

Structure-preserving model reduction for nonlinear port-Hamiltonian systems

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Abstract—Port-Hamiltonian systems result from port-based network modeling of physical systems and constitute an important class of passive nonlinear state-space systems. In this paper, we develop a framework for model reduction of large-scale multi-input/multi-output nonlinear port-Hamiltonian systems that retains the port-Hamiltonian structure in the reduced order models. Within this framework, reduced order models are determined by the selection of two families of approximating subspaces. We consider two approaches deriving from a) a POD-based selection of subspaces, and b) an \mathcal{H}_2 -based quasi-optimal selection of subspaces. We compare performance of the reduced order models on a nonlinear lossy LC ladder network.

I. INTRODUCTION

The modeling of complex physical systems often involves systems of coupled partial differential equations, which upon spatial discretization, lead to dynamical models with very large state-space dimension. This creates a need for *model reduction* methods that can produce (comparatively) low dimensional surrogate models that nonetheless are able to closely mimic the original system's input/output map. Port-based network modeling of the original system will lead directly to a representation as a *port-Hamiltonian system* as well, a representation which encodes structural properties related to the manner in which energy is distributed and flows through the system. When the related Hamiltonian function is non-negative, port-Hamiltonian systems constitute an important class of *passive state-space systems*.

A. Port-Hamiltonian systems

Modeling and simulation often follows a system-theoretic network modeling paradigm that formalizes the interconnection of naturally specified subsystems. If the core dynamic models of subsystem components arise from variational principles, the aggregate system model typically has structural features that characterize it as a *port-Hamiltonian system*. While greater generality is both possible and useful (see, in particular, the review article [12], and the monograph [13]), for our purposes we focus on finite-dimensional port-Hamiltonian systems defined as:

$$\begin{aligned} \dot{\mathbf{x}} &= (\mathbf{J} - \mathbf{R}) \nabla_{\mathbf{x}} H(\mathbf{x}) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y} &= \mathbf{B}^T \nabla_{\mathbf{x}} H(\mathbf{x}), \end{aligned} \quad (1)$$

where $\mathbf{x} \in \mathbb{R}^n$ is the n -dimensional state vector; $H : \mathbb{R}^n \rightarrow [0, \infty)$ is a continuously differentiable scalar-valued vector function - the *Hamiltonian*, describing the distribution of energy in the system; $\mathbf{J} = -\mathbf{J}^T \in \mathbb{R}^{n \times n}$ is the *structure matrix*

describing the interconnection of energy storage elements in the system; $\mathbf{R} = \mathbf{R}^T \geq \mathbf{0}$ is the $n \times n$ *dissipation matrix* describing energy loss in the system; and, $\mathbf{B} \in \mathbb{R}^{n \times m}$ is the *input/output matrix* describing how energy enters and exits the system. We will always assume that the Hamiltonian is positive definite, i.e., $H(\mathbf{x}) > 0$ for all $\mathbf{x} \neq \mathbf{0}$ and $H(\mathbf{0}) = 0$.

The family of systems characterized by (1) generalizes the classical notion of *Hamiltonian systems* which can be expressed in our notation as $\dot{\mathbf{x}} = \mathbf{J} \nabla_{\mathbf{x}} H(\mathbf{x})$. The analog of conservation of energy for Hamiltonian systems becomes for (1): PH systems are always *stable* and *passive*:

$$H(\mathbf{x}(t_1)) - H(\mathbf{x}(t_0)) \leq \int_{t_0}^{t_1} \mathbf{y}(t)^T \mathbf{u}(t) dt,$$

which is to say, the change in the internal energy of the system, as measured by H , is bounded by the total work done on the system. Moreover, the class of port-Hamiltonian systems defined as in (1) is closed under power conserving interconnection - connecting port-Hamiltonian systems together produces an aggregate system that is also port-Hamiltonian, and hence, must be both stable and passive.

This last fact provides compelling motivation to preserve port-Hamiltonian structure when producing reduced order surrogate models intended to mimic closely the input/output response of systems of the sort defined by (1).

B. Petrov-Galerkin reduced models

Most model reduction approaches involve some variation of a Petrov-Galerkin projective approximation to the equations describing the system dynamics. This proceeds by choosing two subspaces of \mathbb{R}^n : an r -dimensional trial subspace, $\mathcal{V}_r \subset \mathbb{R}^n$, and an r -dimensional test subspace, $\mathcal{W}_r \subset \mathbb{R}^n$. It is convenient and generally nonrestrictive in practice to assume additionally that \mathcal{V}_r and \mathcal{W}_r have a "generic orientation" with respect to one another so that neither subspace contains any nontrivial vectors that are orthogonal to all vectors in the other subspace. The evolution of an associated reduced order model may be described in the following (initially indirect) way:

$$\begin{aligned} &\text{Find } \mathbf{v}(t) \text{ contained in } \mathcal{V}_r \text{ such that} \\ \dot{\mathbf{v}}(t) - (\mathbf{J} - \mathbf{R}) \nabla_{\mathbf{x}} H(\mathbf{v}) - \mathbf{B}\mathbf{u}(t) &\perp \mathcal{W}_r; \quad (2) \\ &\text{the associated output is } \mathbf{y}_r(t) = \mathbf{B}^T \nabla_{\mathbf{x}} H(\mathbf{v}). \end{aligned}$$

