the links in the network are uncertain but with only one uncertainty.

Theorem 6 (Single Link Uncertain): Consider the uncertain network system (5) and let the link between the vertices $v_i, v_j \in \mathscr{V}$ be uncertain, where the uncertainty is model as Bernoulli random variable with probability of non-erasure equal to p. Let $\mathscr{H}_j(s(t)) = \frac{\partial f}{\partial x}(s(t)) + \sigma_j b \frac{\partial g}{\partial x}(s(t))$, where σ_j is the j^{th} eigenvalue of the deterministic Laplacian L_1 and $\lambda^k_{exp}(j) = \exp(\Lambda^k_{exp}(j))$ and $\Lambda^k_{exp}(j)$ be the k^{th} Lyapunov exponent of the system

$$y_j(t+1) = \mathscr{H}_j(s(t)) y_j(t), \quad s(t+1) = f(s(t)) \tag{10}$$

We denote by $\bar{\Lambda}^k_{exp}(j)$ the positive Lyapunov exponents of (10) and $\bar{\lambda}^k_{exp}(j) = \exp(\bar{\Lambda}^k_{exp}(j))$ The necessary condition for (5) to synchronize exponentially in mean square sense is given by

$$(1-p)\left(\prod_{j=1}^{M-1}\left(\prod_{i=1}^{N_j}\bar{\lambda}^{k_i}_{exp}(\sigma_j)\right)^2\right) < 1 \tag{11}$$

where $k_i \in \{1, \dots, N_j\}$ and $1 \leq N_j \leq N$ for $j = 1, \dots, M-1$.

The next theorem provides necessary condition for the mean square synchronization for simultaneous switching of all links.

Theorem 7 (All Links Uncertain): Consider the uncertain network system (5) with all links uncertain and with single uncertainty i.e. $L_1 \otimes b + L_2(\Xi(t)) \otimes b = \xi(t) L \otimes b$ where $\xi(t)$ is a Bernoulli random variable with probability of non-erasure equal to p. Then the necessary condition for (5) to synchronize exponentially in mean square sense is given by

$$(1 - p_{network})\left(\prod_{i=1}^{\bar{N}} \bar{\lambda}^{k_i}_{exp}\right)^2 < 1 \tag{12}$$

where $p_{network} = p\frac{\sigma_1}{\sigma_{M-1}}\left(2 - \frac{\sigma_1}{\sigma_{M-1}}\right)$, where σ_i is the i^{th} non-zero eigenvalue of the Laplacian matrix L of the graph $\mathscr{G}(\mathscr{V}, \mathscr{E})$ and $\lambda^k_{exp} = \exp(\Lambda^k_{exp})$ and Λ^k_{exp} is the k^{th} Lyapunov exponent of the system $x(t+1) = f(x(t))$. Here we denote by $\bar{\Lambda}^{k_i}_{exp}$ the positive Lyapunov exponents of $x(t+1) = f(x(t))$ where $k_i \in \{1, \dots, \bar{N}\}$ and $1 \leq \bar{N} \leq N$ and $\bar{\lambda}^k_{exp} = \exp(\bar{\Lambda}^k_{exp})$.

For the synchronization problem, our aim is to get all the systems to converge to the set $\mathscr{S} = \{s(t) \colon s(t+1) = f(s(t)),\ t \in \mathbb{Z}\}$ which is a fixed sequence. Then for this sequence of maps $\mathscr{M}(s(t))$ if $\lim_{t\to\infty}\frac{1}{t}\log(||M(s(t))||) = 0$ and $\lim_{t\to\infty}\frac{1}{t}\log(||M(s(t))^{-1}||) = 0$ we can define Lyapunov exponents as in Definition (5) for sequence $M(s(t))$. These exponents being defined for a sequence of matrices are unique with respect to the initial condition in $\mathbb{R}^N$ as given by Oseledet's Multiplicative Ergodic Theorem [20]. We will make use of this general definition of Lyapunov exponents to define exponents of certain linear maps in the proofs of the main theorems.

