2011 50th IEEE Conference on Decision and Control and
European Control Conference (CDC-ECC)
Orlando, FL, USA, December 12-15, 2011

How Good are the Stochastic Analysis Methods for Stochastic Reachability

Manuela L. Bujorianu

Abstract— **For stochastic hybrid systems, safety verification methods are very little supported mainly because of complexity and difficulty of the associated mathematical problems. Stochastic reachability problem can be treated as an exit problem for a suitable class of Markov processes. Using Newtonian/Martin capacities associated to the Green/Martin kernel of a Markov process, we obtain upper bounds for the reach probabilities.**

Keywords: stochastic hybrid processes, reachability, operator methods, Markov models, capacity, Green kernel.

I. INTRODUCTION

Hybrid systems form a class of systems with behaviors characterized by a non-trivial interaction between discrete and continuous dynamics. Their models are quite useful for technical systems from automotive industry, aeronautics, air traffic control, robotics, and nanotechnology. Hybrid models are also used frequently in system biology and medicine, where their features make controllability and verification more difficult, mostly because of uncertainty, complex continuous nonlinear dynamics, partial information, etc. In the case of systems evolving in dynamical environments, they will exhibit random evolutions that increase the complexity of the corresponding verification and control problems. To address these issues, randomized models have been developed under the name of *stochastic hybrid systems* (SHS). Mathematically, a stochastic hybrid system can be described as an interleaving between a finite family of diffusion processes (or, only deterministic dynamical systems) and a Markov chain. Modeling and analysis of these systems have been proved to be a challenging task, especially from foundational point of view. For studying the SHS properties, we need to handle the appropriate stochastic analysis apparatus that combines purely probabilistic methods (probability distributions, expectations, moments, and so on) with concepts from functional analysis and partial differential equations (PDE). We can not base our study only on transition probabilities as in the case of Markov chains, since the state space of SHS is continuous. Moreover, this state space is hybrid, i.e. the continuous states are 'embedded' in some discrete states called modes or locations. Then, functional analysis operators are needed to capture the continuous transitions and their probabilistic evolution is described by PDEs associated to these operators. The SHS study involves the ability to combine tools available for diffusion processes and jump processes, in order to characterize the executions of such systems. The switching mechanism (governed by a Markov chain) between the continuous dynamics of the modes, together with the interaction between trajectories and the active boundaries, make the studying of the hybrid processes that arise in this way even more complicated.

The *reachability problem* concept in the framework of SHS was set up in [5]. Since then, different authors have studied this problem for particular [17], or general [15] classes of SHS. The mathematical foundations of this concept have been addressed in connection with concepts like hitting operators, occupation measures, first passage problems. Intuitively, the stochastic reachability analysis aims to evaluate the probabilities of those trajectories that visit a target set in finite/infinite horizon time.

In the literature, for deterministic hybrid systems there exist different methods to deal with the reachability problem. The most used methods are based on optimal control, as Hamilton Jacobi Bellman (HJB) equations, such that the computational issues are solved using dynamic programming. As well, reachability problem for hybrid systems can be thought of as an exit problem from a given domain. This also involves solving a standard HJB equation over this set and possibly pieces of its boundary with rather complicated boundary conditions (see the discussions from [18]). In the SHS framework, tackling stochastic reachability can have many facets. We can base the stochastic reachability analysis (SRA) only on probabilistic pillars. Then, it is compulsory to find ways to approximate/abstract SHS by Markov chains, only. In many papers [17], [16], the standard methodology to approach this problem is to approximate the stochastic process that corresponds to the given hybrid system by Markov chains and then to derive convergence results for the reach set probabilities. Also, from a computer science perspective, Markov chain approximations are desirable for probabilistic model checking. Due to the complexity of SHS models, the Markov chain approximations suffer from state space explosion (see [15] and discussions therein). Another perspective on SRA is to employ stochastic control methods. Studying stochastic reachability as an optimal control problem could be a very challenging and difficult task [1]. The main explanation for this difficulty can be found in the structure of the stochastic processes that describe the behaviour of SHS. These processes are Markov models with piecewise continuous paths. Their discontinuities are describe by some Poisson like jumps (spontaneous jumps) and forced jumps (dictated by some interface sets). The presence of the forced transitions leads to some peculiarities of the transition probabilities of the stochastic hybrid processes. The main problem comes from the fact that the dynamic programming theory for the Markov hybrid processes with

This work has been funded by the CICADA project EP/E050441/1.

