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Finite Horizon MPC for Systems in Innovation Form

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Abstract— **System identification and model predictive control have largely developed as two separate disciplines. Nevertheless, the major part of industrial MPC commissioning is generation of data and identification of models. In this contribution we attempt to bridge this gap by contributing some of the missing links. Input-output models (FIR, ARX, ARMAX, Box-Jenkins) as well as subspace models can be represented as state space models in innovation form. These models have correlated process and measurement noise. The correct LQG control law for systems with correlated process and measurement noise is not well known. We provide the correct finite-horizon LQG controller for this system and use this to develop a state space representation of the closed-loop system. This representation is used for closed-loop frequency and covariance analysis. These measures are used in tuning of the unconstrained and constrained MPC. We demonstrate our results on a simulated industrial furnace.**

I. Introduction

Surprisingly, model based control such as model predictive control [1]–[11] and system identification [12]–[22] have evolved almost independently. Even though system identification is the major part of a model predictive control project and accurate models are essential for good performance of the resulting controller, little attention is given to system identification in the model predictive control literature. Modern linear model predictive control is based on constrained linear quadratic regulators, state space models, and output feedback using Kalman filters equipped with disturbance models to have offset free control [23]–[25]. The system identification literature is concerned with identification of input-output models such as finite impulse response models (FIR), auto regressive models with exogenous inputs (ARX), auto regressive moving average models with exogenous inputs (ARMAX), and Box-Jenkins models. Like subspace identification, these input-output parameterizations may be realized as a state space model in innovation form. The state space model in innovation form has correlated process and measurement noise, while the state space models used in [1]–[11], [23]–[25] have uncorrelated process and measurement noise. Most textbooks on linear quadratic Gaussian control (LQG) do not treat the case with correlated noise [26]–[35] with [36], [37] being notable exceptions. The first complete treatments of the LQG with correlated noise structure seems to be [38], [39]. To apply LQG to linear models obtained using methods from system identification, i.e. state space models in innovation form, the case with correlated process and measurement noise is essential.

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In this paper, we state that FIR, ARX, ARMAX, Box-Jenkins, and subspace methods give linear time invariant stochastic state space model in innovation form. To have offset free control in the face of unknown disturbances, the model obtained from system identification procedures is augmented with a disturbance model in ARMA-form. The combined model is also a linear time invariant stochastic state space model in innovation form. This model is a special case of the linear time invariant stochastic state space model with correlated process and measurement noise. We state the optimal filter and predictor for the linear time invariant stochastic state space model with correlated process and measurement noise. Using the separation principle, we apply the predictor for the correlated case to formulate the finite horizon unconstrained predictive controller and develop an explicit expression for the optimal control. Due to the cross correlation of the process and measurement noise, the feedback part in this optimal control is a linear combination of the filtered state as well as the filtered process noise. We use the explicit expression of the feedback control law to develop a state space model for the closed-loop system. This model is used in tuning of both the unconstrained and the constrained predictive controllers. The system has bounds on its inputs and its rate of input movement. Therefore, we develop a finite horizon predictive controller with input constraints using the predictor based on correlated process and measurement noise.

This paper is organized as follows. Section II discusses input-output model parameterizations and their realization as state space models. Section III briefly develops the Kalman filter for the case with correlated process and measurement noise. Section IV presents receding-horizon regulators without constraints and with hard input constraints. Section V demonstrates the tuning of the controller on a simulated industrial furnace. Conclusions are presented in Section VI.

II. Input-Output Models and Realizations

The model used by the predictive controller in this paper is an input-output model with ARMAX structure

$$A(q^{-1})\boldsymbol{y}_k = B(q^{-1})u_k + C(q^{-1})\boldsymbol{e}_k \tag{1}$$

in which $A(q^{-1})$, $B(q^{-1})$, and $C(q^{-1})$ are polynomials in the backward shift operator

$$A(q^{-1}) = I + A_1 q^{-1} + A_2 q^{-2} + \ldots + A_n q^{-n} \tag{2a}$$
$$B(q^{-1}) = B_1 q^{-1} + B_2 q^{-2} + \ldots + B_n q^{-n} \tag{2b}$$
$$C(q^{-1}) = I + C_1 q^{-1} + C_2 q^{-n} + \ldots + C_n q^{-n} \tag{2c}$$

The model (1) may be identified from input-output data $\{y_k, u_k\}$ using system identification technologies [13]–[19]. (1) is sufficiently general to include FIR, ARX, ARMAX, and Box-Jenkins models. Like models identified using subspace methods, (1) can be realized as a state space model in innovation form.

To have offset free control in the face of unmeasured step disturbances, the model (1) is augmented with a disturbance model. The disturbance considered in this paper is an ARMA type model

$$F(q^{-1})\boldsymbol{e}_k = G(q^{-1})\boldsymbol{\varepsilon}_k \quad \boldsymbol{\varepsilon}_k \sim N(0, R_\varepsilon) \tag{3}$$

with

$$F(q^{-1}) = I + F_1 q^{-1} + F_2 q^{-2} + \ldots + F_m q^{-m} \tag{4a}$$
$$G(q^{-1}) = I + G_1 q^{-1} + G_2 q^{-2} + \ldots + G_m q^{-m} \tag{4b}$$

While the time series literature [14], [15] suggests simultaneous identification of the polynomials in (1) and (3), the predictive control literature [6, Section 4.6] recommends that the disturbance model (3) is used to tune the closed-loop properties and robustness of the resulting predictive control system. To have steady-state offset-free control, $F(q^{-1}) = I - Iq^{-1}$, and the closed-loop performance is typically tuned by selection of G_1 in $G(q^{-1}) = I + G_1 q^{-1}$. (3) may be realized as a state space model in innovation form.

