2011 50th IEEE Conference on Decision and Control and
European Control Conference (CDC-ECC)
Orlando, FL, USA, December 12-15, 2011

L^2 Signal Reconstruction with FIR and Steady-State Behavior Constraints

Yaron Levinson and Leonid Mirkin

***Abstract*— The problem of recovering an analog signal from discrete measurements, known as signal reconstruction, can be formulated as a sampled-data L^2 fixed-lag smoothing problem. In this paper the problem is solved under the constraint that the reconstructor is an FIR (finite impulse response) system. An analytic and numerically stable solution, whose computational complexity is linear in the length of the reconstructor impulse response, is derived. The formulation can accommodate steady-state constraints via the use of weights with imaginary axis eigenvalues and imposing the stability requirement on the error system.**

I. Introduction

Signal reconstruction is the problem of recovering an analog signal from a measured discrete sequence. The fundamental result in this area is the Sampling Theorem [1], which establishes that a bandlimited analog signal can be perfectly reconstructed from its ideal samples, provided the signal bandwidth does not exceed the Nyquist frequency $\omega_{\rm N} := \pi/h$ (h stands for the sampling period). The reconstructor (D/A converter or hold) yielding this perfect reconstruction is then the sinc-interpolator having $\mathrm{sinc}_h(t) := \sin(\omega_{\rm N}t)/(\omega_{\rm N}t)$ as its interpolation kernel (hold function). Moreover, the sinc-interpolator generates the L^2-optimal reconstruction even when the original analog signal is not band limited but sampled using the ideal low-pass antialiasing filter [2].

The sinc-interpolator, however, is non-causal, one needs to know all past and *all future* measurements to reconstruct an analog signal at any time instance. Because $\mathrm{sinc}_h(t)$ decays slowly, its truncation is also not practical. For this reason, $\mathrm{sinc}_h(t)$ is normally replaced with "less ideal" hold functions having a faster decay rate, like polynomial or exponential splines [3], [4]. Such hold functions are also typically non-causal, so a common practice is to truncate them using some ad hoc rationale to achieve a required degree of causality, see [5] and the references therein.

An alternative approach to imposing causality constraints on interpolation kernels has been recently proposed in [6]. The idea, following [7], is to cast the reconstruction problem as a sampled-data estimation problem with performance measured by a system norm. When no causality constraints on the reconstruction are imposed, this approach recovers the classical cardinal interpolation splines [8]. By minimizing the very same, L^2, performance index under a causality constraint, [6] effectively derived causal and relaxedly causal (i.e., having a finite preview) versions of several widely used spline functions.

This research was supported by the Israel Science Foundation (grant No. 1238/08).

The authors are with the Faculty of Mechanical Eng., Technion—IIT, Haifa 32000, Israel. E-mails: {*yaronlev, mirkin*}*@technion.ac.il*.

The optimal reconstructors derived in [6] are typically IIR (infinite impulse response), which implies that the current reconstruction is affected by all past measurements. This might be undesirable in some applications. For example, if an abrupt change in the measured signal occurs, one would prefer to "forget" this data as soon as possible. Shift-invariant IIR reconstructors could attain this if their impulse responses decay fast, but this might conflict with what a chosen signal model demands. In such applications FIR (finite impulse response) reconstructors would be preferable. FIR holds are also generally more robust to round-off errors than their IIR counterparts and are easily applicable to finite-horizon cases. Thus, it is important to be able to incorporate constraints on the support length of the hold function into the design of optimal reconstructors.

In this paper we solve the problem of the design of FIR signal reconstructors in the framework of the L^2 (mean-square) optimization. We follow the approach introduced in [9], where a continuous-time L^2 FIR estimation problem was solved. The extension of this result to the discrete-time and, especially, sampled-data (via lifting [10]) settings, however, is not straightforward. Our recent discrete solution in [11], for example, is not readily applicable to sampled-data problems. The reason is that the solution, in [11] hinges on simplifying assumptions that *never hold* in sampled signal reconstruction problems studied in this paper. We, thus, effectively re-derive a discrete counterpart of the solution of [9] using different techniques. As a byproduct, we derive a numerically stable solution, which is based on matrix powers of Schur matrices only and can therefore be computed regardless the support length of the hold function. Following [9], we allow unstable weighting functions, which can be used as a means to impose constraints on steady-state reconstruction performance (via weights with jω-axis poles).

A. Notation

Throughout, signals are represented by lowercase symbols such as $y(t)$ and overbars indicate discrete-time signals, $\bar{y}[k]$. For a set $\mathbb{A}$, the indicator function $\mathbb{1}_{\mathbb{A}}(t)$ equals 1 if $t \in \mathbb{A}$ and zero elsewhere. $\bar{\delta}_k$ is the discrete pulse sequence, equals 1 at $k = 0$ and zero elsewhere. By n_v we understand the number of elements of a vector-valued signal v. To render the exposition more compact, we denote $\mathbb{L} := L^2[0, h)$. Uppercase calligraphic symbols represent continuous-time systems in the time domain. Their corresponding transfer function/frequency responses are then presented by uppercase symbols, like $G(s)$ and $G(\mathrm{j}\omega)$. Discrete-time systems, transfer functions, etcetera, are denoted by overbars, like $\bar{\mathcal{G}}$ and $\bar{G}(z)$. Finally, $\mathbb{Z}_{i_1..i_2} := \{\, i \in \mathbb{Z} : i_1 \le i \le i_2 \}$.