The dynamics described by (2) can be represented directly as a dynamical system evolving in a state-space of reduced dimension r once bases are chosen for the two subspaces \mathcal{V}_r and \mathcal{W}_r . Let $\text{Ran}(\mathbf{M})$ denote the range of a matrix \mathbf{M} . Let $\mathbf{V}_r \in \mathbb{R}^{n \times r}$ and $\mathbf{W}_r \in \mathbb{R}^{n \times r}$ be matrices defined so that $\mathcal{V}_r = \text{Ran}(\mathbf{V}_r)$ and $\mathcal{W}_r = \text{Ran}(\mathbf{W}_r)$. We can represent the

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reduced system trajectories as $\mathbf{v}(t) = \mathbf{V}_r \mathbf{x}_r(t)$ with $\mathbf{x}_r(t) \in \mathbb{R}^r$ for each t and the Petrov-Galerkin approximation (2) can be rewritten as

$$\mathbf{W}_r^T [\mathbf{V}_r \dot{\mathbf{x}}_r(t) - (\mathbf{J} - \mathbf{R}) \nabla_{\mathbf{x}} H(\mathbf{V}_r \mathbf{x}_r) - \mathbf{B} \mathbf{u}(t)] = \mathbf{0}$$

and $\mathbf{y}_r(t) = \mathbf{B}^T \nabla_{\mathbf{x}} H(\mathbf{V}_r \mathbf{x}_r)$.

Since \mathcal{V}_r and \mathcal{W}_r are assumed to have a generic orientation with respect to one another, $\mathbf{W}_r^T \mathbf{V}_r$ is invertible and moreover, we may choose bases for \mathcal{V}_r and \mathcal{W}_r such that, indeed, $\mathbf{W}_r^T \mathbf{V}_r = \mathbf{I}$.

This leads to a state-space representation of a reduced order dynamical system:

$$\begin{aligned} \dot{\mathbf{x}}_r &= \mathbf{W}_r^T (\mathbf{J} - \mathbf{R}) \nabla_{\mathbf{x}} H(\mathbf{V}_r \mathbf{x}_r) + \mathbf{W}_r^T \mathbf{B} \mathbf{u}(t) \\ \mathbf{y}_r &= \mathbf{B}^T \nabla_{\mathbf{x}} H(\mathbf{V}_r \mathbf{x}_r), \end{aligned} \quad (3)$$

Significantly, the definitions of these reduced-order quantities are invariant under a change of basis for the original state space, so the quality of reduced approximations evidently will depend only on effective choices for the subspaces $\mathcal{V}_r = \text{Ran}(\mathbf{V}_r)$ and $\mathcal{W}_r = \text{Ran}(\mathbf{W}_r)$.

As a cautionary note, although the representation in (3) portrays the reduced order model as a dynamical system that evolves in \mathbb{R}^r , one may find that due to the lifting of \mathbf{x}_r to \mathbb{R}^n that is implicit in forming $\mathbf{V}_r \mathbf{x}_r$, naive implementation of a direct simulation of this reduced order model may still have complexity proportional to $n \gg r$, and no savings from reducing the system order are realized. Approaches for resolving this difficulty are well understood and described in [1], for example. We defer consideration of this important aspect of creating efficient reduced order models to a later time and focus only on what constitute choices that result in high fidelity models in our setting. Note, in particular, that (3) does not evidently have the form (1) and so, will not typically be a port-Hamiltonian system.

II. REDUCTION OF PORT-HAMILTONIAN SYSTEMS

The usual heuristic that is applied in model reduction involves identifying and eliminating low-value portions of the state space, or conversely, identifying and preserving high-value portions of the state space. Adapting this approach to port-Hamiltonian systems, suppose we have identified two r -dimensional subspaces, \mathcal{V}_r and \mathcal{W}_r , that are “high-value” in the sense that for “most” input signal profiles, $\mathbf{u}(t)$, in (1)

- the resulting trajectory, $\mathbf{x}(t)$, stays “close” to \mathcal{V}_r , so that $\mathbf{x}(t) \approx \mathbf{V}_r \mathbf{x}_r(t)$ for some trajectory $\mathbf{x}_r \in \mathbb{R}^r$, and
- the internal force, $\nabla_{\mathbf{x}} H(\mathbf{x})$, stays “close” to \mathcal{W}_r , so that $\nabla_{\mathbf{x}} H(\mathbf{x}(t)) \approx \mathbf{W}_r \mathbf{f}_r(t)$, for some choice of $\mathbf{f}_r \in \mathbb{R}^r$.