The combined model (1) and (3) may be realized as a linear time invariant state space model in innovation form

$$\boldsymbol{x}_{k+1} = A\boldsymbol{x}_k + Bu_k + K\boldsymbol{\varepsilon}_k \tag{5a}$$
$$\boldsymbol{y}_k = C\boldsymbol{x}_k + \boldsymbol{\varepsilon}_k \tag{5b}$$

with $\boldsymbol{\varepsilon}_k \sim N(0, R_\varepsilon)$.

III. Filtering and Prediction

In this section, we develop the filter and predictor for the stochastic linear time invariant system [40]

$$\boldsymbol{x}_{k+1} = A\boldsymbol{x}_k + Bu_k + G\boldsymbol{w}_k \tag{6a}$$
$$\boldsymbol{y}_k = C\boldsymbol{x}_k + \boldsymbol{v}_k \tag{6b}$$

with

$$\boldsymbol{x}_0 \sim N(\hat{x}_{0|-1}, P_{0|-1}) \tag{7a}$$
$$\begin{bmatrix} \boldsymbol{w}_k \\ \boldsymbol{v}_k \end{bmatrix} \sim N_{iid}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} R_{ww} & R_{wv} \\ R_{vw} & R_{vv} \end{bmatrix} \right) \tag{7b}$$

It is important to notice that the process noise, $\boldsymbol{w}_k$, and the measurement noise, $\boldsymbol{v}_k$, are correlated, i.e. $R_{wv} = R'_{vw} \neq 0$.

Let $\mathcal{I}_0 = y_0$ and $\mathcal{I}_k = \{\mathcal{I}_{k-1}, u_{k-1}, y_k\}$ for $k = 1, 2, \ldots$. Then the conditional variables are normally distributed, i.e. $\boldsymbol{x}_{k+j}|\mathcal{I}_k \sim N(\hat{x}_{k+j|k}, P_{k+j|k})$, $\boldsymbol{w}_{k+j}|\mathcal{I}_k \sim N(\hat{w}_{k+j|k}, Q_{k+j|k})$, and $\boldsymbol{y}_{k+j}|\mathcal{I}_k \sim N(\hat{y}_{k+j|k}, R_{y,k+j|k})$ for $j = 0, 1, 2, \ldots$. Filtering and prediction in (6)-(7) correspond to computation of the conditional means, $\hat{x}_{k+j|k} = E\{\boldsymbol{x}_{k+j}|\mathcal{I}_k\}$, $\hat{w}_{k+j|k} = E\{\boldsymbol{w}_{k+j}|\mathcal{I}_k\}$, and $\hat{y}_{k+j|k} = E\{\boldsymbol{y}_{k+j}|\mathcal{I}_k\}$, and the conditional co-variances, $P_{k+j|k} = V\{\boldsymbol{x}_{k+j}|\mathcal{I}_k\}$, $Q_{k+j|k} = V\{\boldsymbol{w}_{k+j}|\mathcal{I}_k\}$, and $R_{y,k+j|k} = V\{\boldsymbol{y}_{k+j}|\mathcal{I}_k\}$, for $j = 0, 1, 2, \ldots$.

Remark 1: The model in innovation form (5) is a special case of the linear model (6). (5) may be expressed in the form (6) using $G = K$, $\boldsymbol{w}_k = \boldsymbol{v}_k = \boldsymbol{\varepsilon}_k$. Consequently, the model in innovation form (5) has correlated noise: $R_{ww} = R_{vv} = R_{wv} = R_{vw} = R_\varepsilon$.

Remark 2: FIR, ARX, ARMAX, and Box-Jenkins models may be realized in innovation form (5). State space models obtained using subspace identification are also in innovation form (5). Therefore, all these models have correlated process and measurement noise.

Remark 3: When the process and measurement noise are independent, $R_{wv} = R'_{vw} = 0$, $\boldsymbol{w}_{k+j}|\mathcal{I}_k \sim N(0, R_{ww})$ for $j = 0, 1, 2, \ldots$. This is a special case of the situation treated in this paper.

Theorem 1 (Riccati Equation Convergence): Let $R_{vv} > 0$ and define

$$A_e = A - GR_{wv}R_{vv}^{-1}C \tag{8a}$$
$$Q_e = R_{ww} - R_{wv}R_{vv}^{-1}R'_{wv} \tag{8b}$$

Assume that 1) (C, A_e) is detectable, 2) $(A_e, GQ_e^{1/2})$ is stabilizable, and 3) $P_{0|-1} \geq 0$ (positive semi-definite).

Then the sequence, $\left\{P_{k|k-1}\right\}_{k=0}^{N}$, generated by

$$\begin{aligned} P_{k+1|k} = {} & AP_{k|k-1}A' + GR_{ww}G' \\ & - (AP_{k|k-1}C' + GR_{wv}) \\ & (R_{vv} + CP_{k|k-1}C')^{-1}(AP_{k|k-1}C' + GR_{wv})' \end{aligned} \tag{9}$$

converges to

$$\lim_{k\to\infty} P_{k+1|k} = P \tag{10}$$

in which P is the solution of the discrete algebraic Riccati equation (DARE)

$$\begin{aligned} P = {} & APA' + GR_{ww}G' \\ & - (APC' + GR_{wv})(R_{vv} + CPC')^{-1}(APC' + GR_{wv})' \end{aligned} \tag{11}$$

By assumption 1) and 2), this limit, P, is unique, positive semi-definite, and stabilizing (i.e. $A - K_{px}C$ is stable).