The error equation in (6) resembles the observer error equation with $s(t)$ being the true state and $\bar{x}$ the estimated states. In [23], the problem of nonlinear observation over analog erasure channel, modeled as a Bernoulli random variable, is studied. In particular, the main results of [23] (Theorem 10) provides necessary condition for the mean square exponential stabilziation of the observer error dynamics. This resemblence between the synchronization problem and the observation problem will now be used to prove the two main results of this paper. The first Lemma towards the proof of the main Theorems 6 and 7 provides necessary condition for the synchronization of (6) in terms of the linearization of the error dynamics.

Lemma 8: The necessary condition for the synchronization of the system (6) is that the following linearized error dynamics

$$\begin{aligned} s(t+1) &= f(s(t)) \\ \bar{\eta}(t+1) &= \left(\frac{\partial \bar{f}}{\partial \bar{x}}(s(t)) + ((L_1 + L_2(\Xi(t))) \otimes b)\frac{\partial \bar{g}}{\partial \bar{x}}(s(t))\right)\bar{\eta}(t) \\ &= \Big(I_M \otimes A(s(t)) + (L_1 + L_2(\Xi(t))) \otimes bG(s(t))\Big)\bar{\eta}(t) \end{aligned} \tag{13}$$

is mean square exponentially stable i.e. there exist positive constants $K_2 < \infty$ and $\beta < 1$ s.t.

$$E_{\Xi_0^t}\left[\| \bar{\eta}(t+1) \|^2\right] \leq K_2 \beta^t \| \bar{\eta}(0) \|^2,\ \forall t > 0 \tag{14}$$

for almost all Lebesgue measurable initial condition $\bar{\eta}(0) \in \mathbb{R}^N$ where $E_{\Xi_0^t}[\cdot]$ is the expectation over $\{\Xi(0), \dots, \Xi(t)\}$.

Proof: The proof uses Mean Value Theorem for vector valued functions and Fatou's Lemma and is on similar lines to the proof in [23]. ■

We now use some properties of graph Laplacian to simplify the system equation (13). We choose V to be the orthonormal eigenvectors of the deterministic network Laplacian L_1 as given in (13) to get

$$L_1 = V\begin{bmatrix} 0 & 0 \\ 0 & \tilde{\Lambda}_1 \end{bmatrix}V' = V\Lambda_1 V' \tag{15}$$

Here $\tilde{\Lambda}_1$ is a diagonal matrix of non-zero eigenvalues of L_1. Then we can decompose the Laplacian with uncertain links $L_2(\Xi(t))$ as

$$L_2(\Xi(t)) = V\begin{bmatrix} 0 & 0 \\ 0 & \tilde{\mathscr{L}}(\Xi(t)) \end{bmatrix}V' = V\mathbb{L}(\Xi(t))V' \tag{16}$$

We now give a simplified form of (13) in the following lemma.

Lemma 9: The mean square exponential stability of (13) is equivalent to the mean square exponential stability of

$$\begin{aligned} s(t+1) &= f(s(t)) \\ \tilde{\eta}(t+1) &= (I_{M-1} \otimes A(s(t)))\,\tilde{\eta}(t) \\ &\quad + \left(\left(\tilde{\Lambda}_1 + \tilde{\mathscr{L}}(\Xi(t))\right) \otimes bG(s(t))\right)\tilde{\eta}(t) \\ &:= \mathscr{A}(s(t), \Xi(t))\tilde{\eta}(t) \end{aligned} \tag{17}$$

where $\tilde{\Lambda}_1$ is given in (15) and $\tilde{\mathscr{L}}(\Xi(t))$ is given in (16).