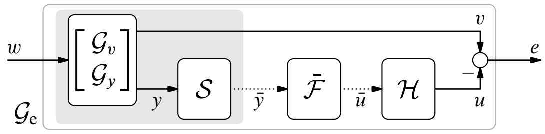


Fig. 1. System-theoretic signal reconstruction setup

II. Problem Formulation

We study the signal reconstruction setup in Fig. 1. The problem is to reconstruct an analog signal v from a discrete sequance $\bar{y}$, which is obtained from another analog signal y by passing it through the ideal sampler $\mathcal{S}$. The signal y might be different from v. The available information about v, y, and their relations is expressed via modeling these signals as outputs of the *signal generator* $\mathcal{G} := \left[\begin{smallmatrix}\mathcal{G}_v\\ \mathcal{G}_y\end{smallmatrix}\right]$ driven by a common normalized signal w.

The reconstructor that we design comprises a discrete filter $\bar{\mathcal{F}}$ and a D/A converter $\mathcal{H}$. This separation is somewhat artificial, without loss of generality we may assume that $\bar{\mathcal{F}}$ is identity. It, however, might be convenient, e.g., from the implementation point of view, to have both these components. We assume that $\mathcal{H}\bar{\mathcal{F}}$ is shift invariant, in the sense that $\mathcal{H}\bar{\mathcal{F}}(\bar{y}[\cdot-1]) = (\mathcal{H}\bar{\mathcal{F}})(\cdot-h)$. In this case, the general form of $\mathcal{H}\bar{\mathcal{F}}$ is

$$u(t) = \sum_{i\in\mathbb{Z}} \phi(t-ih)\bar{y}[i], \quad t \in \mathbb{R}, \tag{1}$$

where $\phi(t)$ is an *interpolation kernel* or *hold function* to be designed. For given $\lambda \in \mathbb{N}$ and $\rho \in [0,1)$, called the *preview ratio*, we say that this reconstructor is (λ,ρ)-FIR if

$$\phi(t) = 0, \quad \text{whenever } t \notin [-\rho\lambda h, (1-\rho)\lambda h). \tag{2}$$

In other words, a (λ,ρ)-FIR $\mathcal{H}\bar{\mathcal{F}}$ produces the reconstruction signal u using $(1-\rho)\lambda-1$ past and $\rho\lambda$ future measurements of $\bar{y}$ according to (we implicitly assume that $\rho\lambda$ is integer)

$$u(kh+\tau) = \sum_{i=-\rho\lambda}^{(1-\rho)\lambda-1} \phi(ih+\tau)\bar{y}[k-i], \quad k\in\mathbb{Z}, \tau\in[0,h).$$

We say that this reconstruction is admissible if ϕ is a bounded function, which guarantees that u is bounded (in whatever sense) whenever so is $\bar{y}$.

As the measure of reconstruction performance we consider the L^2-norm of the error system

$$\mathcal{G}_e := \mathcal{G}_v - \mathcal{H}\bar{\mathcal{F}}\mathcal{S}\mathcal{G}_y, \tag{3}$$

which connects w and the reconstruction error $e = v - u$. The L^2 system norm is a non-causal version of the familiar H^2-norm and shares its deterministic and stochastic interpretations, see [10] and the references therein.

We suppose that the signal generator has a strictly proper transfer function and is given in terms of its minimal state-space realization

$$G(s) = \begin{bmatrix} G_v(s) \\ G_y(s) \end{bmatrix} = \begin{bmatrix} C_v \\ C_y \end{bmatrix}(sI-A)^{-1}B. \tag{4}$$

$G_v(s)$ must be strictly proper to guarantee that the L^2-norm of the error system is finite and $G_y(s)$—to guarantee the boundedness of the sampling operation [10]. Throughout, we also assume that

$\boldsymbol{\mathcal{A}}_1$: C_y has full raw rank,

$\boldsymbol{\mathcal{A}}_2$: the pair (C_y, e^{Ah}) is detectible,

$\boldsymbol{\mathcal{A}}_3$: λ is not smaller than the observability index of the unstable part of (C_y, A);

Assumption $\boldsymbol{\mathcal{A}}_1$ merely rules out redundant measurements and thus entails no loss of generality. Assumptions $\boldsymbol{\mathcal{A}}_{2,3}$ are related to the stabilizability of the error system: $\boldsymbol{\mathcal{A}}_2$ is necessary for the existence of a stabilizing reconstructor even in the IIR case, while $\boldsymbol{\mathcal{A}}_3$ then guarantees that an FIR stabilizing reconstructor of length λ exists. Unlike the analog case, where the detectability of the measurement channel is sufficient for the existence of a stabilizing FIR solution [9], in the sampled-data cases the impulse response of $\mathcal{H}\bar{\mathcal{F}}$ has to be sufficiently long.