Proof: See [40]. ■

In the following, we assume that the system is stochastic stationary, i.e. that $P_{0|-1} = P$. P is computed as specified in Theorem 1. Then the gain matrices used by the filter and the one-step-ahead predictor may be computed as

$$R_{fe} = CPC' + R_{vv} \tag{12a}$$
$$K_{fx} = PC'R_{fe}^{-1} \tag{12b}$$
$$K_{fw} = R_{wv}R_{fe}^{-1} \tag{12c}$$
$$K_{px} = AK_{fx} + GK_{fw} = (APC' + GR_{wv})R_{fe}^{-1} \tag{12d}$$

The filtered estimates, $\hat{x}_{k|k}$ and $\hat{w}_{k|k}$, are computed by

$$e_k = y_k - C\hat{x}_{k|k-1} \tag{13a}$$
$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + K_{fx}e_k \tag{13b}$$
$$\hat{w}_{k|k} = K_{fw}e_k \tag{13c}$$

and the one-step-ahead prediction of the states, $\hat{x}_{k+1|k}$, as well as the $(j+1)$-step-ahead predictions may be obtained using

$$\hat{x}_{k+1|k} = A\hat{x}_{k|k} + Bu_k + G\hat{w}_{k|k} \tag{14a}$$
$$\hat{x}_{k+1+j|k} = A\hat{x}_{k+j|k} + Bu_{k+j} \qquad j = 1, 2, \ldots, N-1 \tag{14b}$$
$$\hat{y}_{k+j|k} = C\hat{x}_{k+j|k} \qquad j = 1, 2, \ldots, N \tag{14c}$$

Combining (13b), (13c) and (14a), the one-step prediction may also be expressed as

$$\hat{x}_{k+1|k} = A\hat{x}_{k|k-1} + Bu_k + K_{px}e_k \tag{15}$$

Remark 4: Models in innovation form with $P_{0|-1} = 0$ converges to $\lim_{k\to\infty} P_{k|k-1} = P = 0$. Therefore, $K_{fx} = 0$, $K_{fw} = I$, $K_{px} = K$, $\hat{x}_{k|k} = \hat{x}_{k|k-1}$, $\hat{w}_{k|k} = e_k$, and $\hat{x}_{k+1|k} = A\hat{x}_{k|k-1} + Bu_k + Ke_k$.

Remark 5: A key observation is that when the process and measurement noise are correlated ($R_{wv} \neq 0$), the gain $K_{fw} \neq 0$ and the process noise term in (14a) are in general non-zero. When using the separation principle and deriving the optimal regulator, it is important to include this term. For the usual case with uncorrelated process and measurement noise, $R_{wv} = 0$ and $K_{fw} = 0$ such that $\hat{w}_{k|k} = 0$. In this case, the term $G\hat{w}_{k|k}$ in (14a) is zero and drops out.

IV. Model Predictive Control

In this section, we use the predictions (14) to develop receding horizon optimal regulators. We state the optimal control law for the case without constraints. We also state the quadratic optimization problem for the case with hard input constraints.

A. Unconstrained MPC

Define the objective function by

$$\phi = \frac{1}{2}\sum_{j=0}^{N-1} \left\|\hat{y}_{k+1+j|k} - r_{k+1+j|k}\right\|_Q^2 + \left\|\Delta u_{k+j}\right\|_S^2 \tag{16}$$

The first term penalizes deviations of the predicted outputs, $\left\{\hat{y}_{k+j+1|k}\right\}_{j=0}^{N-1}$ from the anticipated set-points, $\left\{r_{k+j+1|k}\right\}_{j=0}^{N-1}$. The second term is a regularization term that penalizes excessive movement, $\Delta u_k = u_k - u_{k-1}$, of the manipulated variable. It used to ensure smoothness of the solution.

The finite horizon optimal control problem with the objective function (16) and the predictions (14) may be stated as

$$\min_{\{u_{k+j}\}_{j=0}^{N-1}} \phi \quad s.t.\ (14) \tag{17}$$

The solution to this quadratic program may be stated explicitly as

$$u_k = L_x\hat{x}_{k|k} + L_w\hat{w}_{k|k} + L_R R_k + L_u u_{k-1} \tag{18}$$

$R_k = \begin{bmatrix} r_{k+1|k}; & r_{k+2|k}; & \ldots; & r_{k+N|k}\end{bmatrix}$ is a vector with the anticipated future set-points. The derivation and the expressions for the gains, $\{L_x, L_w, L_R, L_u\}$, are provided in Appendix I. Appendix II derives the corresponding closed-loop properties.

Remark 6: For a model in innovation form with $P_{0|-1} = 0$, the innovation is, $e_k = y_k - C\hat{x}_{k|k-1}$, the controller is

$$u_k = L_x\hat{x}_{k|k-1} + L_w e_k + L_R R_k + L_u u_{k-1} \tag{19}$$

and the one-step prediction is $\hat{x}_{k+1|k} = A\hat{x}_{k|k-1} + Bu_k + Ke_k$.

Remark 7: Note that the optimal control law (18) includes the term $L_w\hat{w}_{k|k}$. This term arises due to the correlation between the process and the measurement noise. In most derivations of the LQG controller this term has been neglected. Therefore, such control laws are not applicable to models with correlated process and measurement noise. Accordingly, they are not applicable to state space models in innovation form.

B. Input Constrained MPC

In practical control applications, the inputs are bounded in size and rate of movement. Consider the input bound constraints

$$u_{\min} \leq u_{k+j} \leq u_{\max} \qquad j = 0, \ldots, N-1 \tag{20}$$

and the rate of input movement constraints

$$\Delta u_{\min} \leq \Delta u_{k+j} \leq \Delta u_{\max} \quad j = 0, \ldots, N-1 \tag{21}$$

Then the open-loop finite horizon optimal control problem with the objective function (16), the predictions (14), and the input constraints (20)-(21) may be stated as the convex quadratic program

$$\min_{\{u_{k+j}\}_{j=0}^{N-1}} \quad \phi \tag{22a}$$
$$s.t. \qquad (14), (20), (21) \tag{22b}$$

Only the first input, u_k, of the optimal input sequence, $\{u_{k+j}\}_{j=0}^{N-1}$, is implemented. Consequently, this regulator can be stated as the function

$$u_k = \mu(\hat{x}_{k|k}, \hat{w}_{k|k}, \left\{r_{k+j|k}\right\}_{j=1}^{N}, u_{k-1}) \tag{23}$$

in which it is understood that the function, μ, involves solution of a convex quadratic program. It should be noted that (14a) is included as a constraint in (22) and that is has a term, $G\hat{w}_{k|k}$, due to the correlation of the process and measurement noise. In most MPC descriptions, this term is neglected. Consequently, these MPCs are not applicable to systems with correlated process and measurement noise. The computational steps in this controller are stated in Algorithm 1.