Exactly what comprises “most” input signal profiles and how one measures “closeness” in this context will vary, and we consider two possibilities in the following sections. For the time being, note that if $\mathbf{x}(t) \approx \mathbf{V}_r \mathbf{x}_r(t)$ then plausibly,

$$\nabla_{\mathbf{x}} H(\mathbf{V}_r \mathbf{x}_r(t)) \approx \nabla_{\mathbf{x}} H(\mathbf{x}(t)) \approx \mathbf{W}_r \mathbf{f}_r(t).$$

Notice that these statements amount to assertions about the *subspaces*, \mathcal{V}_r and \mathcal{W}_r , and not about particular bases for these subspaces, so as long as \mathcal{V}_r and \mathcal{W}_r have a generic

orientation with respect to one another, bases may be chosen so that $\mathbf{W}_r^T \mathbf{V}_r = \mathbf{I}$. With this in mind, note further that

$$\mathbf{f}_r(t) = \mathbf{V}_r^T \mathbf{W}_r \mathbf{f}_r(t) \approx \mathbf{V}_r^T \nabla_{\mathbf{x}} H(\mathbf{V}_r \mathbf{x}_r(t)) = \nabla_{\mathbf{x}_r} H_r(\mathbf{x}_r(t)),$$

where we have introduced the *reduced Hamiltonian*, $H_r(\mathbf{x}_r) = H(\mathbf{V}_r \mathbf{x}_r)$. Our discussion leads to

$$\nabla_{\mathbf{x}} H(\mathbf{V}_r \mathbf{x}_r(t)) \approx \mathbf{W}_r \nabla_{\mathbf{x}_r} H_r(\mathbf{x}_r(t)).$$

Substituting $\mathbf{V}_r \mathbf{x}_r(t)$ for $\mathbf{x}(t)$ and $\mathbf{W}_r \nabla_{\mathbf{x}_r} H_r(\mathbf{x}_r(t))$ for $\nabla_{\mathbf{x}} H(\mathbf{x}(t))$ in (1) and multiplying by \mathbf{W}_r^T leads to a state-space representation of a reduced order port-Hamiltonian approximation:

$$\begin{aligned} \dot{\mathbf{x}}_r &= (\mathbf{J}_r - \mathbf{R}_r) \nabla_{\mathbf{x}_r} H_r(\mathbf{x}_r) + \mathbf{B}_r \mathbf{u}(t), \\ \mathbf{y}_r(t) &= \mathbf{B}_r^T \nabla_{\mathbf{x}_r} H_r(\mathbf{x}_r) \end{aligned} \quad (4)$$

with $H_r(\mathbf{x}_r) = H(\mathbf{V}_r \mathbf{x}_r)$, $\mathbf{J}_r = \mathbf{W}_r^T \mathbf{J} \mathbf{W}_r$, $\mathbf{R}_r = \mathbf{W}_r^T \mathbf{R} \mathbf{W}_r$, and $\mathbf{B}_r = \mathbf{W}_r^T \mathbf{B}$. Note that $H_r : \mathbb{R}^r \rightarrow [0, \infty)$ is a continuously differentiable scalar-valued vector function; $\mathbf{J}_r = -\mathbf{J}_r^T$; and $\mathbf{R}_r = \mathbf{R}_r^T \geq \mathbf{0}$, so (4) retains the structure of (1) and is a port-Hamiltonian system.

[11] and [2] also offer structure-preserving model reduction methods for nonlinear port-Hamiltonian systems. These papers elegantly use the concepts of Kalman decomposition (in [11]) and balanced truncation (in [2]) in deriving reduced order models of nonlinear port-Hamiltonian systems. However, obtaining the Kalman decomposition of the full-original system or balancing it is computationally very demanding for nonlinear systems; see e.g. [3], [11] and references therein. These approaches are infeasible for the problem class we consider — having hundreds to thousands of state-variables.

We consider different approaches here.

A. Structure-preserving POD

A natural notion of “closeness” used in choosing subspaces, \mathcal{V}_r and \mathcal{W}_r , may be found through the Proper Orthogonal Decomposition (POD), a popular approach to (unstructured) model reduction ([7]). We recast POD into our setting: Fix a square integrable input signal, $\mathbf{u}(t)$, for the system (1). The corresponding trajectory, $\mathbf{x}(t)$, will then also be square integrable. Denoting orthogonal projections, \mathbf{P} and \mathbf{Q} , we consider the minimization problem:

$$\mathbf{P}_* = \underset{\text{rank}(\mathbf{P}) = r}{\text{argmin}} \int_0^\infty \|(\mathbf{I} - \mathbf{P}) \mathbf{x}(t)\|^2 dt$$

and

$$\mathbf{Q}_* = \underset{\text{rank}(\mathbf{Q}) = r}{\text{argmin}} \int_0^\infty \|(\mathbf{I} - \mathbf{Q}) \nabla_{\mathbf{x}} H(\mathbf{x}(t))\|^2 dt.$$

We then take $\mathcal{V}_r = \text{Ran}(\mathbf{P}_*)$ and $\mathcal{W}_r = \text{Ran}(\mathbf{Q}_*)$.

Of course, as it stands this is not a computationally tractable approach. However, if we truncate the integrals and approximate them with the Trapezoid Rule, we arrive at a method for obtaining our approximating subspaces that is essentially the same as POD.