Manuela Bujorianu is with the School of Mathematics, University of Manchester, UK, e-mail: {Manuela.Bujorianu@manchester.ac.uk)

predictable jumps is not fully understood and developed. In most cases, dynamic programming methods are applied only locally to these processes where their behavior is continuous [15]. Then, in the stochastic control approach, the main idea is to find the suitable 'smooth' approximations for stochastic hybrid processes. Stochastic hybrid processes are jump type processes. From the perspective of control theory and stochastic analysis, it seems that such processes are better studied using diffusion type approximations [12].

In this paper, we investigate how far stochastic analysis methods can go to provide accurate estimation of the reach probabilities for SRA. We start with known results for Markov chains and Brownian motion. Then we give some extensions for continuous Markov processes and stochastic hybrid processes. These extensions will constitute the basement for further developments and discussions. The scope is to provide a short tutorial about advantages and disadvantages of these approaches, and then to discuss novel ideas that can circumvent the hindrances that appear when we play only with the stochastic analysis tools.

II. PRELIMINARIES

This section provides the necessary background for stochastic hybrid systems and their dynamics, and some stochastic analysis that is used for studying SHS. The presentation is given in a hybrid automata framework. First we present the main elements (the hybrid automaton structure) that will be used then to describe the hybrid state space and the hybrid dynamics. Then we provide the description of the state space and the executions of the hybrid automaton. Finally, under some standard assumptions, we point the fact that the hybrid executions form of a stochastic hybrid process, which is usually a Markov process. The transition probabilities of such a process can be employed to construct functional analysis operators like those that define the transition semigroup, resolvent and the infinitesimal generator of the underlying Markov process. These operators represent the main analysis tool to deal with Markov processes.

A. Stochastic Hybrid Systems

Formally, a stochastic hybrid automaton (SHA) is defined as a tuple $H = (Q, \mathcal{X}, F, R, \lambda)$ where: (i) Q is a finite set of discrete variables; (ii) $\mathcal{X} : Q \to \mathbb{R}^{d(.)}$ maps each $q \in Q$ into a mode (an open subset) X^q of $\mathbb{R}^{d(q)}$, where $d(q)$ is the Euclidean dimension of the corresponding mode; (iii) $F : Q \to 2^{\mathcal{F}_{SDE}}$ specifies the continuous evolution of the automaton in terms of stochastic differential equations (SDE) over the continuous state x^q for each mode; (iv) $R = (R^q)_{q\in Q}$ a family of stochastic kernels defined as $R^q : \overline{X}^q \times \bigcup_{j\in Q\backslash\{q\}} \mathcal{B}(X^j) \to [0,1]$, where $\mathcal{B}(X^j)$ is the Borel σ-algebra of X^j; (v) $\lambda : \bigcup_{j\in Q} \overline{X}^j \to \mathbb{R}^+$ is a transition rate function, which gives the distributions of the jump times.

The hybrid state space is the set $\mathbf{X}(Q, d, \mathcal{X}) = \bigcup_{i\in Q}\{i\} \times X^i$, and the hybrid state is defined as $x = (i, z^i) \in \mathbf{X}(Q, d, \mathcal{X})$. The closure of the hybrid state space will be $\overline{\mathbf{X}} = \mathbf{X} \cup \partial\mathbf{X}$,where $\partial\mathbf{X} = \bigcup_{i\in Q}\{i\} \times \partial X^i$.