Proof: The proof uses the property of the graph Laplacian matrix that it has an eigenvector with zero eigen value which gives rise to average linearized error. This linearized error cannot be stabilized as it signifies the difference of the synchronized system from the trajectory $s(t+1) = f(s(t))$ with initial condition $s(0)$ that we have considered. This error is not of significance to synchronization of the trajectories as our aim is to synchronize to the manifold and not to the specific trajectory we consider. Hence the exponential mean square stability of (13) is equivalent to exponential mean square stability of (17). ■

To study the synchronization of the linear error dynamics we will first give a Lyapunov function condition as a necessary condition for exponential mean square stability

Theorem 10: Let the $\tilde{\eta}(t)$ in the system (17) be exponentially mean square stable (Definition 3). Then there exists a matrix function $P(s(t)$ as a function of $s(t)$ and positive constants α, γ such that, $\alpha I_{N(M-1)} \leq P(s(t)) \leq \gamma I_{N(M-1)}$ and $P(s(t))$ satisfies the following inequality

$$E_{\Xi(t)}\left[\mathscr{A}'(s(t),\Xi(t))P(s(t+1))\mathscr{A}(s,\Xi(t))\right] < P(s(t)) \quad (18)$$

where $s(t+1) = f(s(t))$.

Proof: This follows from equation (17) and necessary condition for exponential mean square stability proved in [23] Lemma 10. ■

We now give the proof for the different cases for the stochastic coupling given in Theorem 6 and 7.

Proof of Theorem 6 for single link switching:

Since only one link is uncertain $L_2(\Xi(t)) = \xi(t)L_2$ for (5). From the equations given in (17) we get

$$\tilde{\eta}(t+1) = \Big(I_{M-1}\otimes A(s(t)) + \big(\tilde{\Lambda}_1 + \xi(t)\tilde{\mathscr{L}}\big)\otimes bG(s(t))\Big)\tilde{\eta}(t) \quad (19)$$

where $\tilde{\mathscr{L}}$ is obtained as follows $\tilde{\mathscr{L}} = \tilde{U}'L_2\tilde{U} = \tilde{U}'\ell_{ij}\ell_{ij}'\tilde{U} = \tilde{\ell}\tilde{\ell}'$, where L_2 is the Laplacian for just the uncertain edge where between vertices $v_i, v_j \in \mathscr{V}$ and $\ell_{ij} \in \mathbb{R}^M$ with 1 in the i^{th} position and -1 in the j^{th} position, rest all being zeros. Thus Laplacian $\tilde{\mathscr{L}}$ is a rank 1 matrix and has only one non-zero eigenvalue $\sigma_U = 2$. Thus we can write (19) as

$$\tilde{\eta}(t+1) = \big(\mathscr{H}(s(t)) + \xi(t)\tilde{b}\tilde{K}(s(t))\big)\tilde{\eta}(t) \quad (20)$$

where $\mathscr{H}(s(t)) = I_{M-1}\otimes A(s(t)) + \tilde{\Lambda}_1\otimes bG(s(t))$, $\tilde{b} = \tilde{\ell}\otimes b$ and $\tilde{K}(s(t)) = \tilde{\ell}\otimes G(s(t))$. We know from Lemma 9 and Theorem 10, that the necessary condition for mean square synchronization can be expressed in terms of the existence of matrix $P(s(t))$ such that following condition is satisfied

$$E_{\xi(t)}\Big[\big(\mathscr{H}(s(t)) + \xi(t)\tilde{b}\tilde{K}(s(t))\big)'P(s(t+1))\times\cdots$$
$$\cdots\big(\mathscr{H}(s(t)) + \xi(t)\tilde{b}\tilde{K}(s(t))\big)\Big] < P(s(t)) \quad (21)$$

for all $t \geq 0$. Expanding the above equation, we get

$$\mathscr{H}(s(t))'P(s(t+1))\mathscr{H}(s(t)) + p\tilde{K}(s(t))'\tilde{b}'P(s(t+1))\mathscr{H}(s(t))$$
$$+ p\mathscr{H}(s(t))'P(s(t+1))\tilde{b}\tilde{K}(s(t))$$
$$+ p\tilde{K}(s(t))'\tilde{b}'P(s(t+1))\tilde{b}\tilde{K}(s(t)) < P(s(t)) \quad (22)$$