Algorithm 1: POD-PH

POD-based reduction of port-Hamiltonian systems

- 1) Generate a trajectory $\mathbf{x}(t)$, and collect snapshots: $\mathbb{X} = [\mathbf{x}(t_0), \mathbf{x}(t_1), \mathbf{x}(t_2), \dots, \mathbf{x}(t_N)]$.
- 2) Truncate an SVD of the snapshot matrix, \mathbb{X} , to get a POD basis, $\tilde{\mathbf{V}}_r$, for a “high-value” subspace of the state space. ($\mathbf{x}(t) \approx \tilde{\mathbf{V}}_r \tilde{\mathbf{x}}_r(t)$)
- 3) Simultaneously collect associated *force* snapshots: $\mathbb{F} = [\nabla_{\mathbf{x}} H(\mathbf{x}(t_0)), \nabla_{\mathbf{x}} H(\mathbf{x}(t_1)), \dots, \nabla_{\mathbf{x}} H(\mathbf{x}(t_N))]$.
- 4) Truncate an SVD of \mathbb{F} to get a second POD basis, $\tilde{\mathbf{W}}_r$, spanning a second “high-value” subspace approximating the range of $\nabla_{\mathbf{x}} H(\mathbf{x}(t)) \approx \tilde{\mathbf{W}}_r \tilde{\mathbf{f}}_r(t)$.
- 5) Change bases $\tilde{\mathbf{W}}_r \mapsto \mathbf{W}_r$ and $\tilde{\mathbf{V}}_r \mapsto \mathbf{V}_r$ such that $\mathbf{W}_r^T \mathbf{V}_r = \mathbf{I}$.
- 6) With \mathbf{V}_r and \mathbf{W}_r determined in this way, the POD-PH reduced port-Hamiltonian system is then specified by (4).

B. Example

We illustrate the POD-PH approach with an M -stage nonlinear ladder network (see Figure 1). The two inputs to the system are a voltage signal applied to the left-hand terminal pair and a current injection across the right-hand terminal pair. The symmetrically paired outputs are the induced current across the left-hand terminal pair and the induced voltage signal across the right-hand terminal pair. For simplicity, we assume that each stage of the ladder network is built from identical components and that the current injection from the right is zero. Resistors and inductors are assumed to behave linearly. The capacitors are assumed to have a nonlinear C-V characteristic of the form

$$C_k(V) = \frac{C_0 V_0}{V_0 + V}.$$

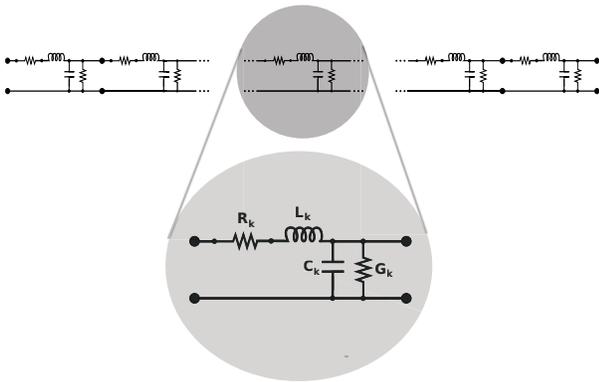


Fig. 1. Ladder network circuit topology (capacitors are nonlinear)

Inductors and capacitors are evidently the energy storage elements of the circuit, so we take as state variables the magnetic fluxes in the inductors, $\{\phi_k(t)\}_{k=1}^M$, and the charges on the capacitors, $\{Q_k\}_{k=1}^M$, with labels referring to the Stages $k = 1, \dots, M$, respectively, where they occur. The energy

stored in the Stage k (linear) inductor may be expressed in terms of its magnetic flux as $\frac{1}{2L_0} \phi_k^2$. To determine the energy stored in the nonlinear capacitors, note first that the charge on a capacitor may be expressed as a function of the voltage, V , held across the capacitor:

$$Q_k(V) = \int_0^V C(v) dv = C_0 V_0 \log \left(1 + \frac{V}{V_0} \right),$$

which may be inverted to find

$$V_k(Q_k) = V_0 \left[\exp \left(\frac{Q_k}{C_0 V_0} \right) - 1 \right].$$

The energy stored in the capacitor at Stage k of the circuit, is then given by

$$\int_0^{Q_k} V_k(q) dq = C_0 V_0^2 \left[\exp \left(\frac{Q_k}{C_0 V_0} \right) - 1 \right] - Q_k V_0,$$

and the total energy stored in Stage k is then

$$H^{[k]}(\phi_k, Q_k) = C_0 V_0^2 \left[\exp \left(\frac{Q_k}{C_0 V_0} \right) - 1 \right] - Q_k V_0 + \frac{1}{2L_0} \phi_k^2.$$

The Hamiltonian for this system is

$$H(Q_1, \dots, Q_M, \phi_1, \dots, \phi_M) = \sum_{k=1}^M H^{[k]}(\phi_k, Q_k).$$

We order the state variables so that $\mathbf{x} = [Q_1, \dots, Q_M, \phi_1, \dots, \phi_M]^T$. Elementary considerations lead to $\mathbf{J} = \begin{bmatrix} \mathbf{0} & \mathbf{S} \\ -\mathbf{S}^T & \mathbf{0} \end{bmatrix}$ where \mathbf{S} is an upper bidiagonal matrix with 1 on the diagonal and -1 on the superdiagonal; $\mathbf{R} = \begin{bmatrix} G_0 \mathbf{I} & \mathbf{0} \\ \mathbf{0} & R_0 \mathbf{I} \end{bmatrix}$; and $\mathbf{B} = [\mathbf{e}_{M+1}, \mathbf{e}_M]$ where \mathbf{e}_k denotes the k^{th} column of the identity.