The executions of an SHA, H, can be described as follows: start with an initial point $x_0 \in X^q$, follow a solution of the SDE associated to X^q, jump when this trajectory hits the boundary or according with the transition rate λ (the jump time is the minimum of the boundary hitting time and the time that is exponentially distributed with the transition rate λ). Under standard assumptions [7], for each initial condition $x \in \bigcup_{j\in Q} X^j$, the possible trajectories starting from x, form a stochastic process. The component diffusion processes are Markov process and the SHA jumping times have the memoryless property. Therefore, for all initial conditions x, the realizations of an SHA can be thought of as a Markov process in a general setting.

Let us consider the stochastic process $\mathbf{M} = (x_t, \mathbb{P}_x)$, which represents the realization of H, i.e. all its possible trajectories. We can define in a standard way the probability space Ω as the set of all trajectories of $\mathbf{M}$. As well, for each time $t > 0$, we may define the history of the process $\mathcal{F}_t$ in the form of a σ-algebra [11]. Under mild assumptions on the parameters of H, $\mathbf{M}$ can be viewed as a Markov process with the state space $(\mathbf{X}, \mathcal{B})$, where $\mathbf{X}$ is the union of modes and $\mathcal{B}$ is its Borel σ-algebra. Let $\mathcal{B}^b(\mathbf{X})$ be the Banach space of bounded positive measurable functions on $\mathbf{X}$ with the norm given by supremum. Here, $(\mathbb{P}_x)_{x\in\mathbf{X}}$ represent the Wiener probabilities on the trajectories, i.e. $\mathbb{P}_x(x_0 = x) = 1$.

B. Hybrid Processes

Stochastic Analysis Elements: For the analysis of stochastic hybrid systems, we need to use the different characterizations of Markov processes. Briefly, in the following the most important functional analysis operators associated to a Markov process are presented. They represent the leveraging to the continuous time, continuous space case of the similar concepts that are popular when studying Markov chains. Their presence in this paper is justified by the fact that these operators are not standard in the theory of discrete processes (like Markov chains), and the reader familiar only with discrete processes might have difficulties in understanding the stochastic analysis characterizations for stochastic reachability developed in the continuous context.

Let us consider the Markov process $\mathbf{M} = (x_t, \mathbb{P}_x)$ with the state (topological) space $\mathbf{X}$. The following mathematical objects can be defined:

The *transition probability function* is given by a family of stochastic kernels $(p_t)_{t>0}$: $p_t(x, A) = \mathbb{P}_x(x_t \in A)$, $A \in \mathcal{B}$. The meaning of $p_t(x, A)$ is the probability that, if $x_0 = x$, x_t will lie in the set A.

The *operator semigroup or transition semigroup* $\mathcal{P} = (\mathbf{P}_t)_{t>0}$ is obtained by integration on the space $\mathcal{B}^b(\mathbf{X})$ (bounded measurable functions) w.r.t. $p_t(x, dy)$:

$$\mathbf{P}_t f(x) := \int f(y)p_t(x, dy) = \mathbb{E}_x f(x_t), \forall x \in \mathbf{X} \qquad (1)$$

where $\mathbb{E}_x$ is the expectation w.r.t. $\mathbb{P}_x$. The operator semigroup $(\mathbf{P}_t)_{t>0}$ is, in fact, the collection of all first order moments, which can be associated with the family of random variables $\{x_t|t > 0\}$.

Taking the Laplace transform the transition semigroup $(\mathbf{P}_t)_{t>0}$, we obtain the *operator resolvent*. The resolvent $\mathcal{V} = (V_\alpha)_{\alpha \geq 0}$ associated with $\mathcal{P}$ is

$$V_\alpha f(x) = \int_0^\infty e^{-\alpha t} \mathbf{P}_t f(x) dt, x \in \mathbf{X}. \quad (2)$$

Let denote by V the initial operator V_0 of $\mathcal{V}$, which is known as the *kernel operator* of the Markov process $\mathbf{M}$.