Algorithm 1 MPC for Systems with Correlated Noise

Require: y_k, $\{r_{k+j|k}\}_{j=1}^{N}$, $\hat{x}_{k|k-1}$, u_{k-1}

Filter:

$e_k = y_k - C\hat{x}_{k|k-1}$

$\hat{x}_{k|k} = \hat{x}_{k|k-1} + K_{fx}e_k$

$\hat{w}_{k|k} = K_{fw}e_k$

Regulator:

$u_k = \mu(\hat{x}_{k|k}, \hat{w}_{k|k}, \{r_{k+j|k}\}_{j=1}^{N}, u_{k-1})$

One-step predictor:

$\hat{x}_{k+1|k} = A\hat{x}_{k|k} + Bu_k + G\hat{w}_{k|k}$

Return: u_k, $\hat{x}_{k+1|k}$

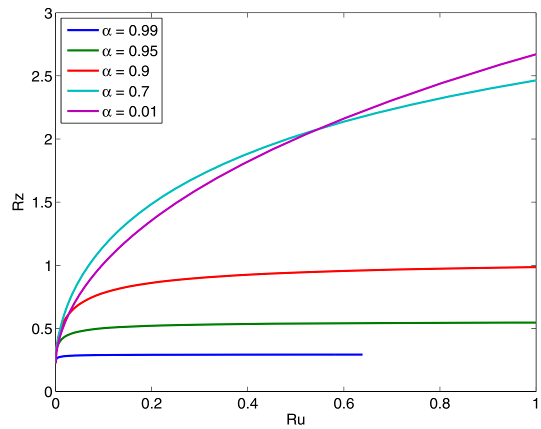


Fig. 1. Output variance, R_z, versus input variance, R_u, for different values of α. The curves are generated by varying S. In all cases the lowest output variance is obtained by turning the controller off (i.e. $S \to \infty$).

V. Simulated Furnace Example

To illustrate the controllers suggested in this paper, we control discrete-time systems

$$\boldsymbol{x}_{k+1} = A\boldsymbol{x}_k + Bu_k + G\boldsymbol{w}_k + Ed_k \tag{24a}$$
$$\boldsymbol{z}_k = C_z\boldsymbol{x}_k \tag{24b}$$
$$\boldsymbol{y}_k = C\boldsymbol{x}_k + \boldsymbol{v}_k \tag{24c}$$

using model predictive controllers designed based on state space models in innovation form

$$\bar{\boldsymbol{x}}_{k+1} = \bar{A}\bar{\boldsymbol{x}}_k + \bar{B}u_k + \bar{K}\boldsymbol{e}_k \tag{25a}$$
$$\boldsymbol{y}_k = \bar{C}\bar{\boldsymbol{x}}_k + \boldsymbol{e}_k \tag{25b}$$

We consider an industrial furnace example [41]. The furnace is described by

$$\boldsymbol{Z}(s) = G(s)U(s) + H(s)(D(s) + \boldsymbol{W}(s)) \tag{26a}$$
$$\boldsymbol{y}(t_k) = \boldsymbol{z}(t_k) + \boldsymbol{v}(t_k) \tag{26b}$$

in which $\boldsymbol{z}$ is the output, u is the manipulated variable, d is the deterministic unknown disturbances, $\boldsymbol{w}$ is a stochastic disturbance, $\boldsymbol{y}$ is the measurement, and $\boldsymbol{v}$ is measurement noise. u, d, and $\boldsymbol{w}$ are assumed to be piecewise constant within a sample period ($T_s = 2$). $[\boldsymbol{w}_k; \boldsymbol{v}_k] \sim N_{iid}([0; 0], [R_{ww}\, R_{wv}; R'_{wv} R_{vv}])$ with $R_{ww} = 0.3^2$, $R_{wv} = 0.01^2$, and $R_{wv} = 0$. The transfer functions are

$$G(s) = \frac{20}{(40s+1)(4s+1)}\, e^{-50s} \tag{27a}$$
$$H(s) = \frac{-5}{(5s+1)^2}\, e^{-10s} \tag{27b}$$

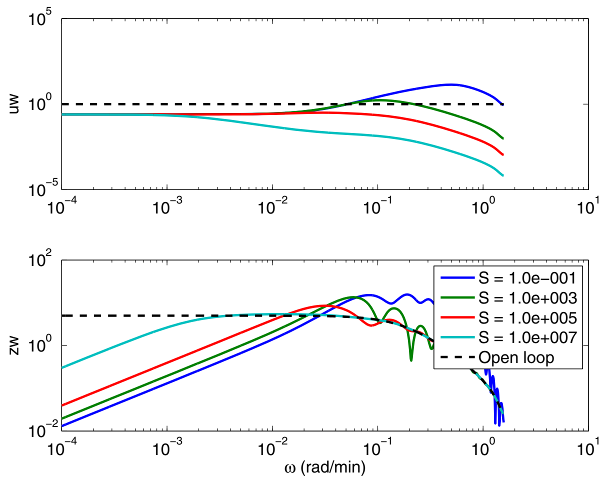
This model of the system is realized in the state space form (24). The model used by the controller is (1) and (3) with

$$A(q^{-1}) = 1 - 1.5578q^{-1} + 0.5769q^{-2} \tag{28a}$$
$$B(q^{-1}) = 0.2094q^{-26} + 0.1744q^{-27} \tag{28b}$$
$$C(q^{-1}) = 1 \tag{28c}$$
$$F(q^{-1}) = 1 - q^{-1} \tag{28d}$$
$$G(q^{-1}) = 1 - \alpha q^{-1} \tag{28e}$$