We consider the particular case of a 50-stage ($M = 50$) circuit with the following parameters:

$L_0 = 2\mu\text{H}$	
$C_0 = 100\text{pF}$	$V_0 = 1\text{V}$
$R_0 = 1\Omega$	$G_0 = 10\mu\text{S}$

We applied a Gaussian voltage pulse with a magnitude of 3V, standard deviation of .5, and a duration of $3\mu\text{sec}$ to the left port of the network and observed the induced voltage at the right port. The output is displayed as a solid green trace in Figure 2. The induced response of the *linearized* network is also displayed as a green dashed line for comparison. Notice that nonlinearity sharpens the peak of the response and significantly reduces dispersion. Notice also that there is a propagation delay in the network of about $0.8\mu\text{sec}$. We took 1000 uniformly spaced samples of the system trajectory and internal forcing, applied Algorithm 1 (POD-PH), and constructed reduced order port-Hamiltonian models of orders $r = 6, 12, \text{ and } 20$. We then forced each reduced order model (ROM) with a) the same Gaussian pulse used to generate the POD bases (Figure 2), and b) a positive sinusoid with a peak-to-peak magnitude of 3V and frequency of 300kHz (Figure 3). The $r = 20$ POD-PH model provides the best fidelity for each case, as expected. Lower order models provide coarser

resolution but interestingly, the $r = 6$ POD-PH model (blue solid line) provides better fidelity for the Gaussian pulse input than what the $r = 12$ POD-PH model (blue dashed line) is able to provide, especially during the initial lag. This discrepancy is even more pronounced for the sinusoidal input. Notice that for the sinusoidal input, all POD-PH models have some difficulty capturing certain response features such as the initial lag period, the asymmetry of the response peaks, and the response ripple following each peak.

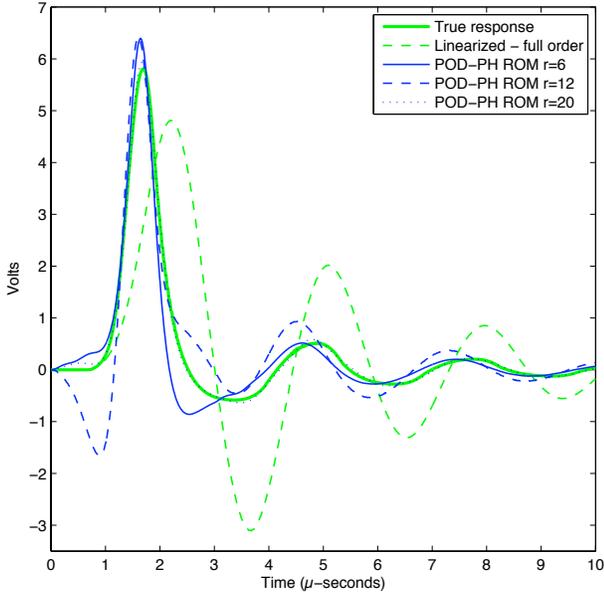


Fig. 2. ROM time response to Gaussian pulse with pulse-generated POD basis

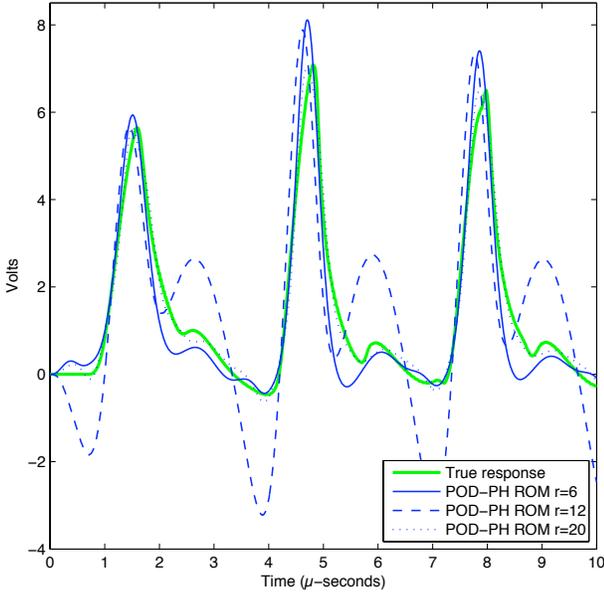


Fig. 3. ROM time response to sinusoid with Gaussian pulse POD basis

III. QUASI- \mathcal{H}_2 -OPTIMAL METHODS FOR PORT-HAMILTONIAN SYSTEMS

POD provides a method for choosing a \mathcal{V}_r and \mathcal{W}_r that can be very effective in extracting features that are present in the original sampled system response. As an empirical method, it is incapable of providing information about dynamic response features that are absent in the sampled system response, but that *could* have been present had a different choice of input profile been made. This is the reason the discrepancies between the original and reduced-order system outputs were magnified for the sinusoid input — a different input from the Gaussian pulse that generated the POD basis.

We consider now choices of the subspaces, \mathcal{V}_r and \mathcal{W}_r , that are *asymptotically optimal* for all possible input profiles that are sufficiently small (and hopefully at least pretty good for other input profiles). In contrast to POD approaches, no choice of inputs is necessary; no simulations actually need to be performed to derive effective approximating subspaces. The proviso that (virtual) inputs be sufficiently small, but otherwise arbitrary, allows us to tailor the subspaces \mathcal{V}_r and \mathcal{W}_r , so as to provide near-optimal reduction for the corresponding linearized port-Hamiltonian models. Note that linearization is used here only as a tool to obtain effective information that will be encoded in the subspaces, \mathcal{V}_r and \mathcal{W}_r , which are then used for reduction of the *nonlinear* system (1).