The gain $\tilde{K}(s(t))$ in the above equation has a special structure, in particular $\tilde{K}(s(t)) = \tilde{\ell}'\otimes G(s(t))$. Since we are interested in deriving necessary condition for mean square synchronization, the necessary condition for inequality (22) to be true can be written as

$$\mathscr{H}(s(t))'P(s(t+1))\mathscr{H}(s(t)) + p\mathscr{K}(s(t))'\tilde{b}'P(s(t+1))\mathscr{H}(s(t))$$
$$+ p\mathscr{H}(s(t))'P(s(t+1))\tilde{b}\mathscr{K}(s(t))$$
$$+ p\mathscr{K}(s(t))'\tilde{b}'P(s(t+1))\tilde{b}\mathscr{K}(s(t)) < P(s(t)) \quad (23)$$

where $\mathscr{K}(s(t))$ is not constrained to be equal to $\tilde{\ell}'\otimes G(s(t))$. The solution to the inequality (23) for unknown gain $\mathscr{K}(s(t))$ and Lyapunov function matrix P proceeds in two steps. In first step, the optimal gain $\mathscr{K}(s(t))$ can be obtained in terms of the unknown Lyapunov function matrix P. The optimal gain $\mathscr{K}(s(t))$ can be obtained by minimizing the r.h.s of (23) w.r.t. $\mathscr{K}(s(t))$ and can be shown to be equal to $\mathscr{K}(s(t)) = -\frac{\tilde{b}'P(s(t+1))\mathscr{H}(s(t))}{\tilde{b}'P(s(t+1))\tilde{b}}$. Substituting $\mathscr{K}(s(t))$ in (23), we get

$$\mathscr{H}'(s(t))P(s(t+1))\mathscr{H}(s(t))$$
$$- p\frac{\mathscr{H}'(s(t))P(s(t+1))\tilde{b}\tilde{b}'P(s(t+1))\mathscr{H}(s(t))}{\tilde{b}'P(s(t+1))\tilde{b}} < P(s(t)) \quad (24)$$

The second step is to search over the optimal (smallest) matrix Lyapunov function P that satisfies the above inequality. Following the proof in [23], it can be shown that the necessary condition for the existence of optimal matrix Lyapunov function $P(s(t))$ that satisfies above inequality is given by

$$(1-p)\left(\prod_{k=1}^{N_M}\bar{\lambda}_{exp}^{k_i}\right)^2 < 1 \quad (25)$$

where $\bar{\lambda}_{exp}^{k_i} = \exp(\bar{\Lambda}_{exp}^{k_i})$, and $\bar{\Lambda}_{exp}^{k_i}$ are the positive Lyapunov exponents for the system $y(t+1) = \mathscr{H}(s(t))y(t)$, $y\in\mathbb{R}^{N(M-1)}$, and $s(t+1) = f(s(t))$, where $k_i \in \{1,\ldots,N_M\}$ and $1 \leq N_M \leq N(M-1)$. Let $\mathscr{H}_j(s(t)) = \frac{\partial f}{\partial x}(s(t)) + \sigma_j b\frac{\partial g}{\partial x}(s(t))$, where σ_j is the j^{th} eigenvalue of the deterministic Laplacian L_1 and $\Lambda_{exp}^{k}(j)$ be the k^{th} Lyapunov exponent of the system

$$y_j(t+1) = \mathscr{H}_j(s(t))y_j(t), \quad s(t+1) = f(s(t)) \quad (26)$$

for $j = \{1,\ldots,M-1\}$. We denote by $\bar{\Lambda}_{exp}^{k_i}(j)$ the positive Lyapunov exponents of (26) and $\bar{\lambda}_{exp}^{k_i}(j) = \exp(\bar{\Lambda}_{exp}^{k_i}(j))$ where $k_i \in \{1,\ldots,N_j\}$ and $1 \leq N_j \leq N$ for $j = 1,\ldots,M-1$. As $y(t+1) = \mathscr{H}(s(t))y(t)$ is just $M-1$ decoupled equations given by (26) the positive Lyapunov exponents $\bar{\Lambda}_{exp}^{k_i}$ in (25) are the positive Lyapunov exponents $\bar{\Lambda}_{exp}^{k_i}(j)$ for $M-1$ systems (26). Hence we get the necessary condition for mean square exponential stability of (5) to be