When we are dealing with continuous processes, the stochastic matrix defined for Markov chains, becomes a linear operator called the *infinitesimal generator*, denoted by $\mathcal{L}$. $\mathcal{L}$ is the derivative of $\mathbf{P}_t$ at $t = 0$. Let $D(\mathcal{L}) \subset \mathcal{B}_b(\mathbf{X})$ be the set of functions f for which $\lim_{t \searrow 0} \frac{1}{t}(\mathbf{P}_t f - f)$ exists and is denoted by $\mathcal{L}f$. $D(\mathcal{L})$ is known as the domain of the generator. Traditionally, Markov processes have been described by their generators and the corresponding evolutions by the operator semigroups/resolvents..

Realization of a stochastic hybrid system: Suppose now $\mathbf{M}$ represents the realization of a stochastic hybrid system H. Under standard assumption $\mathbf{M}$ is a Borel right process [7], i.e. (i) $\mathbf{M}$ is a strong Markov process with right-continuous paths. (ii) $\mathbf{X}$ is a separable metric space homeomorphic to a Borel subset of some compact metric space, equipped with Borel σ-algebra $\mathcal{B}(\mathbf{X})$ or shortly $\mathcal{B}$. That means $\mathbf{X}$ is a Lusin state space. (iii) The operator semigroup of $\mathbf{M}$, given by (1), maps $\mathcal{B}^b(\mathbf{X})$ into itself. Moreover, the sample paths of $\mathbf{M}$ are right continuous with left limit (RCLL), i.e. are cadlags (the French abbreviation for RCLL). We can add to $\mathbf{X}$ a cemetery point Δ where the process is trapped when $p_t(x, \mathbf{X}) < 1$, and define $\mathbf{X}_\Delta = \mathbf{X} \cup \{\Delta\}$. One can take the sample space Ω for $\mathbf{M}$ to be the set of all paths $(0, \infty) \ni t \mapsto \omega(t) \in \mathbf{X}_\Delta$ and the life time of the process as $\zeta(\omega) := \inf\{s > 0 | \omega(s) = \Delta\}$. Assume also that $\mathbf{M}$ is *transient*, i.e. there exists a strictly positive Borel function q such that Vq is bounded. The transience of $\mathbf{M}$ means that any trajectory which visits a Borel set of the state space will leave it after a finite time.

For an appropriate domain, the infinitesimal generator of a stochastic hybrid system has the following integro-differential form

$$\mathcal{L}f(x) = \mathcal{L}_{cont} f(x) + \lambda(x) \int_{\mathbf{X}} (f(y) - f(x)) R(x, dy) \quad (3)$$

where $\mathcal{L}_{cont} f(x)$ has the standard form of the diffusion infinitesimal operator. What makes this generator "special" is its domain that contains at least the set of second order differentiable functions that satisfy the following boundary condition: $f(x) = \int_{\mathbb{X}} f(y) R(x, dy), x \in \partial \mathbf{X}$.

III. Stochastic Reachability

Let us consider $\mathbf{M}$ being a Markov process that describes the realization of a stochastic hybrid system H. For this stochastic hybrid process, we address the *stochastic reachability problem* as follows. Given a target set $A \in \mathcal{B}(\mathbf{X})$, the objective of the reachability problem is to compute the probability that the system trajectories from an arbitrary initial state will reach the target set in finite ($T > 0$) or infinite horizon time. Two sets of trajectories, which reach the set A (the flow that enters A) in the interval of time $[0, T]$ or $[0, \infty)$ can be defined:

$Reach_T(A) = \{\omega \in \Omega \mid \exists t \in [0, T] : x_t(\omega) \in A\}$

$Reach_\infty(A) = \{\omega \in \Omega \mid \exists t \geq 0 : x_t(\omega) \in A\}$.

