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An active set solver for input-constrained robust receding horizon control

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Abstract— **An efficient optimization procedure is proposed for computing a receding horizon control law for linear systems with constrained control inputs and additive disturbances. The procedure uses an active set method to solve the dynamic programming problem associated with the min-max optimization of a predicted cost. The active set at the solution is determined at each sampling instant as a function of the current system state using the first-order necessary conditions for optimality. The computational complexity of each iteration is linear in the length of the prediction horizon. We discuss conditions for stability and bounds on state and input l^2-norms in closed loop operation.**
Keywords: **Dynamic programming, robust control, constrained model predictive control, min-max optimization.**

I. INTRODUCTION

The aim of robust control is to provide guarantees of stability and of performance with respect to a suitable measure, despite uncertainty in the model of the controlled system. Model Predictive Control (MPC) uses a receding horizon strategy to derive robust control laws by repeatedly solving a constrained optimization problem online, and consequently the approach is effective for systems with constraints and bounded disturbances [1].

Robust receding horizon control based on a worst-case optimization was first proposed in [2]. The approach employed a min-max optimization, which was subsequently adopted in [3] to derive an MPC law for linear systems with uncertain impulse response coefficients. In this strategy, and in the related contribution of [4], an open loop predicted future input sequence was used to minimize the worst-case predicted performance. It was argued in [5] that by optimizing instead over closed loop predicted input sequences, control laws with improved performance and larger regions of attraction could be obtained. However, unless a degree of optimality is sacrificed through the use of sub-optimal controller parameterizations (such as, for example, those proposed in [6], [7] and [8]), strategies that involve a receding horizon optimization over predicted feedback policies generally require impractically large computational loads. For example [9] and [10] apply a scenario-based approach to constrained linear systems with bounded additive uncertainty, which leads to an optimization problem in a number of variables which grows exponentially with the prediction horizon length.

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Parametric solution methods aim to avoid the explosion in computational complexity of robust dynamic programming with horizon length by characterizing the solution of the receding horizon optimization problem offline, typically as a feedback law that is a piecewise affine function of the model state. In [11] and [12] this method was applied to linear systems with polytopic parametric uncertainty. However, whereas MPC typically solves an optimization problem for a unique initial condition at each time-step, this approach requires the solution at all points in state space, and moreover relies on being able to efficiently determine which of a large number of polyhedral regions contains the current state. Although efficient point location techniques have been proposed (e.g. [13]), the method is generally applicable only to small problems and horizon lengths.

The contribution of this paper is to extend the methodology developed in [14], [15], [16] to the case of linear systems with bounded additive uncertainty and input constraints in order to derive a robust dynamic programming solver. An online active set approach avoids the need to compute the solution over the entire state space, and it also forms the basis of an efficient line-search-based point location technique. The resulting algorithm has complexity per iteration that grows only linearly with the horizon length. We use a quadratic cost, and to derive a convex-concave min-max problem, an $\mathcal{H}_\infty$ performance index is therefore employed. The algorithm ensures closed loop stability and a specified l^2-disturbance gain bound.

II. PROBLEM STATEMENT

We consider linear discrete time systems with model

$$x_{t+1} = Ax_t + Bu_t + Dw_t, \quad t = 0, 1, \ldots \tag{1}$$

with state $x_t \in \mathbb{R}^{n_x}$, control input $u_t \in \mathbb{R}^{n_u}$ and disturbance input $w_t \in \mathbb{R}^{n_w}$. Here u_t and w_t are subject to constraints: $u_t \in \mathcal{U}$, $w_t \in \mathcal{W}$, in which $\mathcal{U}$ and $\mathcal{W}$ are assumed to be convex polytopic sets defined by

$$\mathcal{U} = \{u \in \mathbb{R}^{n_u} : Fu \leq \mathbf{1}\} \tag{2}$$

$$\mathcal{W} = \{w \in \mathbb{R}^{n_w} : Gw \leq \mathbf{1}\} \tag{3}$$

for $F \in \mathbb{R}^{n_F \times n_u}$, $G \in \mathbb{R}^{n_G \times n_w}$, and where $\mathbf{1} = [1 \ \cdots \ 1]^T$ denotes a vector of conformal dimensions.

We define the following closed loop robust optimal control problem (see e.g. [11], [17]):

$$\big(u_k^*(x_k), w_k^*(x_k, u_k)\big) = \arg \min_{u_k \in \mathcal{U}} \max_{w_k \in \mathcal{W}} J_k(x_k, u_k, w_k) \tag{4a}$$

subject to $x_{k+1} = Ax_k + Bu_k + Dw_k$, where J_k is defined for $k = 0, 1, \dots N-1$ by

$$J_k(x_k, u_k, w_k) = \frac{1}{2}\big(\|x_k\|_Q^2 + \|u_k\|_R^2 - \gamma^2\|w_k\|^2\big) + J_{k+1}^*(x_{k+1}), \tag{4b}$$