Remark 2.1: We do not restrict the signal generator to be stable but do require that the error system (3) is stable (in the L^2 sense). The sole reason for considering unstable signal generators is the possibility to cast steady-state requirements in the form of the stability requirement. To illustrate this point, consider a simple analog problem in which a stable filter $F(s)$ is designed for the the error system $G_e(s) = 1 - F(s)G_y(s)$. Zero steady-state error for a step input is equivalent to the condition that $G_e(0) = 0$. This condition can be cast as the stability of $G_e(s)\frac{s+a}{s} = \frac{s+a}{s} - F(s)\big(G_y(s)\frac{s+a}{s}\big)$ for any $a > 0$. In other words, by introducing an (artificial) unstable mode at the origin into the signal generator we can impose the requirement of zero steady-state error for a constant measured signal via the stability of the error system. This approach can be extended to ramp inputs (a double integrator), a sine wave with a known frequency (a harmonic oscillator), etc. In the open-loop context of Fig. 1, the stability requirement effectively implies that all unstable modes of $\mathcal{G}$ must be canceled in the error system, $\mathcal{G}_e$. ∇

To conclude, the reconstruction problem addressed in this paper is formulated as follows:

$\mathbf{RP}_{\lambda,\rho}$: Let $\mathcal{G}$ be given by (4), assumptions $\boldsymbol{\mathcal{A}}_{1\text{–}3}$ hold, $\mathcal{S}$ be the ideal sampler, and $\lambda \in \mathbb{N}$ and $\rho \in [0, (\lambda-1)/\lambda]$ be such that $\rho\lambda \in \mathbb{Z}$. Find an admissible (λ,ρ)-FIR reconstructor $\mathcal{H}\bar{\mathcal{F}}$, which stabilizes the error system (3) and minimizes $\mathcal{J}_{\lambda,\rho} := \|\mathcal{G}_e\|_2^2$.

III. Problem Solution

To formulate the solution of $\mathbf{RP}_{\lambda,\rho}$, we need to introduce the following matrix valued function of a real argument:

$$\Lambda(t) = \begin{bmatrix} \Lambda_{11}(t) & \Lambda_{12}(t) \\ 0 & \Lambda_{22}(t) \end{bmatrix} := \exp\left(\begin{bmatrix} A & BB' \\ 0 & -A' \end{bmatrix} t\right). \tag{5}$$

We skip the argument whenever $t = h$, so we write Λ_{ij} instead of $\Lambda_{ij}(h)$. We shall also need the matrix Δ defined via

$$\begin{bmatrix} \Lambda & \Delta \\ 0 & \Lambda \end{bmatrix} := \exp\left(\begin{bmatrix} A & BB' & 0 & 0 \\ 0 & -A' & C_v'C_v & 0 \\ 0 & 0 & A & BB' \\ 0 & 0 & 0 & -A' \end{bmatrix} h\right). \quad (6)$$

Define the discrete algebraic Riccati equation (DARE)

$$Y = \Lambda_{11} Y \Lambda_{11}' - \Lambda_{11} Y C_y' (C_y Y C_y')^{-1} C_y Y \Lambda_{11}' + \Lambda_{12} \Lambda_{11}'. \quad (7)$$

Its solution is said to be stabilizing if $C_y Y C_y'$ is non-singular and the matrix

$$\bar{A}_1 := \Lambda_{11} + L C_y,$$

in which $L := -\Lambda_{11} Y C_y' (C_y Y C_y')^{-1}$, is Schur. It is known [6] that if assumptions $\mathcal{A}_{1,2}$ hold, the stabilizing solution of (7) exists and $Y = Y' > 0$. We also need the *maximal* solution $X_c = X_c' \geq 0$ of the DARE

$$X_c \Lambda_{11} - \Lambda_{22} X_c + X_c \Lambda_{12} X_c = 0 \quad (8)$$

for which the matrix $\Lambda_{22} - X_c \Lambda_{12}$ has no eigenvalues outside the *closed* unit disc.

Remark 3.1: The Riccati equation with a zero free term, (8), which actually verifies the continuous ARE (also known as the *algebraic Bernoulli equation*)

$$X_c A + A' X_c + X_c B B' X_c = 0,$$

is effectively required to extract the unstable spectral subspace of Λ_{11} (equivalently, A), which has to be canceled by stabilizing reconstructors. Indeed, it can be shown that $\ker X_c$ coincides with this space. If A is stable, X_c is invertible and X_c^{-1} is the controllability Gramian of (4). ∇

Finally, introduce three matrix sequences:

$$\bar{A}_{k+1} = \bar{A}_1 \bar{A}_k, \quad \bar{A}_0 = I$$

(so, obviously, $\bar{A}_k = \bar{A}_1^k$, we actually need this sequence just to simplify the formulae below),

$$W_k = \bar{A}_1' W_{k+1} \bar{A}_1 + C_y' (C_y Y C_y')^{-1} C_y, \quad W_\lambda = 0$$

and

$$V_{k+1} = \bar{A}_1 V_k \bar{A}_1', \quad V_0 = (X_c + (I - X_c Y) W_0)^{-1} (I - X_c Y),$$

The main result of this paper, whose proof is outlined in Section V, is as follows:

Theorem 1: Let $\mathcal{A}_{1\text{–}3}$ hold. Then $X_c + (I - X_c Y) W_0$ is invertible and the unique solution of $\mathbf{RP}_{\lambda,\rho}$ is as shown in Fig. 2, where, denoting $k_\rho := (1 - \rho)\lambda$,

$$\bar{F}_c(z) := \sum_{j=0}^{k_\rho - 1} \bar{A}_j (\bar{A}_1 V_{k_\rho - j - 1} (\bar{A}_1' W_{k_\rho - j} - I) - I) L z^{-j}$$