Any input profile, $\mathbf{u}(t)$, may be scaled to have sufficiently small magnitude so that the resulting trajectory, $\mathbf{x}(t)$, is small as well. Then linear terms in the internal forcing dominate, and $\nabla_{\mathbf{x}}H(\mathbf{x}) \approx \mathbf{Q}\mathbf{x}$ for a symmetric positive definite matrix $\mathbf{Q} \in \mathbb{R}^{n \times n}$ (indeed, the Hessian matrix for $H(\mathbf{x})$ evaluated at $\mathbf{x} = \mathbf{0}$). This approximation leads to the *linear port-Hamiltonian system*:

$$\dot{\mathbf{x}} = (\mathbf{J} - \mathbf{R})\mathbf{Q}\mathbf{x} + \mathbf{B}\mathbf{u}(t), \quad \mathbf{y}(t) = \mathbf{B}^T \mathbf{Q}\mathbf{x}. \quad (5)$$

A. Model Reduction of Linear Port-Hamiltonian Systems

In the case of a linear port-Hamiltonian system as in (5), the goal is to construct a reduced-order system of the form

$$\dot{\mathbf{x}}_r = (\mathbf{J}_r - \mathbf{R}_r)\mathbf{Q}_r\mathbf{x}_r + \mathbf{B}_r\mathbf{u}(t), \quad \mathbf{y}_r(t) = \mathbf{B}_r^T \mathbf{Q}_r\mathbf{x}_r \quad (6)$$

which is also port-Hamiltonian, i.e. $\mathbf{J}_r = -\mathbf{J}_r^T$, $\mathbf{R}_r = \mathbf{R}_r^T \geq 0$ and $\mathbf{Q}_r = \mathbf{Q}_r^T > 0$. In the linear case, we can associate (5) and (6) with their transfer functions

$$\mathcal{G}(s) = \mathbf{B}^T \mathbf{Q}(s\mathbf{I} - (\mathbf{J} - \mathbf{R})\mathbf{Q})^{-1} \mathbf{B} \quad (7)$$

and

$$\mathcal{G}_r(s) = \mathbf{B}_r^T \mathbf{Q}_r(s\mathbf{I} - (\mathbf{J}_r - \mathbf{R}_r)\mathbf{Q}_r)^{-1} \mathbf{B}_r, \quad (8)$$

respectively. Our model reduction problem reduces to a rational approximation problem: Find a degree- r rational function $\mathcal{G}_r(s)$ that approximates $\mathcal{G}(s)$ well with respect to an appropriate norm. Here, we focus on interpolatory approaches: Given r interpolation points $\sigma_1, \dots, \sigma_r$ in the complex plane with corresponding tangential directions $\{\mathbf{b}_1, \dots, \mathbf{b}_r\} \in \mathbb{C}^m$, the goal is to construct \mathcal{G}_r so that \mathcal{G}_r not only is port-Hamiltonian but also tangentially interpolates \mathcal{G} , i.e.

$$\mathcal{G}_r(\sigma_i)\mathbf{b}_i = \mathcal{G}(\sigma_i)\mathbf{b}_i \quad \text{for } i = 1, \dots, r. \quad (9)$$

A solution to this problem was recently given in [6] (note that [6] focusses on *linear* port-Hamiltonian systems):

Theorem 1: Given interpolation points $\sigma_1, \dots, \sigma_r$ and tangential directions $\mathbf{b}_1, \dots, \mathbf{b}_r$, construct

$$\tilde{\mathbf{V}}_r = [(\sigma_1 \mathbf{I} - (\mathbf{J} - \mathbf{R})\mathbf{Q})^{-1} \mathbf{B}\mathbf{b}_1, \dots, (\sigma_r \mathbf{I} - (\mathbf{J} - \mathbf{R})\mathbf{Q})^{-1} \mathbf{B}\mathbf{b}_r].$$

Let $\mathbf{V}_r = \tilde{\mathbf{V}}_r \mathbf{L}^{-1}$ and $\mathbf{W}_r = \mathbf{Q}\mathbf{V}_r$ where $\tilde{\mathbf{V}}_r^T \mathbf{Q} \tilde{\mathbf{V}}_r = \mathbf{L}^T \mathbf{L}$.

$$\text{Set } \begin{aligned} \mathbf{J}_r &= \mathbf{W}_r^T \mathbf{J} \mathbf{W}_r, & \mathbf{Q}_r &= \mathbf{V}_r^T \mathbf{Q} \mathbf{V}_r = \mathbf{I}_r, \\ \mathbf{R}_r &= \mathbf{W}_r^T \mathbf{R} \mathbf{W}_r, & \mathbf{B}_r &= \mathbf{W}_r^T \mathbf{B}. \end{aligned}$$

Then the reduced model, \mathcal{G}_r :

$$\dot{\mathbf{x}}_r = (\mathbf{J}_r - \mathbf{R}_r) \mathbf{Q}_r \mathbf{x}_r + \mathbf{B}_r \mathbf{u}, \quad \mathbf{y}_r = \mathbf{B}_r^T \mathbf{Q}_r \mathbf{x}_r \quad (10)$$

is port-Hamiltonian, passive, and satisfies the interpolation conditions (9).

For information on transfer function interpolation in the special case of single-input/single-output port-Hamiltonian systems, see [9], [5], [10]. For an overview of model reduction methods for linear port-Hamiltonian systems, see [8].