$$J_k^*(x_k) = J_k\big(x_k, u_k^*(x_k), w_k^*(x_k, u^*(x_k))\big), \tag{4c}$$

and is subject to initial and terminal conditions over a finite horizon of N time-steps:

$$x_0 = x_t^p \tag{4d}$$

$$J_N^*(x_N) = \frac{1}{2}\|x_N\|_P^2. \tag{4e}$$

Here x_t^p denotes the current plant state (at time t), R is a positive-definite matrix (denoted $R \succ 0$), Q is positive semidefinite ($Q \succeq 0$), $\|x\|_Q^2$ denotes x^TQx, and the scalar γ is chosen (as explained in Section III-A) to be sufficiently large that (4) is strictly concave in w_k. We assume that P is chosen so that $J_N^*(x_N)$ is equal to the infinite horizon cost: $J_N^*(x_N) = \sum_{k=N}^{\infty} \frac{1}{2}(\|x_k\|_Q^2 + \|u_k^*\|_R^2 - \gamma^2\|w_k^*\|^2)$ that is obtained under the optimal linear feedback law defined by (4a-c) in the absence of the constraints $u_k \in \mathcal{U}$, $w_k \in \mathcal{W}$, and with terminal condition $\lim_{k\to\infty} J_k^*(x_k) = 0$. Furthermore, problem (4) is assumed to be feasible for the initial state x_0^p.

The problem formulated in (4) defines a closed loop optimal control problem (see e.g. [5]), since the optimal control u_k depends on x_k, while the worst-case disturbance w_k depends on u_k and x_k. The sequential nature of this min-max problem and the fact that the optimization is performed over a set of arbitrary feedback laws $\{u_k(x), w_k(x, u),\ k = 0, \dots, N-1\}$ imply that, unlike open-loop formulations of robust MPC, (4) cannot be solved exactly by a single quadratic program.

For given x_0, we denote the optimal state, input and disturbance sequences as

$$\begin{aligned}
\mathbf{x}(x_0) &= \{x_0, \dots, x_N\}\\
\mathbf{u}(x_0) &= \{u_0^*(x_0), \dots, u_{N-1}^*(x_{N-1})\}\\
\mathbf{w}(x_0) &= \{w_0^*(x_0, u_0^*(x_0)), \dots, w_{N-1}^*(x_{N-1}, u_{N-1}^*(x_{N-1}))\}.
\end{aligned}$$

A receding horizon control law is defined by implementing at each time $t = 0, 1, \dots$ the initial optimal control law evaluated at the current plant state, $u_t = u_0^*(x_t^p)$. The process of solving the optimization problem is then repeated at each subsequent time instant.

III. ACTIVE SET SOLUTION VIA RICCATI RECURSION

The objectives of this section are threefold. Firstly the Karush-Kuhn-Tucker (KKT) conditions providing first-order necessary conditions for optimality [18], [19] are stated for the problem (4). We derive Riccati recursions to determine the solution of an associated problem involving only equality constraints, using a sweep method (as in [20]). Thus we obtain the sequence of optimal control and disturbance policies for an equality constrained problem corresponding to a given active constraint set as a sequence of affine state feedback functions. We give necessary and sufficient conditions for these control policies to be optimal with respect to the original problem (4).

Secondly, having determined the optimal feedback laws at each point on the horizon via backwards recursion of the KKT conditions, we then determine state, input, disturbance, costate and multiplier sequences as functions of the initial state x_0 by forward simulation using the system model (1). We show that it is possible to devise a line-search through polyhedral partitions of x_0-space, starting from an estimate of the optimal solution, and successively updating the active set as a function of x_0 until $x_0 = x^p$, as in [14]. Finally we determine how the computation required by this approach depends on the problem size.

A. KKT conditions and Riccati recursion

Let λ_k denote the vector of Lagrange multipliers associated with the constraints $x_{k+1} = Ax_k + Bu_k + Dw_k$ and let μ_k and η_k denote the Lagrange multipliers of constraints $u_k \in \mathcal{U}$ and $w_k \in \mathcal{W}$ respectively, for $k = 0, \dots, N-1$. Then, using standard results (see e.g. [19]), the KKT conditions defining first order necessary conditions for the optimal solution of problem (4) can be expressed as follows.

$$x_{k+1} = Ax_k + Bu_k + Dw_k \quad \text{for } k = 0, \dots, N-1 \tag{5}$$

$$\lambda_{k-1} = A^T\lambda_k + Qx_k \quad \text{for } k = 1, \dots, N-1 \tag{6}$$

$$Ru_k = -B^T\lambda_k - F^T\mu_k \tag{7a}$$

$$\mu_k \geq 0, \quad \mu_k^T(\mathbf{1} - Fu_k) = 0, \quad \mathbf{1} - Fu_k \geq 0 \tag{7b}$$

$$\gamma^2 w_k = D^T\lambda_k - G^T\eta_k \tag{8a}$$

$$\eta_k \geq 0, \quad \eta_k^T(\mathbf{1} - Gw_k) = 0, \quad \mathbf{1} - Gw_k \geq 0 \tag{8b}$$