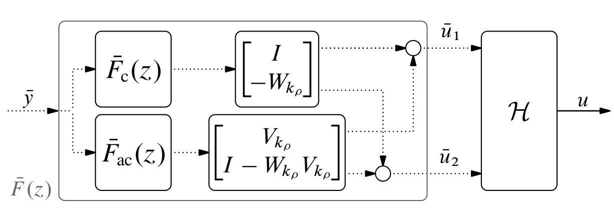


Fig. 2. The optimal $\rho\lambda$-FIR reconstructor

is causal,

$$\bar{F}_{ac}(z) := \sum_{j=-\rho\lambda}^{-1} \bar{A}_{-j-1}' (\bar{A}_1' W_{k_\rho - j} - I) L z^{-j}$$

is anti-causal, and $\mathcal{H}$ is a D/A converter with the $(n_v \times 2n)$-valued zero-order generalized hold function

$$\phi_h(\tau) = \begin{bmatrix} C_v & 0 \end{bmatrix} \Lambda(\tau - h) \begin{bmatrix} I & Y \\ 0 & I \end{bmatrix} \mathbb{1}_{[0,h)}.$$

The optimal $\|\mathcal{G}_e\|_2^2$ equals then

$$\frac{1}{h} \operatorname{tr}\left(\begin{bmatrix} W_{k_\rho} & I - W_{k_\rho} Y \end{bmatrix} \Delta \Lambda^{-1} \begin{bmatrix} Y + V_{k_\rho} - Y W_{k_\rho} V_{k_\rho} \\ I - W_{k_\rho} V_{k_\rho} \end{bmatrix} \right),$$

which is a nonincreasing function of λ for any fixed ρ. ∇

Remark 3.2: It is worth emphasizing that the solution of Theorem 1 uses matrix powers of stable matrix $\bar{A}_1$ only, either directly or through stable recursions for W_k and V_k. This implies that the solution can be safely calculated for an arbitrarily large λ. ∇

Remark 3.3: Note that

$$\bar{A}_1 = \Lambda_{11} (I - Y C_y' (C_y Y C_y')^{-1} C_y)$$

is always singular, because $\bar{A}_1 Y C_y' = 0$. This means that $\bar{A}_1$ has n_y eigenvalues at the origin and the formulae for $\bar{F}_c(z)$ and $\bar{F}_{ac}(z)$ can be simplified further. ∇

Remark 3.4: As k_ρ, which is the length of the impulse response of the causal part of $\mathcal{H}\bar{\mathcal{F}}$, increases, the matrix $V_{k_\rho} = \bar{A}_{k_\rho} V_0 \bar{A}_{k_\rho}'$ vanishes. In the limit, $k_\rho \to \infty$, the optimal performance reads then

$$\mathcal{J}_{\infty,\rho} = \frac{1}{h} \operatorname{tr}\left(\begin{bmatrix} W_{k_\rho} & I - W_{k_\rho} Y \end{bmatrix} \Delta \Lambda^{-1} \begin{bmatrix} Y \\ I \end{bmatrix} \right).$$

It can be shown that $W_{k_\rho} = X - \bar{A}_{\rho\lambda}' X \bar{A}_{\rho\lambda}$, where $X = X' \geq 0$ is the solution to the Lyapunov equation

$$X = \bar{A}_1' X \bar{A}_1 + C_y' (C_y Y C_y')^{-1} C_y$$

Taking into account that $\rho\lambda$ is the length of the impulse response of the non-causal part of $\mathcal{H}\bar{\mathcal{F}}$, it can be seen that $\mathcal{J}_{\infty,\rho}$ is exactly the optimal performance of the IIR reconstruction with preview derived in [6]. If, in addition, $\rho\lambda \to \infty$, we have that $W_{k_\rho} = X$, so that

$$\mathcal{J}_{\infty,\rho} = \mathcal{J}_\infty := \frac{1}{h} \operatorname{tr}\left(\begin{bmatrix} X & I - XY \end{bmatrix} \Delta \Lambda^{-1} \begin{bmatrix} Y \\ I \end{bmatrix} \right),$$

which is the performance of non-causal reconstruction. ∇

IV. FIR Cardinal Cubic B-Splines

To illustrate the proposed approach we consider in this section the $\mathbf{RP}_{\lambda,\rho}$ for

$$G_v = G_y = \frac{1}{s^2}. \quad (9)$$

This signal generator can be thought of as reflecting the low-frequency dominance of the signal to be reconstructed and the requirement for zero steady-state error for step and ramp components of v. In the non-causal IIR case, this problem yields the cardinal cubic B-spline of [3], see [8, Thm. 3.3]. In [6] the optimal IIR l-causal reconstructor was derived for this $\mathcal{G}$.