B. Quasi-optimal port-Hamiltonian rational approximation

Even though Theorem 1 shows how to construct a structure-preserving interpolatory reduced-order port-Hamiltonian approximation once $\{\sigma_i\}$ and $\{\mathbf{b}_i\}$ are given, it does not give any information on how to choose $\{\sigma_i\}$ and $\{\mathbf{b}_i\}$. We discuss issues related to finding effective approximations with respect to the \mathcal{H}_2 norm: The \mathcal{H}_2 norm of \mathcal{G} is defined as

$$\|\mathcal{G}\|_{\mathcal{H}_2} = \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} \|\mathcal{G}(i\omega)\|_{\mathbb{F}}^2 d\omega \right)^{1/2}.$$

Let $\mathcal{G}_r(s)$ minimize the \mathcal{H}_2 error $\|\mathcal{G} - \mathcal{G}_r\|_{\mathcal{H}_2}$ over all possible degree- r rational functions and suppose that $\mathcal{G}_r(s)$ has a partial fraction expansion $\mathcal{G}_r(s) = \sum_{k=1}^r \frac{c_k \mathbf{b}_k^T}{s - \hat{\lambda}_k}$. Then, as shown in [4], the interpolation conditions

$$\mathcal{G}(-\hat{\lambda}_k) \mathbf{b}_k = \mathcal{G}_r(-\hat{\lambda}_k) \mathbf{b}_k, \quad \text{for } k = 1, \dots, r \quad (11)$$

are necessary conditions for \mathcal{H}_2 optimality (there are additional conditions that must also be satisfied in general). In other words, the optimal \mathcal{H}_2 approximant \mathcal{G}_r is a tangential interpolant to \mathcal{G} at the mirror images of the reduced-order poles. For the full set of necessary conditions required for optimality, see [4].

Based on (11), a method was introduced in [6] that produces an interpolatory reduced-order port-Hamiltonian system satisfying the necessary conditions given in (11). Since the interpolation points $-\hat{\lambda}_k$ and the tangential directions \mathbf{b}_k depend on the reduced-model to be computed, an iterative process is used to correct the interpolation points and tangential directions until the desired conditions in (11) are obtained. A brief sketch of the algorithm is given below in Algorithm 2. We call this method “quasi- \mathcal{H}_2 -optimal” since it only satisfies a subset of the necessary conditions required for optimality. The remaining degrees of freedom are used in maintaining port-Hamiltonian structure. For details, see [6].

Algorithm 2: $Q\mathcal{H}_2$ OPT-PH

Quasi-optimal reduction of linear port-Hamiltonian systems ((6))

- 1) Make an initial shift selection $\{\sigma_i\}_1^r$, and tangent directions $\{\mathbf{b}_i\}_1^r$.
- 2) while (not converged)
 - a) $\hat{\mathbf{V}}_r = [(\sigma_1 \mathbf{I} - (\mathbf{J} - \mathbf{R})\mathbf{Q})^{-1} \mathbf{B}\mathbf{b}_1, \dots, \dots, (\sigma_r \mathbf{I} - (\mathbf{J} - \mathbf{R})\mathbf{Q})^{-1} \mathbf{B}\mathbf{b}_r]$.
 - b) Set $\mathbf{V}_r = \hat{\mathbf{V}}_r \mathbf{L}^{-1}$ with $\hat{\mathbf{V}}_r^T \mathbf{Q} \hat{\mathbf{V}}_r = \mathbf{L}^T \mathbf{L}$ (so $\mathbf{Q}_r = \mathbf{V}_r^T \mathbf{Q} \mathbf{V}_r = \mathbf{I}_r$).
 - c) Set $\mathbf{W}_r = \mathbf{Q} \mathbf{V}_r$. (so $\mathbf{V}_r^T \mathbf{W}_r = \mathbf{I}_r$).
 - d) Set $\mathbf{J}_r = \mathbf{W}_r^T \mathbf{J} \mathbf{W}_r$, $\mathbf{R}_r = \mathbf{W}_r^T \mathbf{R} \mathbf{W}_r$, and $\mathbf{B}_r = \mathbf{W}_r^T \mathbf{B}$.
 - e) Calculate left eigenvectors: $\mathbf{z}_i^T (\mathbf{J}_r - \mathbf{R}_r) = \lambda_i \mathbf{z}_i^T$.
 - f) Set $\sigma_i \leftarrow -\lambda_i$ and $\mathbf{b}_i \leftarrow \mathbf{B}_r^T \mathbf{z}_i$ for $i = 1, \dots, r$.
- 3) Calculate final $Q\mathcal{H}_2$ OPT-PH bases:
Find $\tilde{\mathbf{V}}_r = [(\sigma_1 \mathbf{I} - (\mathbf{J} - \mathbf{R})\mathbf{Q})^{-1} \mathbf{B}\mathbf{b}_1, \dots, \dots, (\sigma_r \mathbf{I} - (\mathbf{J} - \mathbf{R})\mathbf{Q})^{-1} \mathbf{B}\mathbf{b}_r]$.
Set $\mathbf{V}_r = \tilde{\mathbf{V}}_r \mathbf{L}^{-1}$ with $\hat{\mathbf{V}}_r^T \mathbf{Q} \hat{\mathbf{V}}_r = \mathbf{L}^T \mathbf{L}$.
Set $\mathbf{W}_r = \mathbf{Q} \mathbf{V}_r$.
- 4) With \mathbf{V}_r and \mathbf{W}_r determined in this way, the $Q\mathcal{H}_2$ OPT-PH reduced port-Hamiltonian system is then specified by (4).