for $k = 0, \dots, N-1$, with the initial and terminal conditions:

$$x_0 = x^p \tag{9}$$

$$\lambda_{N-1} = Px_N. \tag{10}$$

An active set approach solves the optimization problem (4) by solving a sequence of problems involving only equality constraints. Let $\mathbf{s} = (\mathbf{s}^u, \mathbf{s}^w)$ define a set of active constraints in (4), namely a set of constraints that are satisfied with equality at a solution of (4) for some initial state x_0. Specifically, let $\mathbf{s}^u = \{s_{0,i}^u, \dots, s_{N-1,i}^u,\ i = 1, \dots, n_u\}$ and $\mathbf{s}^w = \{s_{0,i}^w, \dots, s_{N-1,i}^w,\ i = 1, \dots, n_w\}$, where $s_{i,k}^u, s_{i,k}^w$ can take values of 0 or 1, and rewrite (7b) and (8b) as

$$\left.\begin{aligned} e_i^TFu_k &= 1\\ e_i^T\mu_k &\geq 0\end{aligned}\right\} \text{ if } s_{k,i}^u = 1, \qquad \left.\begin{aligned} e_i^TFu_k &\leq 1\\ e_i^T\mu_k &= 0\end{aligned}\right\} \text{ if } s_{k,i}^u = 0 \tag{11}$$

$$\left.\begin{aligned} e_i^TGw_k &= 1\\ e_i^T\eta_k &\geq 0\end{aligned}\right\} \text{ if } s_{k,i}^w = 1, \qquad \left.\begin{aligned} e_i^TGw_k &\leq 1\\ e_i^T\eta_k &= 0\end{aligned}\right\} \text{ if } s_{k,i}^w = 0 \tag{12}$$

where e_i denotes the ith column of an identity matrix of conformal dimensions. Furthermore, let F_k, G_k denote the matrices that consist of the rows of F, G corresponding to the active sets indicated by $s_{k,i}^u = 1$, $i = 1, \dots, n_u$, and $s_{k,i}^w = 1$,

$i = 1, \dots, n_w$ respectively. Also denote the multipliers of these active constraints as $\mu_{a,k}$ and $\eta_{a,k}$. Then the equality constraints in (7a),(11) and (8a),(12) are equivalent to

$$Ru_k = -B^T \lambda_k - F_k^T \mu_{a,k} \quad (13a)$$
$$F_k u_k = \mathbf{1} \quad (13b)$$

$$\gamma^2 w_k = D^T \lambda_k - G_k^T \eta_{a,k} \quad (14a)$$
$$G_k w_k = \mathbf{1}. \quad (14b)$$

For any $\mathbf{s} \in \Sigma$, where Σ denotes the set of all $\mathbf{s}$ such that the equality constrained problem admits a solution for some x_0, the equality constraints of the KKT conditions, namely (5)-(6) and (13)-(14), together with (9)-(10), define a two-point boundary value problem.

In order to solve this equality constrained problem for given $\mathbf{s} \in \Sigma$ using a Riccati recursion, we first express the costate variables as

$$\lambda_k = P_k x_{k+1} + q_k. \quad (15)$$

Then, using (5), (14) gives

$$\begin{bmatrix} \gamma^2 I - D^T P_k D & G_k^T \\ G_k & 0 \end{bmatrix} \begin{bmatrix} w_k \\ \eta_{a,k} \end{bmatrix} = \begin{bmatrix} D^T P_k (Ax_k + Bu_k) + D^T q_k \\ \mathbf{1} \end{bmatrix}. \quad (16)$$

Under the assumption that (16) has a unique solution:

$$\begin{bmatrix} w_k \\ \eta_{a,k} \end{bmatrix} = \begin{bmatrix} M_k^w \\ M_k^\eta \end{bmatrix} (Ax_k + Bu_k) + \begin{bmatrix} m_k^w \\ m_k^\eta \end{bmatrix}, \quad (17)$$

(15) gives $\lambda_k = \hat{P}_k (Ax_k + Bu_k) + \hat{q}_k$, where $\hat{P}_k, \hat{q}_k$ are defined

$$\hat{P}_k = P_k (I + DM_k^w) \quad (18a)$$
$$\hat{q}_k = q_k + P_k D m_k^w. \quad (18b)$$

Hence (13) gives

$$\begin{bmatrix} R + B^T \hat{P}_k B & F_k^T \\ F_k & 0 \end{bmatrix} \begin{bmatrix} u_k \\ \mu_{a,k} \end{bmatrix} = \begin{bmatrix} -B^T \hat{P}_k A x_k - B^T \hat{q}_k \\ \mathbf{1} \end{bmatrix}. \quad (19)$$

Assuming that (19) has a unique solution:

$$\begin{bmatrix} u_k \\ \mu_{a,k} \end{bmatrix} = \begin{bmatrix} L_k^u \\ L_k^\mu \end{bmatrix} x_k + \begin{bmatrix} l_k^u \\ l_k^\mu \end{bmatrix}, \quad (20)$$

equation (6) yields $\lambda_{k-1} = P_{k-1} x_k + q_{k-1}$, where

$$P_{k-1} = Q + A^T \hat{P}_k (A + BL_k^u) \quad (21a)$$
$$q_{k-1} = \hat{q}_k + A^T \hat{P}_k B l_k^u \quad (21b)$$

Finally, since (10) has the form of (15), with $P_{N-1} = P$ and $q_{N-1} = 0$, it follows by induction that (15) holds for $k = 0, \dots, N-1$.