The reachability problem consists of determining the probabilities of such sets. The probabilities of reach events are

$$\mathbb{P}(T_A < T) \text{ or } \mathbb{P}(T_A < \zeta) \quad (4)$$

where T_A is the first hitting time of A

$$T_A = \inf\{t > 0 | x_t \in A\}, \quad (5)$$

and $\mathbb{P}$ is a probability on the measurable space $(\Omega, \mathcal{F})$ of the elementary events associated to $\mathbf{M}$. $\mathbb{P}$ can be chosen to be $\mathbb{P}_x$ (if we want to consider the trajectories that start in x).

When defining the stochastic reachability in this way, it is clear that these definitions are related with the first passage/exit problems studied in the literature for Markov chains, diffusion processes, and jump processes. In the following, we summarize classical results for such problems available for Markov chains and diffusion processes.

First Passage Time Distributions for Markov chains: Let $\{X_n : n \geq 0\}$ be a discrete time Markov chain (DTMC) with state space $\mathbb{S} = \{0, 1, 2, ...\}$, the transition probability matrix P and the initial distribution p. Denote $T = \inf\{n \geq 0 | X_n = 0\}$. The random variable T is called the *first passage time* into state 0 (or the *first hitting time* of $\{0\}$). We can associate to T two probability distributions: $\alpha_i(n) = \mathbb{P}(T = n | X_0 = i)$; $u_i(n) = \mathbb{P}(T \leq n | X_0 = i)$. The following theorem provides a recursive method of computing $u_i(n)$.

Theorem 1: $u_i(n) = p_{i0} + \sum_{j=1}^\infty p_{ij} u_j(n-1)$, for all $i, n \geq 1$ with $u_i(0) = 0$ for all $i \geq 1$.

Consider a finite stationary time-homogeneous irreducible continuous time Markov chain (CTMC) with n states $\{1, 2, ..., n\}$ and $n \times n$ generator matrix $Q = (q_{ij})$. The chain is called irreducible if for any pair i, j of states we have that $p_{ij}(t) > 0$ for some t. If $X(t)$ denotes the state of the CTMC at time $t \geq 0$, then the first passage time from a source state i into a different state j is $T_j = \inf\{u > 0 : X(u) = j | X(0) = i\}$. Let S be the first time when the chain leaves the state i. Recall that S is exponentially distributed, i.e. $S \sim \exp(-q_{ii})$. Then, by the Markov property we have:

$\mathbb{E}(T_j | X(0) = i) = \mathbb{E}(S | X(0) = i) +$

$+ \sum_{k \neq i,j} \mathbb{E}(T_j | X(0) = k) \mathbb{P}(X(S) = k | X(0) = k)$.

The sum does not include i because the chain can not leave i to arrive also in i, and it does not include j because $\mathbb{E}(T_j | X(0) = j) = 0$.

We know that $\mathbb{E}(S | X(0) = i) = 1/(-q_{ii})$, and $\mathbb{P}(X(S) = k | X(0) = k) = \frac{q_{ik}}{(-q_{ii})}$, $k \neq i$. Then $\mathbb{E}(T_j | X(0) = i) = \{1 + \sum\limits_{k \neq i,j} q_{ik} \mathbb{E}(T_j | X(0) = k)\}(-q_{ii})^{-1}$. Using the notation $u_i := \mathbb{E}(T_j | X(0) = i)$,we obtain $u_i(-q_{ii}) = 1 + \sum_{k \neq i,j} q_{ik} u_k$, or $1 + \sum_{k \neq, j} q_{ik} u_k = 0$. Let us denote the matrix by $Q(j)$ obtained from Q by deleting the row and column corresponding to state j. Then for all possible starting values $i \neq j$, the equation above can be written in matrix form as

$$\mathbf{1} + Q(j)\mathbf{u} = \mathbf{0} \tag{6}$$

where $\mathbf{1} = (1, 1, ..., 1)^\intercal$ and $\mathbf{u} = (u_1, u_2, ..., u_n)^\intercal$. Then the solution for (6) is $\mathbf{u} = [-Q(j)]^{-1}\mathbf{1}$. A very nice introductory presentation for the first passage time for Markov chains can be found in [20]. We have inserted here only the "classical" methods available for DTMC and CTMC.