A. Solution

Bring in the minimal state-space realization

$$G(s) = \left[\begin{array}{cc} 1 & 0 \\ \hdashline 1 & 0 \end{array}\right] \left(sI - \begin{bmatrix} 0 & 1/h \\ 0 & 0 \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 \\ h \end{bmatrix}.$$

Clearly, $\boldsymbol{\mathcal{A}}_1$ holds true. As $\mathrm{e}^{Ah} = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$, the observability matrix of (C_y, e^{Ah}) is non-singular and thus $\boldsymbol{\mathcal{A}}_2$ holds as well. Since $G(s)$ has no stable poles, $\boldsymbol{\mathcal{A}}_3$ holds whenever $\lambda \geq 2$ (and $X_\mathrm{c} = 0$).

Analytic expressions for the parameters of the optimal reconstructor are quite lengthy for general preview ratios. To keep the exposition simple, we present below only the formulae for the symmetric reconstructor, i.e., for $\rho = 1/2$ (consequently, only even window lengths are considered). Define

$$\alpha := \sqrt{3} - 2 \approx -0.2679.$$

Using the formulae of Theorem 1, we can calculate the causal FIR filter,

$$\bar{F}_\mathrm{c}(z) = \begin{bmatrix} 1/\alpha \\ 1/\alpha - 1 \end{bmatrix} + \begin{bmatrix} 6 \\ 6 \end{bmatrix} \sum_{j=0}^{\lambda/2-1} \frac{\alpha^j - \alpha^{\lambda-j-2}}{1 - \alpha^{2\lambda-2}} z^{-j},$$

the anti-causal FIR filter,

$$\bar{F}_\mathrm{ac}(z) = \begin{bmatrix} 1-\alpha \\ -1 \end{bmatrix} \frac{6\alpha}{h^3} \sum_{j=-\lambda/2}^{-1} \frac{\sqrt{3}(1 - \alpha^{\lambda+2j}) + \bar{\delta}_{j+\lambda/2}}{\alpha^j} z^{-j},$$

the matrices

$$W_{\lambda/2} = \frac{3(\alpha+1)}{h^3} \left(\begin{bmatrix} 2 & -1 \\ -1 & \frac{1}{\sqrt{3}} \frac{1}{\alpha+1} \end{bmatrix} - \begin{bmatrix} \sqrt{3} & -1 \\ -1 & \frac{1}{\alpha+1} \end{bmatrix} \alpha^{\lambda-2} \right),$$

$$V_{\lambda/2} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \frac{h^3}{\sqrt{3}\alpha} \frac{\alpha^{\lambda-1}}{1 - \alpha^{2(\lambda-1)}},$$

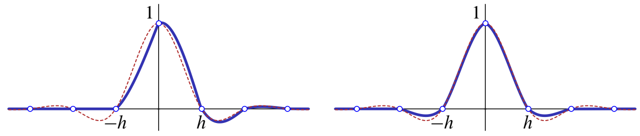
and two components of the interpolation kernels of $\mathcal{H}$ (which are exactly as in [6]):

$$\phi_{h1}(\tilde{\tau}) = \begin{bmatrix} 1 & \tilde{\tau} - 1 \end{bmatrix} \mathbb{1}_{[0,h]},$$

$$\phi_{h2}(\tilde{\tau}) = \frac{h^3 \tilde{\tau}}{6} \begin{bmatrix} -\tilde{\tau}^2 + 3\tilde{\tau} + \sqrt{3} & 3\tilde{\tau} + \sqrt{3} \end{bmatrix} \mathbb{1}_{[0,h]},$$

where $\tilde{\tau} := \tau/h$ is the normalized intersample time. Note that all factors depending on h^3 are canceled in the final reconstructor.

The interpolation kernels $\phi(t)$ of the optimal reconstructors with $(\lambda, \rho) = (4, 1/2)$ and $(4, 1/4)$ are depicted in Fig. 3 by solid lines. Dashed lines there represent the interpolation kernel of the non-causal IIR reconstructor, which is the cardinal cubic B-spline of [3]. Similarly to the latter, the optimal FIR hold functions satisfy $\phi(kh) = \bar{\delta}_k$ for all $k \in \mathbb{Z}$.



(a) Preview ratio $\rho = 1/4$ (b) Preview ratio $\rho = 1/2$

Fig. 3. Interpolation kernels $\phi(t)$ of $(4, \rho)$-FIR optimal solutions

This implies that the reconstructed signal always interpolates the samples. In other words, if the reconstructed signal is injected back into the sampler, the resulting discrete signal will coincide with $\bar{y}$. This property is called the *consistency*, see [2] for details.

B. Optimal performance

The optimal performance for this problem is

$$\mathcal{J}_{\lambda,\rho} = \frac{(8-\alpha)\alpha + (12+\alpha)\alpha^{2\lambda-1} - 11(\alpha^{2\lambda(1-\rho)} + \alpha^{2\lambda\rho})}{840\sqrt{3}\alpha(1 - \alpha^{2(\lambda-1)})} h^3$$

if $1/\lambda \leq \rho \leq (1-\lambda)/\lambda$ or

$$\mathcal{J}_{\lambda,0} = \frac{-(1 + \alpha^{2\lambda})}{12\sqrt{3}\alpha(1 - \alpha^{2(\lambda-1)})} h^3$$

if $\rho = 0$. It is readily seen that $\mathcal{J}_{\lambda,\rho}$ is symmetric around the midpoint $\rho = 1/2$, i.e., that $\mathcal{J}_{\lambda,\lambda/2-i} = \mathcal{J}_{\lambda,\lambda/2+i}$ for every $i \in \mathbb{Z}_{1..\lambda/2-1}$. Moreover, the function $\psi(\rho) = \alpha^{2\lambda(1-\rho)} + \alpha^{2\lambda\rho}$ attains its minimum at $\rho = 1/2$, which implies that

$$\arg\min_\rho \mathcal{J}_{\lambda,\rho} = 1/2$$

for every even λ (if λ is odd, the minimum is attained by $\rho = 1/2 \pm 1/(2\lambda)$). Figs. 4(a) and 4(b) depict the optimal cost for $\lambda = 4$ and $\lambda = 8$, respectively. One can see that the reconstruction performance visibly improves when at least one step of preview is available.