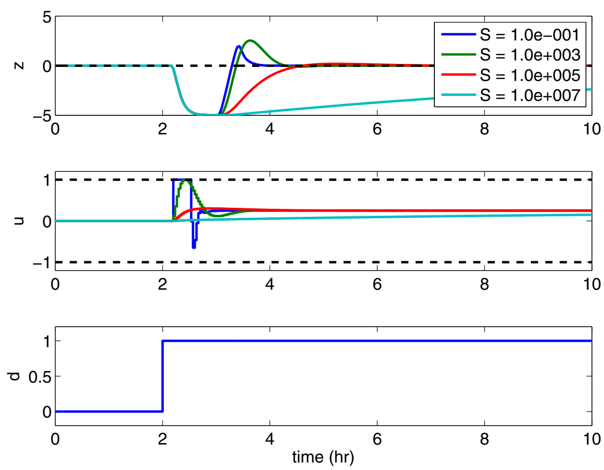
$A(q^{-1})$ and $B(q^{-1})$ are obtained by an exact discretization of $G(s)$ with the sampling time $T_s = 2$. The filter (3) is used to ensure steady-state offset free control. The tuning parameter α is selected to trade off sensitivity to noise versus speed of estimation of unknown disturbances. The model (1) and (3) with (28) is realized as a state space model in innovation form (25). $|u_k| \leq 1$ and the control and prediction horizon, $N = 150$, is long to emulate an infinite horizon controller and thereby eliminate the effect of discrepancies between open-loop and closed-loop profiles. The penalty on set-point deviation is $Q = 1$ such that the only tuning parameters of the controller are $S \in [0, \infty[$ and $\alpha \in [0, 1[$.

Fig. 1 illustrates the closed-loop variance of $\boldsymbol{z}$ and $\boldsymbol{u}$ for the system. In contrast to systems without long time delay, the output variance cannot be reduced by control. The lowest output (and input) variance is obtained from a completely detuned controller. The filter (3) with (28d)-(28e) provides steady-state offset free control for $\alpha < 1$. $\alpha = 0$ corresponds to a pure integrator while $\alpha = 1$ implies no filter, i.e. $G(q^{-1})/F(q^{-1}) = 1$. No or low filtering ($\alpha \approx 1$) provides the lowest output variance but at the expense of steady-state offset or rejecting step disturbances very slowly.

Fig. 2 and 3 illustrate the effect of a sinusoidal disturbance, w, with different frequencies on the magnitudes of u and z for two values of α and different values of S. These computations are for the unconstrained controller. The step responses of the constrained controller for the same tuning parameters are also illustrated in Fig. 2 and 3. While the responses to step disturbances indicate that a low value of S should be selected, the frequency response analysis indicates that an aggressive tuning (low values of S) magnifies disturbances at high frequencies dramatically. Therefore, one should detune the controller as suggested by the frequency response analysis and the closed-loop variance analysis (Fig. 1). From the step response simulation, the best tuning seems to be $(S, \alpha) = (0.1, 0.95)$, while the combined frequency response analysis and step response simulation suggest the tunings $(S, \alpha) = (10^5, 0.7)$ or $(S, \alpha) = (10^3, 0.95)$. The frequency response analysis for all these tunings reveal that the closed-loop magnitude of the output at certain frequencies is larger that output magnitude of the open-loop system. To avoid this,

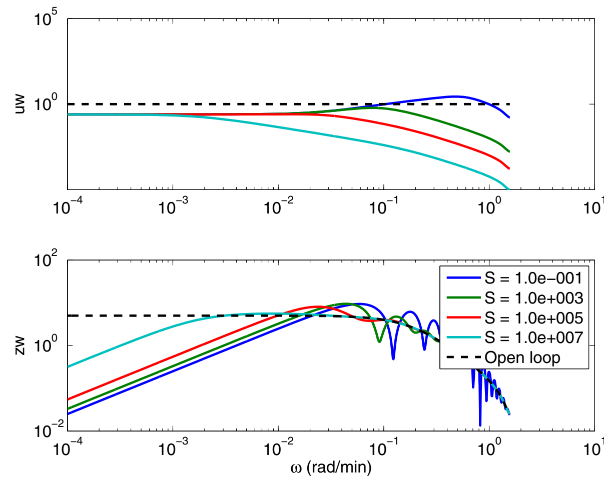


(a) Closed-loop frequency response analysis. Amplitude of u and z.

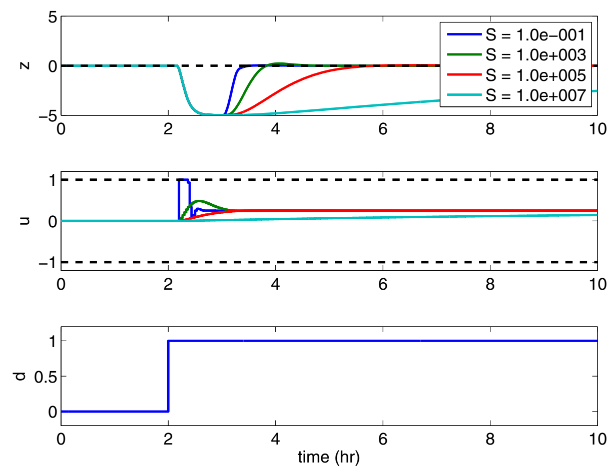


(b) Closed-loop response of input constrained MPC.