The following result gives conditions for optimality of the Riccati recursion (17)-(18) and (20)-(21).

Lemma 1: The optimal solution of problem (4) is given by

$$w_k^*(x_k, u_k) = M_k^w (Ax_k + Bu_k) + m_k^w \quad (22a)$$
$$u_k^*(x_k) = L_k^u x_k + l_k^u \quad (22b)$$

if and only if

$$G_{k,\perp}^T (\gamma^2 I - D^T P_k D) G_{k,\perp} \succ 0 \quad (23a)$$
$$F_{k,\perp}^T (R + B^T \hat{P}_k B) F_{k,\perp} \succ 0 \quad (23b)$$

(where the columns of $F_{k,\perp}$ and $G_{k,\perp}$ form bases for the kernels of F_k and G_k respectively), and

$$G\big(M_k^w (Ax_k + Bu_k) + m_k^w\big) \leq 1, \quad F(L_k^u x_k + l_k^u) \leq 1 \quad (24a)$$
$$M_k^\eta (Ax_k + Bu_k) + m_k^\eta \geq 0, \qquad L_k^\mu x_k + l_k^\mu \geq 0. \quad (24b)$$

Proof: Problem (4) is respectively strictly concave in w_k and strictly convex in u_k iff conditions (23a) and (23b) hold. In this case the KKT conditions have a unique solution and are sufficient as well as necessary for optimality. Conditions (24a,b) ensure that the solution of the equality constraint problem (5)-(6), (13)-(14), (9), coincides with that of the KKT conditions (5)-(9), for the given active set $\mathbf{s}$ and x_0. □

Remark 2: It is easy to show that the conditions in (23a,b) ensure that (16) and (19) admit unique solutions.

Remark 3: Condition (23b) is necessarily satisfied since $R \succ 0$ by assumption and $Q \succeq 0$ implies $P_k, \hat{P}_k \succ 0$ for all k. However (23a) is very difficult to verify in practice, since this requires checking all active sets $\mathbf{s} \in \Sigma$. In this paper we simply assume that γ is sufficiently large to satisfy (23a) for all active sets likely to be encountered.

B. Active set method

Using the feedback law (22) in conjunction with (5) to simulate forward over the prediction horizon, we obtain:

$$x_k = \Phi_k x_0 + \phi_k \quad \text{for } k = 1, \dots, N \quad (25)$$

where $\Phi_k \in \mathbb{R}^{n \times n}$ and $\phi_k \in \mathbb{R}^n$ are defined by

$$\Phi_{k+1} = (I + DM_k^w)(A + BL_k^u)\Phi_k \quad (26a)$$
$$\phi_{k+1} = (I + DM_k^w)\big((A + BL_k^u)\phi_k + Bl_k^u\big) + Dm_k^w \quad (26b)$$

with inital conditions $\Phi_0 = I$ and $\phi_0 = 0$. Therefore the input, disturbance and costate sequences $\mathbf{u}(x_0)$, $\mathbf{w}(x_0)$ and $\boldsymbol{\lambda}(x_0) = \{\lambda_0, \dots, \lambda_{N-1}\}$, as well as the corresponding multiplier sequences $\boldsymbol{\mu}(x_0) = \{\mu_0, \dots, \mu_{N-1}\}$ and $\boldsymbol{\eta}(x_0) = \{\eta_0, \dots, \eta_{N-1}\}$ can be determined as affine functions of x_0 by substituting (25) into (17), (20) and (15). Hence, for a given active set $\mathbf{s}$, we can define a region of state space $\mathcal{X}(\mathbf{s}) \subset \mathbb{R}^{n_x}$ in which the KKT conditions hold:

$$\mathcal{X}(\mathbf{s}) = \big\{x_0 : \mathbf{x}(x_0) \text{ satisfies (24a,b)}\big\}. \quad (27)$$

Lemma 4: The sets $\mathcal{X}(\mathbf{s})$ defined by (27) are convex polyhedra, and the collection $\{\mathcal{X}(\mathbf{s}) : \mathbf{s} \in \Sigma\}$ is a complex with the properties (see e.g. [21]):

$$\partial \mathcal{X}(\mathbf{s}) \subset \mathcal{X}(\mathbf{s}) \quad (28a)$$
$$\mathcal{X}(\mathbf{s}_1) \cap \mathcal{X}(\mathbf{s}_2) = \partial \mathcal{X}(\mathbf{s}_1) \cap \partial \mathcal{X}(\mathbf{s}_2) \quad (28b)$$

for any $\mathbf{s}_1, \mathbf{s}_2 \in \Sigma$ (where $\partial \mathcal{X}(\mathbf{s})$ denotes the boundary of $\mathcal{X}$). Furthermore the union $\bigcup_{\mathbf{s} \in \Sigma} \mathcal{X}(\mathbf{s})$ of all admissible active sets covers the set of feasible initial conditions for (4).