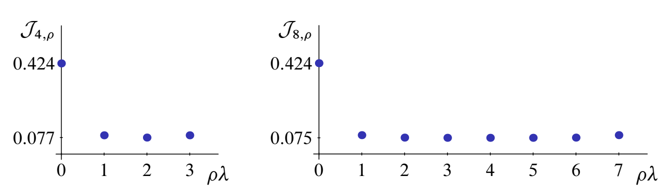
It is of interest to quantify the deterioration of the performance achievable with FIR reconstruction with respect to the performance of the non-causal IIR reconstruction (cardinal cubic B-splines), which is $\mathcal{J}_\infty = \frac{8-\alpha}{840\sqrt{3}} h^3$, see [6]. The table below shows this quantification for several even λ and the optimal $\rho = 1/2$:

λ	2	4	6	8
$\mathcal{J}_{\lambda,1/2}/\mathcal{J}_\infty - 1$	0.95524	0.05210	0.00368	0.00026

While close to 100% at $\lambda = 2$, the deterioration with respect to $\mathcal{J}_\infty$ at $\lambda = 4$ is already about 5% and drops below 1‰ at $\lambda \geq 8$.

C. Frequency Power Response

The signal generator in (9) is not a real system, but rather a shaping filter used in the design of $\phi(t)$ to reflect our assumption that v is a lowpass signal and our requirement that the reconstruction error for step and ramp v is zero in steady state. Once designed, however, the reconstructor will operate on any sequence $\bar{y}$. It is therefore of interest to understand how the reconstruction error e is connected



(a) $\lambda = 4$ (b) $\lambda = 8$

Fig. 4. Optimal cost $\mathcal{J}_{\lambda,\rho}$ vs. preview length $\rho\lambda$

with v itself, rather than with an artificial signal w. One possibility to shed light on this connection is through a frequency domain analysis.

The system connecting v and e is

$$\mathcal{G}_{ev} := I - \mathcal{H}\bar{\mathcal{F}}\mathcal{S}\mathcal{F}_{\rm a}, \tag{10}$$

where $\mathcal{F}_{\rm a}$ is an antialiasing filter, which may be used to render the ideal sampling stable. This system is periodically time varying, so its frequency-domain analysis is more complicated than that of LTI systems. There are generalization of the notion of the frequency response to sampled-data (and other periodic) systems, see [12] and the references therein. To account for possible folding effects, these generalizations determine the frequency response at each frequency as an infinite-dimensional operator, which is hard to visualize. The magnitude frequency response, defined then as a norm of such operators, has all aliased frequencies of input signals mixed up in it. This is fine for the analysis of system norms, but less appropriate for a harmonic analysis. In fact, if applied to $\mathcal{G}_{ev}$ defined above, the methods from [12] would result in the frequency response gain larger than one at each frequency, which makes no sense.

To circumvent this problem, we adopt the *frequency power response* (FPR), defined in [6] as the power of the response of a system to the harmionic signal $e^{j\omega t}$:

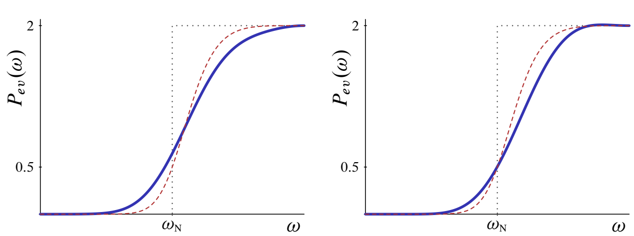
$$\begin{aligned} P_{ev}(\omega) &:= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |(\mathcal{G}_{ev} e^{j\omega t})(t)|^2 dt \\ &= \frac{1}{h}\int_0^h |(\mathcal{G}_{ev} e^{j\omega t})(t)|^2 dt \end{aligned}$$

(no limit is required because $\mathcal{G}_{ev}$ is h-periodic). As shown in [6, Thm. 4.1], the FPR of the error system defined by (10) (we assume hereafter, for simplicity, that there is no antialiasing filter, i.e., that $\mathcal{F}_{\rm a} = I$) equals

$$P_{ev}(\omega) = \left|1 - \frac{1}{h}\Phi(j\omega)\right|^2 + \frac{1}{h^2}\sum_{i\in\mathbb{Z}\setminus\{0\}} |\Phi(j(\omega + 2\tfrac{\pi}{h}i))|^2,$$

where $\Phi(j\omega)$ is the Fourier transform of the hold function $\phi(t)$ (the impulse response of $\mathcal{H}\bar{\mathcal{F}}$). The second term in the right-hand side above contains aliased terms and can be though of as reflecting the deterioration of the reconstruction error due to sampling.