Fig. 2. The closed loop effects on u and z of process disturbances, w, at different frequencies (a) and a step disturbance, d (b) for different tunings. $\alpha = 0.7$.



(a) Closed-loop frequency response analysis. Amplitude of u and z.



(b) Closed-loop response of input constrained MPC.

Fig. 3. The closed loop effects on u and z of process disturbances, w, at different frequencies (a) and a step disturbance, d (b) for different tunings. $\alpha = 0.95$

S must be increased significantly at the price of very slow disturbance rejection.

Fig. 4 illustrates the response to an unknown step for the closed loop system with stochastic process and measurement noise for the three tunings $(S,\alpha) = \{(0.1, 0.95), (10^5, 0.7), (10^3, 0.95)\}$. The closed loop variances of the three tunings are $(R_z, R_u) = \{(0.52, 0.23), (0.28, 0.00071), (0.34, 0.0026)\}$. While the outputs of three different tunings are similar, the input of the first tuning is very plant unfriendly. Simulations not shown also demonstrate that this tuning is very sensitive to plant-model mismatch. Consequently, what from a deterministic step response simulation point of view seemed to be a very attractive tuning is useless from a practical point of view when stochastic noise and model-plant mismatch are considered. The tunings $(S,\alpha) = \{(10^5, 0.7), (10^3, 0.95)\}$ give similar performance. The first tuning has a lower variance at the price of a slightly longer time to reject a step disturbance. Fig. 5 illustrates the closed-loop response to a deterministic step disturbance at time $t = 2$ for the tuning $(S,\alpha) = (10^5, 0.7)$ when the model used by the controller (28a)-(28b) corresponds to the gains $K = 40$ (blue), $K = 20$ (red, plant gain), and $K = 10$ (green). The tuning chosen by the combined frequency response analysis, closed-loop variance analysis, deterministic- and stochastic simulations gives a controller that is robust to large gain uncertainties. Similar, studies have been performed for the time delay and the dominant time constant. The controller with the tuning $(S,\alpha) = (10^5, 0.7)$ is robust to plant-model mismatch. The controller which from a deterministic simulation point of view appears the most promising, $(S,\alpha) = (0.1, 0.95)$, is useless in the face of stochastic noise and is very sensitive to model-plant mismatch.

VI. Conclusion

In this paper, we derived and provided the correct finite-horizon control law for linear quadratic systems with correlated Gaussian process and measurement noise. The cor-

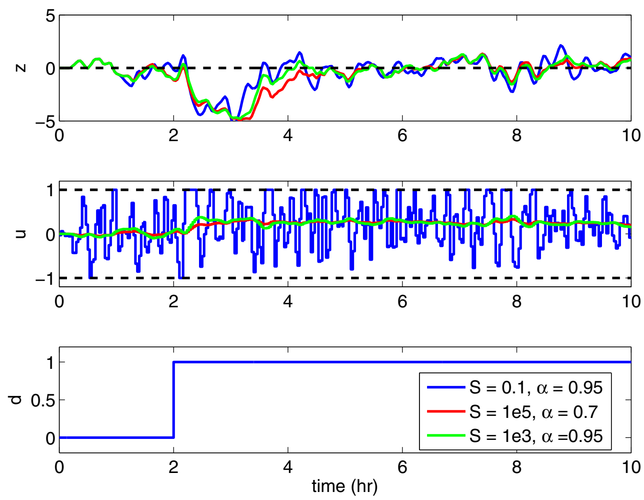


Fig. 4. Closed-loop response to a step disturbance as well as process and measurement noise for three different tunings.

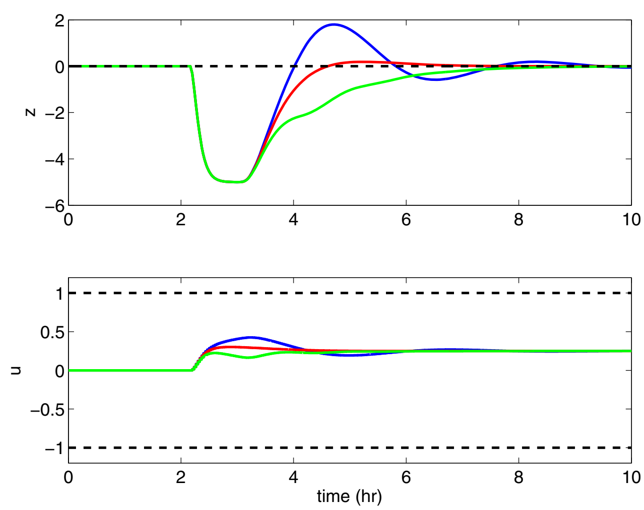


Fig. 5. The closed-loop response of the constrained MPC with $(S, \alpha) = (10^5, 0.7)$ and a model corresponding to the gains $K = 40$ (blue), $K = 20$ (red), and $K = 10$ (green). The plant has gain $K = 20$. This controller is robust to plant-model mismatch.

relation gives a term, $L_w \hat{w}_{k|k}$, in the optimal control law that is usually not considered. We use the correct optimal control law to develop a state-space model for the closed-loop system. This closed-loop state space model explicitly allows the controller model to be different from the plant model. Such plant-model mismatch is almost always present, due to the introduction of a disturbance model (filter) with an integrator in the controller model to ensure that the controller yields steady-state offset free control in the face of unknown step disturbances. Similarly, we develop the correct input constrained predictive controller for correlated process and measurement noise. The closed-loop expression for the unconstrained controller facilitates tuning of the input constrained MPC controller.

FIR, ARX, ARMAX, Box-Jenkins may be realized as state space models in innovation form. Similarly, subspace methods yield state space models in innovation form. State space models in innovation form have correlated process and measurement noise.