$$(1-p)\left(\prod_{j=1}^{M-1}\left(\prod_{i=1}^{N_j}\bar{\lambda}_{exp}^{k_i}(\sigma_j)\right)^2\right) < 1 \quad (27)$$

for $k_i \in \{1,\ldots,N_j\}$ and $1 \leq N_j \leq N$ for $j = 1,\ldots,M-1$. ■

Proof of Theorem 7 for all links switching in unison:

We now consider the case in Theorem 7 where the entire network is blinking in unison. For the system $s(t+1) = f(s(t))$ we can define the Lyapunov exponents from Definition 5 for the matrices $M(s(t)) = \frac{\partial f}{\partial x}(s(t))$. For the case of all links switching, we assume that either all the systems are coupled or none are coupled at a time instant. In this case we have $\Xi(t) = \xi(t)$.Therefore we get $L_1 = 0$ and $L_2(\Xi(t)) = \xi(t)L$, where L is the Laplacian matrix of the network $\mathscr{G}$. Let

$L = V_1 \Lambda V_1'$. Let $\bar{\zeta}(t) = (V_1' \otimes I_N)\, \bar{\eta}(t)$. We can then write the linearized error dynamics as follows

$$\bar{\zeta}(t+1) = (I_M \otimes A(s(t)) + \xi(t)\Lambda \otimes bG(s(t)))\, \bar{\zeta}(t) \qquad (28)$$

where Λ is diagonal matrix of eigenvalues of L. Thus we can write each of these equations in a decoupled form as

$$\zeta_i(t+1) = (A(s(t)) + \sigma_i bG(s(t)))\, \zeta_i(t), \qquad (29)$$

$\forall i = \{1, \ldots, M-1\}$ where σ_i are non-zero eigenvalues of Λ. From (29) we can construct a sample system of the kind

$$\zeta(t+1) = (A(s(t)) + \xi(t)\tau bG(s(t)))\, \zeta(t) \qquad (30)$$

From Theorem 10, we knwo that the necessary condition for the mean square exponential stability of (30) can be expressed in terms of the existence of matrix Lyapunov function P such that

$$E_{\xi(t)}\left[(A(s(t)) + \xi(t)\tau bG(s(t)))' P(s(t+1)) \times \cdots \right.$$
$$\left. \cdots (A(s(t)) + \xi(t)\tau bG(s(t))) \right] < P(s(t))$$

Expanding the above equation and minimizing the L.H.S. of the above inequality w.r.t. G to determine the optimal gain G, we get $G(s(t)) = \frac{-1}{\tau} \frac{b'P(s(t+1))A(s(t))}{b'P(s(t+1))b}$. We choose the strictest possible $\tau = \sigma_{M-1}$ for the gain so as to at least satisfy the equation for the maximum eigenvalue σ_{M-1} of Λ_1 Now we use this Lyapunov function $P(s(t))$ and gain $G(s(t)) = \frac{-1}{\sigma_{M-1}} \frac{b'P(s(t+1))A(s(t))}{b'P(s(t+1))b}$ for each of the systems in (29). This gives us the necessary condition for mean square stability of each decoupled linearized error to be equal to

$$A(s(t))'P(s(t+1))A(s(t))$$
$$- p_{network} \frac{A(s(t))'P(s(t+1))bb'P(s(t+1))A(s(t))}{b'P(s(t+1))b} < P(s(t))$$

for all $i = \{1, \ldots, M-1\}$, where $p_{network} := p\left(\frac{2\sigma_1}{\sigma_{M-1}} - \frac{\sigma_1^2}{\sigma_{M-1}^2}\right)$. The search for the optimal matrix Lyapunov function P will proceed similar to the proof from [23], where it can be shown that the necessary condition for the existence of optimal matrix Lyapunov function P is that following inequlaity is satisfied.