Proof: The convexity property and (28a) follow from the linear inequality constraints of (24a,b). Property (28b) results from the piecewise continuity of the trajectories $\mathbf{x}(x_0), \boldsymbol{\lambda}(x_0), \mathbf{u}(x_0), \mathbf{w}(x_0), \boldsymbol{\mu}(x_0)$, and $\boldsymbol{\eta}(x_0)$. Furthermore, since a solution of (4) exists for all feasible x^p, $\bigcup_{\mathbf{s}\in\Sigma} \mathcal{X}(\mathbf{s})$ necessarily covers the set of feasible initial conditions x^p. □

The algorithm we propose solves (4) by solving the equality constrained problem for an estimate of the optimal active set, and then updates this active set at successive iterations. At each iteration $i = 0, 1, \ldots$ the algorithm determines $\mathbf{s}^{(i+1)}$ from $s^{(i)}$ by performing a line search over $x_0 \in \mathcal{X}(\mathbf{s}^{(i)})$ in the direction of the current plant state x^p. This results in a sequence of dual-feasible iterates $x^{(k)}$ that generate trajectories satisfying (24a,b) but not necessarily (9).

Algorithm 1: Initialize with $x_0^{(0)}$ and an active set $\mathbf{s}^{(0)}$ such that $x_0^{(0)} \in \mathcal{X}(\mathbf{s}^{(0)})$, and set $i = 0$. At iteration $i = 0, 1, \ldots$:

(i) Compute $\{P_k, q_k\}$ for $k = N-1, \ldots, 0$, and $\{\Phi_k, \phi_k\}$ for $k = 0, \ldots, N-1$, and hence $\mathcal{X}(\mathbf{s}^{(i)})$.

(ii) Perform the line search:
$$\alpha^{(i)} = \max_{\alpha\in(0,1]} \{\alpha : x_0^{(i)} + \alpha(x^p - x_0^{(i)}) \in \mathcal{X}(\mathbf{s}^{(i)})\}.$$

(iii) If $\alpha^{(i)} < 1$, then set $x_0^{(i+1)} := x_0^{(i)} + \alpha^{(i)}(x^p - x_0^{(i)})$, $i := i+1$, and update $\mathbf{s}^{(i)}$ on the basis of the new set of active constraints. Return to step (i).

(iv) Otherwise set $\mathbf{s}^* := \mathbf{s}^{(i)}$, compute $u_0^*(x^p)$ and stop.

Theorem 5: Algorithm 1 converges to $\mathbf{s}^*$ such that (4) is minimized by the trajectories for $\mathbf{x}$, $\mathbf{u}$ and $\mathbf{w}$ that are generated by (5) and (22) with $\mathbf{s} = \mathbf{s}^*$.

Proof: The line search in step (ii) of Algorithm 1 implies that each iterate $x_0^{(i)}$ lies on the line segment defined by $x_0^{(0)} + \beta^{(i)}(x^p - x_0^{(0)})$ with $\beta^{(i)} \in [0, 1]$. Since the sequence $\{\beta^{(i)},\ i = 0, 1, \ldots\}$ is non-decreasing and each iterate $x^{(i)}$ lies either at an intersection of the line with the boundary $\partial\mathcal{X}(\mathbf{s}^{(i)})$ or at $x_0^{(i)} = x^p$, the sequence $\{\beta^{(i)},\ i = 0, 1, \ldots\}$ must converge to 1 after a finite number of iterations (due to the finite number of admissible active sets $\mathbf{s} \in \Sigma$). It follows that Algorithm 1 terminates with $x_0^{(i)} = x^p$ after a finite number of iterations. □

Remark 6: x^p is assumed to be a feasible initial condition for (4), therefore a trivial initialization for Algorithm 1 is the choice $x_0^{(0)} = 0$ and $\mathbf{s}^{(0)} = \{0, \ldots, 0\}$. In the context of MPC, further computational savings can be achieved by warm-starting Algorithm 1. This can be done by choosing $x_0^{(0)}$ at time $k+1$ equal to the second element of the optimal sequence at time k, i.e. $x_1^*(k)$. Correspondingly, under the assumption that the state enters a terminal set after N time-steps (as discussed in section IV), the active set can be initially chosen as $\mathbf{s}^{(0)} = \{s_1^*(k), \ldots, s_N^*(k), s_{N+1}^*(k)\}$, i.e. by applying a time-shift to the optimal active set at time k, $\mathbf{s}^*(k)$. Here $s_{N+1}^*(k)$ is determined from the dynamics of (5) under the terminal feedback law described in section IV.