The FPR for our example are presented in Fig. 5 for $\lambda = 4$ and two different preview ratios. The dashed lines there represent the FPR of the non-causal IIR solution



(a) $\rho = 1/4$ (b) $\rho = 1/2$

Fig. 5. Frequency power responses for $\mathcal{G}$ as in (9) and $\lambda = 4$

(cardinal cubic B-splines) and the dotted lines—that of the classical sinc-interpolator. We shall be mostly interested in the frequency range below the Nyquist frequency $\omega_{\rm N}$. In this range the error power is a monotonically increasing function of ω. This is a result of our choice of weighing function, $1/s^2$, which places more emphasis on low frequencies. The requirement for stabilizing the error function translates to $P_{ev}(0) = \dot{P}_{ev}(0) = 0$.

V. Proof (outline)

Because of space limitations, we only outline the proof of our main result, Theorem 1.

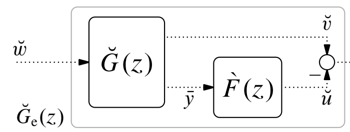


Fig. 6. Reconstruction setup in the lifted domain

By applying the lifting technique [10], the problem can be converted to a pure discrete L^2 estimation problem for the system in Fig. 6. This is an LTI problem for the plant

$$\breve{G}(z) = \begin{bmatrix} \breve{G}_v(z) \\ \acute{G}_y(z) \end{bmatrix} = \left[\begin{array}{c|c} \bar{A} & \acute{B} \\ \hline \grave{C}_v & \breve{D}_v \\ C_y & 0 \end{array}\right], \tag{11}$$

where $\bar{A} = e^{Ah} \in \mathbb{R}^{n\times n}$,

$$\begin{aligned} \acute{B} &: \mathbb{L} \to \mathbb{R}^n & \breve{v} &\mapsto \int_0^h e^{A(h-\sigma)} B\breve{v}(\sigma) d\sigma \\ \grave{C}_v &: \mathbb{R}^n \to \mathbb{L} & \bar{\xi} &\mapsto C_v e^{A\tau}\bar{\xi} \\ \breve{D}_v &: \mathbb{L} \to \mathbb{L} & \breve{v} &\mapsto C_v \int_0^\tau e^{A(\tau-\sigma)} B\breve{v}(\sigma) d\sigma \end{aligned}$$

and FIR

$$\grave{F}(z) = \sum_{i=-\rho\lambda}^{k_\rho - 1} \grave{F}_{k_\rho,i}\, z^{-i}$$

where the coefficients $\grave{F}_{k_\rho,i} : \mathbb{R}^{n_{\bar{y}}} \to \mathbb{L}$ are to be determined and $k_\rho := (1-\rho)\lambda \in \mathbb{Z}_{1..\lambda}$ is the length of the causal part of $\grave{F}(z)$. It is shown in [11] that this problem can be reduce to a static optimization problem as follows.

Let $\Pi_{\rm u} \in \mathbb{R}^{n\times n}$ be an orthogonal projection matrix such that its kernel coincides with the unstable spectral subspace of A, i.e., $\ker X_{\rm c} = \operatorname{Im}\Pi_{\rm u}$ and the matrix $X_{\rm c} + \Pi_{\rm u}$ is invertible. Consider now the optimization problem

$$\min_{\boldsymbol{F}} \operatorname{tr}\left(\begin{bmatrix} I & -\boldsymbol{F} \end{bmatrix} \boldsymbol{\Phi} \begin{bmatrix} I \\ -\boldsymbol{F}' \end{bmatrix}\right) \tag{13a}$$

subject to the (stability) constraint

$$(\boldsymbol{C}_v - \boldsymbol{F}\boldsymbol{C}_y)\Pi_{\rm u} = 0, \tag{13b}$$

where

$$\boldsymbol{\Phi} := \begin{bmatrix} \boldsymbol{D_v} \\ \boldsymbol{D_y} \end{bmatrix} \begin{bmatrix} \boldsymbol{D}_v' & \boldsymbol{D}_y' \end{bmatrix} + \begin{bmatrix} \boldsymbol{C}_v \\ \boldsymbol{C}_y \end{bmatrix} (X_{\rm c} + \Pi_{\rm u})^{-1} \begin{bmatrix} \boldsymbol{C}_v' & \boldsymbol{C}_y' \end{bmatrix}$$

and

$$\left[\begin{array}{c:c} \boldsymbol{C}_v & \boldsymbol{D}_v \end{array}\right] := \left[\begin{array}{c:cccc} \grave{C}_v & \breve{D}_v & 0 & \cdots & 0 \\ \grave{C}_v \bar{A} & \grave{C}_v \acute{B} & \breve{D}_v & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \grave{C}_v \bar{A}^{\lambda-1} & \grave{C}_v \bar{A}^{\lambda-2} \acute{B} & \grave{C}_v \bar{A}^{\lambda-3} \acute{B} & \cdots & \breve{D}_v \end{array}\right],$$

$$\left[\begin{array}{c:c} \boldsymbol{C}_y & \boldsymbol{D}_y \end{array}\right] := \left[\begin{array}{c:cccc} C_y & 0 & 0 & \cdots & 0 \\ C_y \bar{A} & C_y \acute{B} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ C_y \bar{A}^{\lambda-1} & C_y \bar{A}^{\lambda-2} \acute{B} & C_y \bar{A}^{\lambda-3} \acute{B} & \cdots & 0 \end{array}\right].$$