$$(1 - p_{network}) \left(\prod_{i=1}^{\bar{N}} \bar{\lambda}_{exp}^{k_i} \right)^2 < 1 \qquad (31)$$

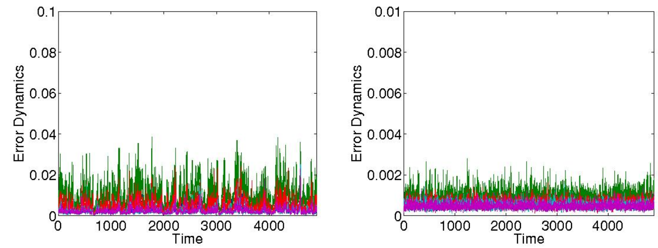
where $p_{network} = p\frac{\sigma_1}{\sigma_{M-1}}\left(2 - \frac{\sigma_1}{\sigma_{M-1}}\right)$, where σ_i is the i^{th} non-zero eigenvalue of the Laplacian matrix L of the graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ and $\lambda_{exp}^k = \exp(\Lambda_{exp}^k)$ where Λ_{exp}^k is the k^{th} Lyapunov exponent of the system $x(t+1) = f(x(t))$. $\bar{\Lambda}_{exp}^{k_i}$ denotes the positive Lyapunov exponents of $x(t+1) = f(x(t))$ where $k_i \in \{1, \ldots, \bar{N}\}$, $1 \leq \bar{N} \leq N$ and $\bar{\lambda}_{exp}^{k_i} = \exp(\bar{\Lambda}_{exp}^{k_i})$. ■

IV. Simulation Results

A. All Links Switching in Unison

In this case the whole network blinks with a given probability p simultaneously. We choose the system to synchronize as the one dimensional logistic map given by

$$x_i(t+1) = a x_i(t)\,(1 - x_i(t)) := f(x_i(t)) \qquad (32)$$



(a) (b)

Fig. 1. (a) Error dynamics at all link non-erasure probability $p = 0.45$, (b) Error dynamics at all link non-erasure probability $p = 0.8$

where $a = 3.6$. We consider a simple network of $N_V = 6$ nodes with edge set $\mathcal{E} = \{e_{12}, e_{15}, e_{16}, e_{24}, e_{34}, e_{36}, e_{45}, e_{46}, e_{56}\}$. We assume that all the links switch in unison and the probability of switching is p. We set the connection over the network as in [24] such that, $k(x_i(t)) = \frac{b}{aN_V} f(x_i(t)) = \frac{b}{N_V} x_i(t)\,(1 - x_i(t))$ where we choose $b = 4$. This coupling ensures that each logistic map evolves in $[0,1]$ even after coupling. The Lyapunov exponent of the system for is $\lambda_{exp} = 1.203$. The second smallest eigenvalue of the Laplacian matrix is given by $\sigma_1 = 1.6072$. The maximum eigenvalue of the Laplacian is $\sigma_{M-1} = 5.5869$. This gives a critical probability of synchronization as $p^* = \left(1 - \frac{1}{\lambda_{exp}^2}\right) \frac{\sigma_{M-1}^2}{\sigma_1(2\sigma_{M-1} - \sigma_1)} = 0.6148$. We add uniform noise with variance $\sigma^2 = 0.01$ to be able to visualize the stability of the interconnection. In Figure (1a) and (1b) we plot the error dynamics between adjacent nodes $v_i - v_{i+1}$ for $i \in \{1, \ldots, V_N - 1\}$ at switching probabilities $p_1 = 0.45$ and $p_2 = 0.8$, below and above the critical probability respectively. As we can see the plots in Figure (1a) the error dynamics is shows much more fluctuation as compared to those in Figure (1b) by an order of magnitude difference.