We use the linearized port-Hamiltonian model only to obtain the quasi-optimal model reduction subspaces. Once \mathbf{V}_r and \mathbf{W}_r are obtained using $Q\mathcal{H}_2$ OPT-PH as outlined in Algorithm 2, we use them to reduce the original nonlinear system as shown in (4). See [?], [?] for other approaches that compute the projection matrices using a linearized system for reduction of nonlinear systems.

C. Example

We repeated the same computational experiments on the ladder network described above using instead quasi- \mathcal{H}_2 -optimal subspaces obtained with Algorithm 2. Notice that no simulations are performed and so, unlike POD-PH, the $Q\mathcal{H}_2$ OPT-PH subspaces are not tied to any particular input profiles.

We applied Algorithm 2, and obtained quasi- \mathcal{H}_2 -optimal bases of orders $r = 6, 12$, and 20 . We used these subspaces to construct reduced order $Q\mathcal{H}_2$ OPT-PH port-Hamiltonian models and then performed reduced-order simulations, forcing each ROM with a) the same Gaussian pulse used previously (Figure 4), and b) the same sinusoid used previously (Figure 5). The $r = 6$ POD-PH results are displayed on the same axes for easy comparison (the $r = 12$ POD-PH case is omitted since $r = 6$ POD-PH performed better). The $r = 20$ $Q\mathcal{H}_2$ OPT-PH model gave results comparable to the $r = 20$ POD-PH case (both quite good) and neither is displayed to reduce clutter. The most significant feature that emerges from examination of these results is that the $Q\mathcal{H}_2$ OPT-PH models were able to capture response features of true response better at lower order than POD-PH for the two input profiles applied. Notably for the sinusoid input, $Q\mathcal{H}_2$ OPT-PH models capture the asymmetry of the response peaks and the response ripple following each peak even at low resolution. All models had some difficulty capturing the initial lag, but the $Q\mathcal{H}_2$ OPT-PH models generally did

better than the POD-PH models. For the example shown here, QH_2OPT -PH exhibited generally monotone improvement in fidelity as r increased in sharp contrast with POD-PH on this example. The advantages of QH_2OPT -PH seem most often to occur at low order on signal profiles that did not participate in generation of a POD basis; this was the main motivation to employ input-independent quasi-optimal model reduction techniques. We have presented an example where POD bases can give poor results even with precisely the same signal from which it was generated.

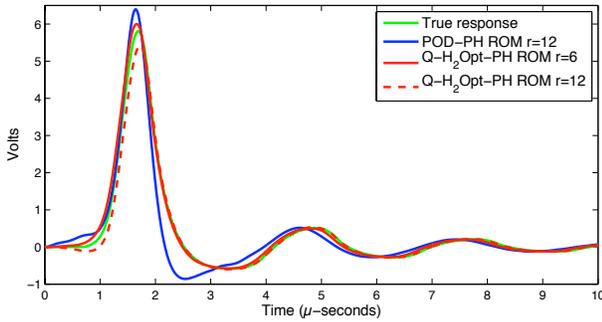


Fig. 4. ROM time response to Gaussian pulse using QH_2OPT -PH bases

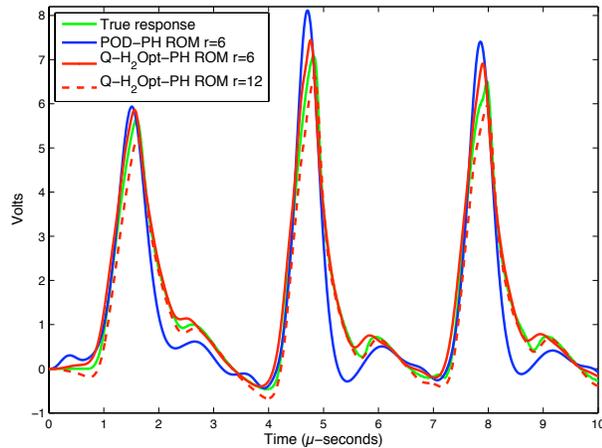


Fig. 5. ROM time response to sinusoid using QH_2OPT -PH bases

IV. CONCLUSIONS AND FUTURE WORK

We have introduced a structure-preserving projection framework for model reduction of large-scale multi-input/multi-output nonlinear port-Hamiltonian systems leading to reduced-order port-Hamiltonian models. Projecting subspaces are constructed with two different approaches: one is POD-based; the other is related to an \mathcal{H}_2 -based approach

originally targeted for linear port-Hamiltonian systems. Numerical experiments on a nonlinear ladder network illustrate better performance for the latter method, especially with low order approximation and in circumstances where simulations are driven by input profiles that do not participate in the generation of the POD basis.

Improving the model reduction subspaces further by combining strengths of both approaches considered here, for example, by selectively merging the two families of subspaces, is the focus of on-going research. Analysis and experimentation of related methods for the fully nonlinear case where \mathbf{J} and \mathbf{R} themselves carry a dependence on state variables is also under study.

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