C. Computation

In order to estimate how the computational complexity of Algorithm 1 depends on the problem size, we make the assumption that (16) and (19) are solved using the null space method commonly employed by QP active set solvers (see e.g. [22]). This approach, applied to (16), involves computing the QR decomposition of F_k, which requires $O(n_w^2)$ floating point operations (assuming that incremental rank-1 updates are employed), as well as calculating the inverse of the matrix on the LHS of (23a), which requires $O\big((n_w - n_F)^3\big)$ operations (assuming Cholesky decomposition is used, where $n_F \le n_w$ is the number of rows of F_k). Applying the same approach to the solution of (19) requires $O(n_u^2)$ operations (for the QR decomposition of G_k) plus $O\big((n_u - n_G)^3\big)$ operations (for the Cholesky decomposition of the LHS of (23b), where $n_G \le n_u$ is the number of rows of G_k). The other significant contribution to the computation in (17)-(21) is due to the matrix multiplications in (18) and (21), which require $O\big((2n_x^3 + (3n_u + 2n_w)n_x^2 + n_u^2 n_x)\big)$ operations.

Combining these estimates, and noting that the computation required for the forward simulation is $O(n_x^2 N)$ (since only the projection, $\Phi_k(x^p - x_0^{(i)})$, of Φ_k in (26a,b) is needed), and also that the computation involved in the line search in step (ii) is comparatively insignificant, we estimate the computation per iteration of Algorithm 1 to grow as

$$O\Big(\big(2n_x^3 + n_x^2(3n_u + 2n_w) + c_1(n_w^3 + n_u^3) + c_2(n_w^2 + n_u^2)\big)N\Big).$$

Here c_1, c_2 are constants that depend on the implementations of Cholesky and QR decompositions, and we have used conservative approximations: $n_u - n_G \approx n_u$, $n_w - n_F \approx n_w$.

Thus the dependence of computation per iteration on the horizon length, N is linear. The required number of iterations is problem-dependent, but empirical evidence (see e.g. the example of Section V) suggests that this also grows roughly linearly with N. Furthermore the number of iterations can be minimized using warm-starts, as described in Remark 6. This is in stark contrast to existing schemes for min-max receding horizon control, which, for the case of optimal approaches that are based on dynamic programming, have computational loads that depend exponentially on N (see e.g. [9], [10]). Likewise, approaches such as [7], [8] based on suboptimal controller parameterizations require the solution of a QP in a number of optimization variables that grows quadratically with the horizon length; hence these are likely to have much higher computational load than Algorithm 1.

IV. Closed Loop stability and l^2-gain Bound

This section discusses the stability and disturbance attenuation properties of the feedback law defined by the receding horizon implementation of Algorithm 1. The problem description (4) does not include explicit state constraints, and hence does not allow terminal state constraints to be included in the definition of the receding horizon policy. However we are able to ensure robust stability and a specified l^2-gain bound within a given set of initial conditions in state space.

The analysis involves three steps. Firstly, assuming dual mode predictions, we calculate an unconstrained optimal feedback law as the terminal control law and also determine a robust, controlled, positive invariant set $\mathcal{X}^f$ as a corresponding terminal set [1]. A specified l^2-gain bound is thus ensured if $x_0 \in \mathcal{X}^f$. Secondly, we find an N-step backwards-reachable set, $\mathcal{X}^N$, from which from the terminal set $\mathcal{X}^0 = \mathcal{X}^f$ is reachable in N steps under the receding horizon application of the feedback law of Algorithm 1. This gives $\mathcal{X}^N$ as a region of attraction for the receding horizon control law. Finally we show that the receding horizon application of Algorithm 1 ensures that the specified l^2-gain bound holds for all initial conditions in $\mathcal{X}^N$.

Lemma 7: If the matrix inequality:

$$\begin{bmatrix} P & 0 & (A+BK_f)^T P & [Q \;\; K_f^T R] \\ \star & \gamma I & D^T P & 0 \\ \star & \star & P & 0 \\ \star & \star & \star & \gamma I \end{bmatrix} \succ 0 \tag{29}$$

holds for some P and K_f, then in the absence of constraints on u_t, w_t, the system $x_{t+1} = Ax_t + Bu_t + Dw_t$, with $u_t = K_f x_t$, satisfies the following bound on disturbance l^2-gain

$$\frac{\sum_{t=0}^{\infty} (\|x_t\|_Q^2 + \|u_t\|_R^2)}{\sum_{t=0}^{\infty} \|w_t\|^2} \leq \gamma^2. \tag{30}$$

Proof: This follows from standard results (see e.g. [23]). □

Remark 8: Condition (29) can be formulated as a linear matrix inequality in $S = P^{-1}$, $Y = K_f P^{-1}$ and γ:

$$\begin{bmatrix} S & 0 & (AS+BY)^T & [SQ \;\; Y^T R] \\ \star & \gamma I & D^T & 0 \\ \star & \star & S & 0 \\ \star & \star & \star & \gamma I \end{bmatrix} \succ 0. \tag{31}$$

Values for P and K_f can be computed for given (sufficiently large) γ by minimizing the trace of $P = S^{-1}$ subject to (31). The optimal P is the solution of the corresponding steady-state $\mathcal{H}_\infty$ Riccati equation [24], and hence defines the terminal cost in (4e) and also determines the gain K_w in the unconstrained optimal disturbance function $w_t = K_w x_t$.