First Passage Time for Diffusions: The First Passage time (FPT) problem has more than a century history starting with Bachelier in 1900, who was examining the first passage of the Wiener process to the constant boundary. For general diffusion problems, such problems have been investigated for first time in the work of A. Khinchine [13], A.N. Kolmogorov [14] and I.G. Petrowsky [19]. Foundations of the general theory of Markov processes were set up by Kolmogorov [14], in 1931. His work clarified the deep connection between probability theory and mathematical analysis and initiated the partial differential equations approach to the FPT problem. For diffusion processes, the main tools for dealing with the first passage time problems are: partial differential equations (PDE), space and time change, measure change and the martingale approach via the optional sampling theorem. For the FPT problem of diffusions, the most popular approach is based on PDEs. The formulation in the PDE setting is done using the (Fokker-Planck) Kolmogorov forward equation.

Suppose that A is a measurable target set in the state space $\mathbf{X}$ of a diffusion process (x_t). Note that the first hitting time of A is equal with the first exit time from the complementary set of A, $E := A^c = \mathbf{X} \backslash A$. Then, the stochastic reachability problem can be formulated also as an exit problem from E. In case of a continuous diffusion with the infinitesimal generator L, for a closed target set A, it is known that if the Dirichlet problem: $\frac{\partial u}{\partial t} = Lu$ on $E \times (0, T]$, with the boundary conditions $u = 0$ on $E \times \{0\}$ and $u = \mathbf{1}_{\partial E}$ on $\partial E \times (0, T]$, has a bounded solution, then $u(x, t) = \mathbb{P}_x\{T_A \leq t, x_{T_A} \in \partial E\}, 0 \leq t \leq T$. If we consider the infinite horizon time reachability, the function $u(x) = \mathbb{P}_x\{T_A < \infty, x_{T_A} \in \partial E\}$ is solution for the following boundary value problem: $Lu = 0$, on E; with the boundary condition $u = 1$ on ∂E. In particular, for the Brownian motion, the solutions of this problem are known and they are connected with the Newtonian potential and Newtonian capacity. If $\mathbb{P}_x$ denote the probability when all paths issued from the point x for the standard Wiener process in $\mathbb{R}^3$; Λ is a compact set; $T_\Lambda(\omega)$ is the first hitting time of Λ by the path ω then, a classical result says that

$$\mathbb{P}_x(T_\Lambda < \infty) = \int_{\partial\Lambda} g(x, \beta)\mu_\Lambda(d\beta) \tag{7}$$

where $\partial\Lambda$ is the boundary of Λ; $g(x, y)$ is the associated *Green kernel* $g(x, y) = \frac{1}{2\pi||x-y||}$;and μ_Λ is called *equilibrium measure*. On the other hand, the probability that a Brownian motion will ever visit a given set Λ, which appears in the left hand side of (7) can be estimated using the capacity of Λ w.r.t. the Green kernel $g(x, y)$. The right hand side of (7) is known as Newtonian potential of Λ, i.e. $U_\Lambda(x) := \int_{\partial\Lambda} g(x, \beta)\mu_\Lambda(d\beta)$, and its Newtonian capacity is $cap\Lambda = \mu_\Lambda(\Lambda) = \int U_\Lambda(x) d\mu_\Lambda(x)$. Similar results can be stated for the Brownian motion in $\mathbb{R}^d$, $d > 3$. This incursion shows us that the reach probabilities are intimately related with concepts like Green kernel, and Newtonian capacity. For an arbitrary Markov process, the expression of the Green kernel is not always available, but this kernel is closely related with the infinitesimal generator of the process.

For a Markov process, the reach probabilities can be expressed using the concept of hitting operator. For a target set A, we denote by P_A the *hitting operator* defined for the underlying Markov process (x_t), i.e. $P_A v = \mathbb{E}_x\{v \circ x_{T_A} | T_A < \zeta\}$ and T_A is given by (5).