Appendix I
Finite-Horizon LQ Regulator

The finite horizon predictive linear quadratic regulation problem may be stated as

$$\min_{\{u_{k+j}\}} \phi = \frac{1}{2} \sum_{j=0}^{N-1} \left\| \hat{y}_{k+1+j|k} - r_{k+1+j|k} \right\|_Q^2 + \left\| \Delta u_{k+j} \right\|_S^2 \tag{29a}$$

$$s.t. \quad \hat{x}_{k+1|k} = A\hat{x}_{k|k} + Bu_k + G\hat{w}_{k|k} \tag{29b}$$

$$\hat{x}_{k+1+j|k} = A\hat{x}_{k+j|k} + Bu_{k+j} \qquad j = 1, ..., N-1 \tag{29c}$$

$$\hat{y}_{k+j|k} = C\hat{x}_{k+j|k} \qquad j = 1, \dots, N \tag{29d}$$

By including the possible non-zero filtered process noise, $\hat{w}_{k|k}$ in (29b), this LQ regulator allows for possible cross-couple process and measurement noise in the stochastic linear state space model describing the system. In most text-books and treatments, the LQ regulator is developed for $\hat{w}_{k|k} = 0$, and such LQ regulators are not valid for systems with cross-coupled noise. In particular, such LQ controllers are not valid for systems in innovation form.

Define the vectors as

$$Y_k = \begin{bmatrix} \hat{y}_{k+1|k} \\ \hat{y}_{k+2|k} \\ \vdots \\ \hat{y}_{k+N|k} \end{bmatrix} \quad R_k = \begin{bmatrix} r_{k+1|k} \\ r_{k+2|k} \\ \vdots \\ r_{k+N|k} \end{bmatrix} \quad U_k = \begin{bmatrix} u_k \\ u_{k+1} \\ \vdots \\ u_{k+N-1} \end{bmatrix}$$

Then the constraints (29b)-(29d) may be used to express the outputs by the affine relation

$$Y_k = \Gamma U_k + b_k \tag{30}$$

with

$$b_k = \Phi_x \hat{x}_{k|k} + \Phi_w \hat{w}_{k|k} \tag{31}$$

and

$$\Gamma = \begin{bmatrix} H_1 & 0 & \dots & 0 \\ H_2 & H_1 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ H_N & H_{N-1} & \dots & H_1 \end{bmatrix} \tag{32}$$

$$\Phi_x = \begin{bmatrix} CA \\ CA^2 \\ \vdots \\ CA^N \end{bmatrix} \qquad \Phi_w = \begin{bmatrix} CG \\ CAG \\ \vdots \\ CA^{N-1}G \end{bmatrix} \tag{33}$$

The impulse response coefficients (Markov parameters) used to assemble Γ are

$$H_i = CA^{i-1}B \qquad i = 1, 2, \dots, N \tag{34}$$

Define

$$\Delta U_k = \begin{bmatrix} \Delta u_k \\ \Delta u_{k+1} \\ \vdots \\ \Delta u_{k+N-1} \end{bmatrix} = \begin{bmatrix} u_k - u_{k-1} \\ u_{k+1} - u_k \\ \vdots \\ u_{k+N-1} - u_{k+N-2} \end{bmatrix}$$

such that

$$\Delta U_k = \Phi_u U_k - I_0 u_{k-1} \tag{35}$$

with

$$\Phi_u = \begin{bmatrix} I & 0 & 0 & \dots & 0 & 0 \\ -I & I & 0 & \dots & 0 & 0 \\ 0 & -I & I & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -I & I \end{bmatrix} \quad I_0 = \begin{bmatrix} I \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{36}$$

Using (30) and (35), the objective function in (29) may be expressed as the quadratic function

$$\begin{aligned} \phi &= \frac{1}{2} \sum_{j=0}^{N-1} \left\| \hat{y}_{k+1+j|k} - r_{k+1+j|k} \right\|_Q^2 + \left\| \Delta u_{k+j} \right\|_S^2 \\ &= \frac{1}{2} \left\| Y_k - R_k \right\|_{\mathcal{Q}}^2 + \frac{1}{2} \left\| \Delta U_k \right\|_{\mathcal{S}}^2 \\ &= \frac{1}{2} \left\| \Gamma U_k + b_k - R_k \right\|_{\mathcal{Q}}^2 + \frac{1}{2} \left\| \Phi_u U_k - I_0 u_{k-1} \right\|_{\mathcal{S}}^2 \\ &= \frac{1}{2} U_k' H U_k + g_k' U_k + \rho_k \end{aligned} \tag{37}$$

with $\mathcal{Q} = I_N \otimes Q$, $\mathcal{S} = I_N \otimes S$, and

$$H = \Gamma' \mathcal{Q} \Gamma + \Phi_u' \mathcal{S} \Phi_u \tag{38a}$$
$$g_k = -\left(\Gamma' \mathcal{Q} (R_k - b_k) + \Phi_u' \mathcal{S} I_0 u_{k-1} \right) \tag{38b}$$
$$\rho_k = \frac{1}{2} \left\| c_k \right\|_{\mathcal{Q}}^2 + \frac{1}{2} \left\| u_{k-1} \right\|_S^2 \tag{38c}$$

Consequently, by state elimation (29) may be expressed as an unconstrained convex quadratic optimization problem

$$\min_{U_k} \quad \phi = \frac{1}{2} U_k' H U_k + g_k' U_k + \rho_k \tag{39}$$

Provided the weights (Q, S) are chosen such that H is positive definite, this unconstrained convex quadratic optimization problem has the solution

$$U_k = -H^{-1} g_k = \bar{L}_x \hat{x}_{k|k} + \bar{L}_w \hat{w}_{k|k} + \bar{L}_R R_k + \bar{L}_u u_{k-1} \tag{40}$$

with the gain matrices defined as

$$\bar{L}_x = -H^{-1} \Gamma' \mathcal{Q} \Phi_x \tag{41a}$$
$$\bar{L}_w = -H^{-1} \Gamma' \mathcal{Q} \Phi_w \tag{41b}$$
$$\bar{L}_R = H^{-1} \Gamma' \mathcal{Q} \tag{41c}$$
$$\bar{L}_u = H^{-1} \Phi_u' \mathcal{S} I_0 \tag{41d}$$