Definition 1: A set $\mathcal{X}_f \subseteq \mathbb{R}^{n_x}$ is robust positive invariant for $x_{t+1} = (A + BK_f)x_t + Dw_t$ if: (i). $\mathcal{X}_f \subseteq \mathcal{X}_u$, where $\mathcal{X}_u = \{x \in \mathbb{R}^{n_x} : K_f x \in \mathcal{U} \text{ and } K_w x \in \mathcal{W}\}$; and (ii). $(A + BK_f)x + Dw \in \mathcal{X}^f$ for all $x \in \mathcal{X}^f$ and all $w \in \mathcal{W}$.

Definition 2: For $t = 1, \ldots, N$, the set $\mathcal{X}^{t-1}$ is the preimage of $\mathcal{X}^t$ under the receding horizon application of the optimal feedback policy, $u_0^*(x)$, of Algorithm 1 if $\mathcal{X}^0 = \mathcal{X}^f$ and $\mathcal{X}^t = \{x \in \mathbb{R}^{n_x} : Ax + Bu_0^*(x) + Dw \in \mathcal{X}^{t-1}, \forall w \in \mathcal{W}\}$.

Theorem 9: If $x_0 \in \mathcal{X}^N$, then for all admissible disturbance sequences $\{w_t \in \mathcal{W},\ t = 0, 1, \ldots\}$, the state of the closed loop system, $x_{t+1} = Ax_t + Bu_t + Dw_t$ under the receding horizon application of the control law $u_t = u_0^*(x_t)$ of Algorithm 1, enters the terminal set $\mathcal{X}^f$ in N time-steps, i.e. $x_t \in \mathcal{X}^f$ for some $t \leq N$. Furthermore, the disturbance l^2-gain of the closed loop system is bounded by

$$\frac{\sum_{t=0}^{\infty} (\|x_t\|_Q^2 + \|u_t\|_R^2)}{\sum_{t=0}^{\infty} \|w_t\|^2} \leq \gamma^2. \tag{32}$$

Proof: If $x_0 \in \mathcal{X}^N$, then $x_t \in \mathcal{X}^f$ for $t \leq N$ follows directly from Definitions 1 and 2. To demonstrate the bound on l^2-gain, we first show that the optimal cost $J_0^*(x_t)$ is finite for all $t \geq 0$. Consider the evolution of the optimal cost, $J_k^*(\hat{x}_k)$, after k steps along the optimal state sequence predicted at time t, which is denoted here as $\mathbf{x}(x_t) = \{\hat{x}_0, \hat{x}_1, \ldots\}$, with $\hat{x}_0 = x_t$, in order to distinguish predicted states $\hat{x}_k$ from the actual plant state x_t. Clearly $J_k^*(\hat{x}_k)$ in (4e) is finite for $k = N$ since (29) is assumed to be feasible. Furthermore, if $J_k^*(\hat{x}_k)$ is finite, then $J_{k-1}^*(\hat{x}_{k-1})$ must also be finite because the optimization in (4a) is necessarily feasible. Since the horizon length N is by assumption finite, it follows by induction that $J_0^*(x_t)$ is bounded. Next, from the definition of problem (4), at time t we obtain:

$$J_0^*(x_t) - J_1^*(\hat{x}_1) = \tfrac{1}{2}(\|x_t\|_Q^2 + \|u_t^*\|_R^2 - \gamma^2 \|w_t^*\|^2)$$

and therefore by optimality the following inequality holds:

$$J_0^*(x_t) - J_0^*(x_{t+1}) \geq \tfrac{1}{2}(\|x_t\|_Q^2 + \|u_t^*\|_R^2 - \gamma^2 \|w_t\|^2)$$

and since $J_0^*(x_t)$ is bounded for all t, this implies:

$$\lim_{m\to\infty} \frac{1}{m}(J_0^*(x_0) - J_0^*(x_{m+1})) = 0$$

which implies the l^2-gain bound stated in (32). □

V. Numerical Example

The proposed robust MPC algorithm was applied to (1) with

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad B = \tfrac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

and with constraint sets $\mathcal{U} = \{u \in \mathbb{R} : -1 \leq u \leq 1\}$ and $\mathcal{W} = \{w \in \mathbb{R}^2 : -0.1 \leq w_i \leq 0.1 \text{ for } i = 1, 2\}$, and cost weights $Q = I$, $R = 1$. Using Remark 8, P, K_f, and K_w were computed with γ chosen sufficiently large to satisfy conditions (23a,b) of Lemma 1. For $N = 4$ this gives

$$P = \begin{bmatrix} 2.422 & 1.160 \\ 1.160 & 3.770 \end{bmatrix}, \quad K_f = \begin{bmatrix} -0.580 & -1.385 \end{bmatrix}$$

$$K_w = \begin{bmatrix} 0.0178 & 0.0145 \\ -0.0033 & 0.0201 \end{bmatrix}, \quad \gamma^2 = 80.$$

A robust positive invariant set $\mathcal{X}^f$ was obtained using the procedure of [25] and the preimage sets $\mathcal{X}^k$ for $k = 1, \ldots, 4$ were determined numerically (Fig. 1). By Theorem 10, under $u_t = u_0^*(x_t)$, $\mathcal{X}^4$ is therefore a region of attraction of $\mathcal{X}^f$.