It was proved in [11] (see also [9]) that the solution of this problem, if it exists, produces the solution to the lifted estimation problem in Fig. 6. Namely, each (block) row of the optimal $\boldsymbol{F}$ contains the coefficients of the optimal $\grave{F}(z)$ for different ρ:

$$\boldsymbol{F} = \begin{bmatrix} \grave{F}_{1,0} & \grave{F}_{1,-1} & \cdots & \grave{F}_{1,1-\lambda} \\ \grave{F}_{2,1} & \grave{F}_{2,0} & \cdots & \grave{F}_{2,2-\lambda} \\ \vdots & \vdots & \ddots & \vdots \\ \grave{F}_{\lambda,\lambda-1} & \grave{F}_{\lambda,\lambda-2} & \cdots & \grave{F}_{\lambda,0} \end{bmatrix}.$$

In [11] the optimization problem (13) was solved exploiting the state-space realization of $\breve{G}(z)$ so that the computational burden is linear in λ. The idea in [11] is to treat $\boldsymbol{D}_\bullet$ and $\boldsymbol{C}_\bullet$ as responses of dynamical systems with two-point boundary conditions, which simplifies manipulations over these matrices. This reduction, however, required that $\acute{G}_y(\infty)$ has full column rank. This assumption is clearly void for (11) as $\acute{G}_y(\infty) = 0$.

We thus cannot use the approach and formulae of [11]. Instead, we represent $\boldsymbol{D}_\bullet$ and $\boldsymbol{C}_\bullet$ via *implicit descriptor* systems with two-point boundary conditions as proposed in [13]. This representation facilitates the treatment of systems with singular "D" and "A" matrices and leads, via some lenghtly manipulation, to the closed-form solution of Theorem 1.

VI. Concluding Remarks

In this paper the problem of reconstructing an analog signal from sampled measurements has been formulated as an L^2 sampled-data estimation problem with the constraint that the estimator (reconstructor) has prescribed lengths of its impulse response and preview. A closed-form solution to this problem in the form of an FIR discrete filter and a zero-order generalized hold has been derived. The solution is based on the standard sampled-data Kalman filter Riccati equation and two Lyapunov recursions over the length of the impulse response of the reconstructor. The solution is numerically stable and can accomodate asymptotic behavior constraints by the use of unstable weighing functions. The procedure has been illustrated by designing an FIR version of the cardinal cubic B-splines.

References

[1] A. J. Jerri, "Shannon sampling theorem — its various extensions and applications — tutorial review," *Proc. IEEE*, vol. 65, no. 11, pp. 1565–1596, 1977.

[2] M. Unser, "Sampling—50 years after Shannon," in *Proc. IEEE*, 2000, pp. 569–587.

[3] M. Unser, A. Aldroubi, and M. Eden, "B-spline signal-processing I: Theory," *IEEE Trans. Signal Processing*, vol. 41, no. 2, pp. 821–833, 1993.

[4] M. Unser and T. Blu, "Cardinal exponential splines: Part I — theory and filtering algorithms," *IEEE Trans. Signal Processing*, vol. 53, pp. 1425–1438, 2005.

[5] D. Petrinović, "Causal cubic splines: Formulations, interpolation properties and implementations," *IEEE Trans. Signal Processing*, vol. 56, no. 11, pp. 5442–5453, 2008.

[6] G. Meinsma and L. Mirkin, "L^2 sampled signal reconstruction with causality constraints: Part I—setup and solutions" (submitted for publication), 2010.

[7] P. P. Khargonekar and Y. Yamamoto, "Delayed signal reconstruction using sampled-data control," in *Proc. 35th IEEE Conf. Decision and Control*, vol. 2, 1996, pp. 1259–1263.

[8] G. Meinsma and L. Mirkin, "Sampling from a system-theoretic viewpoint: Part II—noncausal solutions," *IEEE Trans. Signal Processing*, vol. 58, no. 7, pp. 3591–3606, 2010.

[9] L. Mirkin and G. Tadmor, "Imposing FIR structure on H^2 preview tracking and smoothing solutions," *SIAM J. Control Optim.*, vol. 48, no. 4, pp. 2433–2460, 2009.

[10] T. Chen and B. A. Francis, *Optimal Sampled-Data Control Systems*. London: Springer-Verlag, 1995.

[11] Y. Levinson and L. Mirkin, "L^2 optimization in discrete FIR estimation: exploiting state-space structure," in *Proc. 49th IEEE Conf. Decision and Control*, 2010, pp. 7700–7705.

[12] Y. Yamamoto and M. Araki, "Frequency responses for sampled-data systems — their equivalence and relationships," *Linear Algebra and its Applications*, vol. 205–206, pp. 1319–1339, 1994.

[13] A. C. Kahane, L. Mirkin, and Z. J. Palmor, "Discrete-time lifting via implicit descriptor systems," in *Proc. ECC'99*, Karlsruhe, Germany, 1999.