The optimal control, u_k, to implement on the system is

$$u_k = I_0' U_k = L_x \hat{x}_{k|k} + L_w \hat{w}_{k|k} + L_R R_k + L_u u_{k-1} \tag{42}$$

with

$$L_x = I_0' \bar{L}_x = -I_0' H^{-1} \Gamma' \mathcal{Q} \Phi_x \tag{43a}$$
$$L_w = I_0' \bar{L}_w = -I_0' H^{-1} \Gamma' \mathcal{Q} \Phi_w \tag{43b}$$
$$L_R = I_0' \bar{L}_R = I_0' H^{-1} \Gamma' \mathcal{Q} \tag{43c}$$
$$L_u = I_0' \bar{L}_u = I_0' H^{-1} \Phi_u' \mathcal{S} I_0 \tag{43d}$$

The term $L_w \hat{w}_{k|k}$ in the optimal control law (42) should be noticed. In many textbooks, this term is missing. Therefore, the corresponding control laws are not valid for system with cross-correlated noise. In addition, the stated control law depends on the anticipated reference trajectory, $\left\{ r_{k+j|k} \right\}_{j=1}^N$. This implies that the current control, u_k, may change due to future anticipated reference changes.

Appendix II
Closed-Loop Properties

Consider the stochastic linear time invariant system

$$x_{k+1} = A x_k + B u_k + G w_k + E d_k \tag{44a}$$
$$z_k = C_z x_k \tag{44b}$$
$$y_k = C x_k + v_k \tag{44c}$$

and the corresponding time invariant output feedback LQ controller

$$e_k = y_k - \hat{C} \hat{x}_{k|k-1} \tag{45a}$$
$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + K_{fx} e_k \tag{45b}$$
$$\hat{w}_{k|k} = K_{fw} e_k \tag{45c}$$
$$u_k = L_x \hat{x}_{k|k} + L_w \hat{w}_{k|k} + L_R R_k + L_u u_{k-1} \tag{45d}$$
$$\hat{x}_{k+1|k} = \hat{A} \hat{x}_{k|k} + \hat{B} u_k + \hat{G} \hat{w}_{k|k} \tag{45e}$$

Note that we allow the model $(\hat{A}, \hat{B}, \hat{G}, \hat{C})$ used by the controller and in the design of $(K_{fx}, K_{fw}, L_x, L_w, L_R, L_u)$ to be different from the system (A, B, G, E, C). In particular, the disturbances, d_k, are unknown to the controller and the controller model may be augmented with a disturbance model to ensure offset free control.

Define

$$\Lambda = L_x K_{fx} + L_w K_{fw} \tag{46a}$$
$$\hat{\Lambda} = \hat{A} K_{fx} + \hat{G} K_{fw} + \hat{B} \Lambda \tag{46b}$$
$$A_{cl} = A + B \Lambda C \tag{46c}$$

Then the closed-loop evolution, i.e. the evolution of (44) and (45), may be described by

$$\begin{bmatrix} x_{k+1} \\ \hat{x}_{k+1|k} \\ u_k \end{bmatrix} = \begin{bmatrix} A_{cl} & B(L_x - \Lambda \hat{C}) & B L_u \\ \hat{\Lambda} C & \hat{A} + \hat{B} L_x - \hat{\Lambda} \hat{C} & \hat{B} L_u \\ \Lambda C & L_x - \Lambda \hat{C} & L_u \end{bmatrix} \begin{bmatrix} x_k \\ \hat{x}_{k|k-1} \\ u_{k-1} \end{bmatrix} + \begin{bmatrix} G & B\Lambda & B L_R & E \\ 0 & \hat{\Lambda} & \hat{B} L_R & 0 \\ 0 & \Lambda & L_R & 0 \end{bmatrix} \begin{bmatrix} w_k \\ v_k \\ R_k \\ d_k \end{bmatrix} \tag{47a}$$

and the resulting output, y_k, and input, u_k, are given by

$$\begin{bmatrix} z_k \\ y_k \\ u_k \end{bmatrix} = \begin{bmatrix} C_z & 0 & 0 \\ C & 0 & 0 \\ \Lambda C & L_x - \Lambda \hat{C} & L_u \end{bmatrix} \begin{bmatrix} x_k \\ \hat{x}_{k|k-1} \\ u_{k-1} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & \Lambda & L_R & 0 \end{bmatrix} \begin{bmatrix} w_k \\ v_k \\ R_k \\ d_k \end{bmatrix} \tag{47b}$$

A. Variance and Transfer Function

The state space representation (47) of the closed loop system is of the form

$$x_{k+1} = Ax_k + Bu_k \tag{48a}$$

$$y_k = Cx_k + Du_k \tag{48b}$$

The discrete transfer function model

$$Y(z) = G(z)U(z) \tag{49}$$

with the transfer function

$$G(z) = C(zI - A)^{-1}B + D \tag{50}$$

may be used to compute the gains from various inputs to various outputs at different frequencies. (50) is used to compute $|G(z)|$ for $z = e^{i\omega T_s}$ for various frequencies, ω. T_s is the sampling time. (50) may also be used for computation of the spectral density.

Assuming that A is stable, the stationary variance of the outputs, R_y, may be computed by

$$R_x = AR_xA' + BR_uB' \tag{51a}$$

$$R_y = CR_xC' + DR_uD' \tag{51b}$$

R_u is the variance of the input signal. R_x is computed by solving a discrete Lyapunov equation.

$G(z)$ and R_y are useful in analyzing the properties of the closed loop system (47) for different values of the controller tuning parameters.

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