Figure 2 demonstrates that the computational complexity per iteration of Algorithm 1 scales linearly with horizon length N, in agreement with Section III-C. Average computation times (shown in blue) are given for two sets of 180 plant states $\|x^p\|$, which are equispaced around circles of radius 1.5 (with $\gamma^2 = 80$) and radius 2.5 ($\gamma^2 = 480$) respectively.

In each case Algorithm 1 was initialized with $x_0^{(0)} = 0$. For this example, the number of iterations required to reach convergence (shown in green) initially increases linearly with N, but reaches a maximum when N is sufficiently large that an increase in N causes no more constraints to become active. Thus, for example, the circle of radius 1.5 lies entirely within $\mathcal{X}^4$, so the number of iterations is constant for $N \geq 4$ for the case of $\|x^p\| = 1.5$, and hence the computation time increases quadratically for $N < 4$ and linearly for $N > 4$.

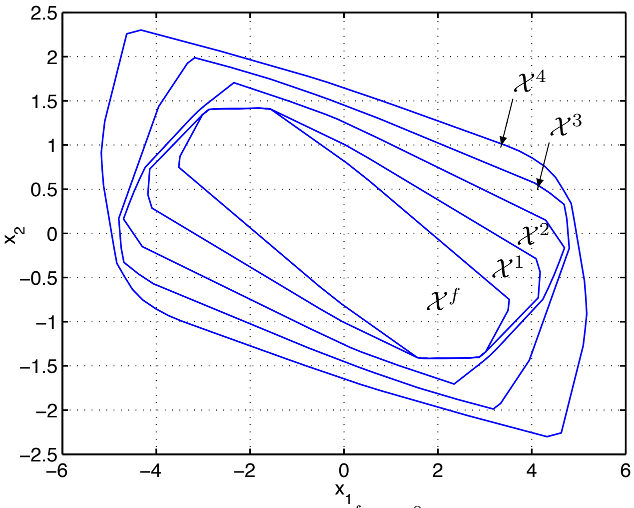

Fig. 1. Robust Positive Invariant Set $\mathcal{X}^f = \mathcal{X}^0$ and numerically computed backwards-reachable sets for $N = 4$

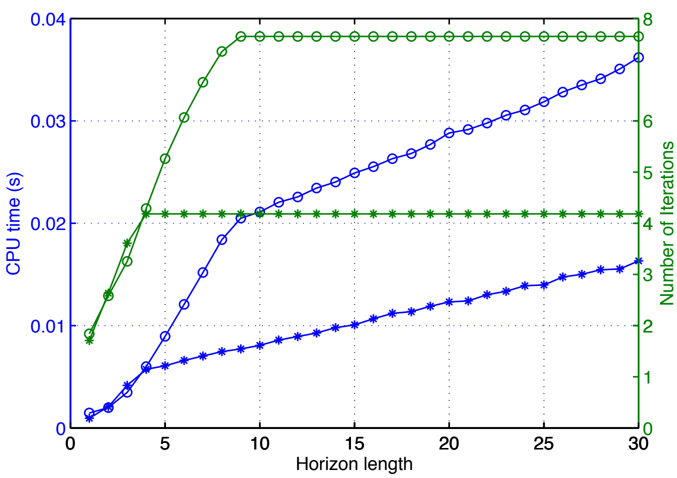

Fig. 2. CPU time (blue) and number of iterations (green) for convergence of Algorithm 1 vs horizon length, averaged over 180 plant states with $||x^p|| = 2.5$ (marked –○–) and 180 states with $||x^p|| = 1.5$ (marked –∗–).

VI. Conclusions

This paper considers a robust min-max MPC problem for input constrained linear systems with bounded disturbances. We give necessary and sufficient conditions for optimality and propose an algorithm based on Riccati recursions which can be warm started. In addition, a guaranteed l^2-gain bound is derived under the condition that the initial system state is inside the region of attraction of the closed loop system.

Future work will incorporate linear state constraints. Recursive feasibility will be ensured through the use of polyhedral backwards reachable sets guaranteeing that state predictions enter a robust positive invariant terminal set. In the presence of state constraints, the optimal value of the cost can be discontinuous in x_t^p [17], and the subproblems analogous to (16) and (19) are degenerate at such points; handling degenerate constraints is one of the main challenges of this work. The inclusion of state constraints is however expected to lead to an active set algorithm with complexity per iteration that remains linearly dependent on horizon length.

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