B. Single Link Switching

In this case a single link blinks with a given probability p at a given time. We choose coupled periodically forced Duffing's oscillators (33) linearly coupled over the network $\mathcal{G}_D(\mathcal{V}_D, \mathcal{E}_D)$ in Figure 2

$$\dot{x}_{i1}(t) = x_{i2}(t)$$
$$\dot{x}_{i2}(t) = -0.1x_{i2}(t) + x_{i1}(t) - 0.25x_{i1}(t)^3 + 1.4\cos(2t)$$
$$+ \sum_{e_{ij} \in \mathcal{E}_D} K(x_{j1}(t) - x_{i1}(t)) + r_{i2}(t) \qquad (33)$$

where the coupling constant is chosen as $K = 2.5$ and $r_{ij}(t)$ is zero mean gaussian noise with variance $R = 0.1$. The noise is added to the system to help us visualize the mean square unstable nature of the system. We take iterations of this system after every π seconds to obtain the Poincaré map of the system. We make the link between oscillator 3 and 4 uncertain with probability p. We compute the Lyapunov exponents for the duffing oscillator with coupling scaled by each of the 6 Laplacian eigenvalues. Then using the fact that the Lyapunov exponents for the Poincaré map are simply the exponents of the flow scaled by the average return time

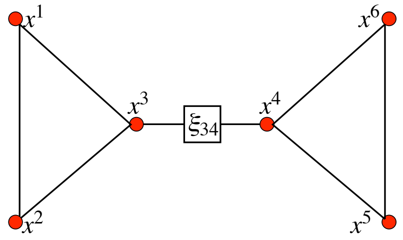


Fig. 2. Graph network $\mathcal{G}$ with on-off link between node 3 and 4.

[25], we obtain the exponents for the Poincaré map. Only two of these exponents are positive and they give the critical probability to be $p^* = 0.88$ as in Theorem 6.

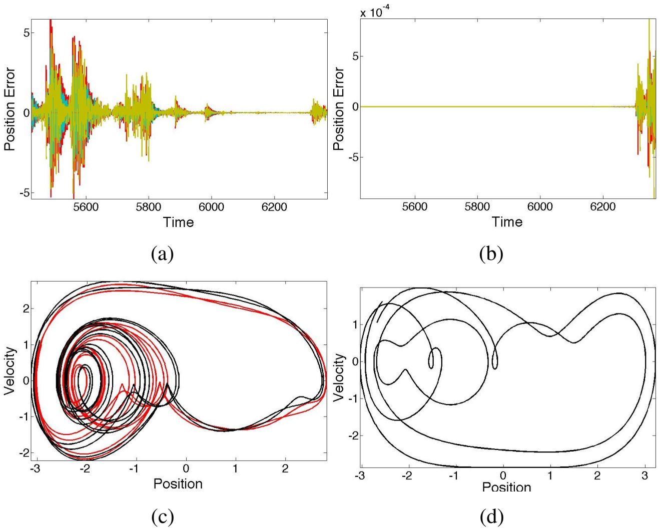


Fig. 3. (a) Position error as a function of time for $p = 0.89$, (b) Position error as a function of time at $p = 0.91$, (c) Phase space trajectories for $p = 0.89$, (d) Phase space trajectories for $p = 0.91$

In Figure (3a) and Figure (3b) we give the error dynamics between neighboring oscillators at probabilities $p = 0.89$ and $p = 0.91$. We see that at $p = 0.89$ the error doesn't converge to zero while for $p = 0.91$ the error dynamics almost converges to zero. In Figure (3c) and Figure (3d) we plot the phase space of all the coupled oscillators for non-erasure probabilities $p = 0.89$ and $p = 0.91$ respectively. We clearly see that for $p = 0.91$ the oscillators are synchronized, while for $p = 0.06$ in Figure (3c) the oscillators are not able to synchronize. Thus we see that the critical transition probability $p^* = 0.88$ is necessary for synchronization.

V. Conclusion and Future Work

We studied the problem of robust synchronization in nonlinear network, where the interconnection link between the individual subsystems is uncertain. The main result of this paper provides critical value of non-erasure probability p^*, below which the mean square synchronization among the subsystem states is not possible. The critical probability is shown to be the function of the individual components dynamics and the network property. Future research efforts will focus on generalizing this work to consider arbitrary uncertain links with heterogenous components dynamics.

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