Proposition 2: [8] For any $x \in X$ and Borel set $A \in \mathcal{B}(\mathbf{X})$, we have $\mathbb{P}_x[Reach_\infty(A)] = P_A 1(x)$.

The stochastic hybrid processes may be viewed as piecewise continuous jump diffusions, where the jumps are allowed to be spontaneous, or forced (predictable). For continuous pure diffusions processes, it is sufficient to consider the time when the process hits the boundary of E or A. However, when the stochastic processes also includes jumps, then it is possible that the process overshoots the boundary and ends up in the exterior of the domain E (i.e. in the interior of A). Therefore, the Dirichlet problem becomes

$$\begin{cases} \mathcal{L}u = 0 & \text{on } E \\ u = 1 & \text{on } A = \mathbf{X} \backslash E; \end{cases} \tag{8}$$

where $\mathcal{L}$ is the infinitesimal generator of the hybrid process. Note that (8) is a Dirichlet problem corresponding to an integro-differential operator. Then solving such an equation would be a difficult task to accomplish, but numerical and analytical methods are under development [2].

Our goal in this paper is to investigate deeper the connections between reachability measures and capacities. We will consider not only Newtonian type capacities, but also capacities defined using some "pay-off" kernels.

IV. Reachability Estimation via Martin Capacities

In this section, we estimate reach set probabilities by a capacity function with respect to a scale-invariant modification of the kernel operator. We start by defining the Martin capacity, which is an imprecise probability measure defined with respect to an energy form. Then we present some classical results regarding the estimation of the hitting probabilities for Markov chains and Brownian motion based on the Martin capacity. This capacity is defined with respect an energy form derived from the kernel operator. Finally, we show how these results can be extended to more general Markov processes, like stochastic hybrid processes.

Martin Capacity: Let Λ be a set and $\mathcal{B}$ a σ-algebra of Λ. Given a measurable function $F : \Lambda \times \Lambda \to [0, \infty]$, and a finite measure μ on $(\Lambda, \mathcal{B})$, the *F-energy* of μ is $F(\mu) := F(\mu, \mu) = \int_\Lambda \int_\Lambda F(\alpha, \beta) d\mu(\alpha) d\mu(\beta)$. $F(\mu)$ can be viewed as a 'value function' when we apply the strategy μ. The Martin *capacity w.r.t.* F is

$$Cap_F(\Lambda) := [\inf F(\mu)]^{-1} \tag{9}$$

where the infimum is over the probability measures μ on $(\Lambda, \mathcal{B})$ and by the convention, $\infty^{-1} = 0$. Then the capacity $Cap_F(\Lambda)$ can be interpreted as the inverse of an optimal value function. If Λ is included in an Euclidean space, then $\mathcal{B}$ can be taken as the Borel σ-algebra. If Λ is countable, then $\mathcal{B}$ will be the σ-algebra of all its subset.

The Case of a Markov Chain: Let (X_n) be a DTMC with the state space $\mathbf{S} = \{0, 1, 2, 3, ...\}$ and $P = (p_{ij})_{i,j\in\mathbf{S}}$ its *one-step transition matrix*. Then its kernel operator (or Green operator) is $U(i,j) = \sum_{n=0}^{\infty} p_{ij}^{(n)} = \sum_{n=0}^{\infty} \mathbb{P}_i[X_n = j]$, where $p_{ij}^{(n)}$ are the n-step transition probabilities and $\mathbb{P}_i$ is the law of the chain when the initial state is i.

We want to estimate the reach probability of a target set Λ for the given DTMC. Assume that the Markov chain is transient, i.e. $U(i,j) < \infty$, for $i, j \in \mathbf{S}$.

Proposition 3: [3] Let (X_n) be a DTMC with the state space $\mathbf{S}$ with the initial state i_0 and the transition probabilities (p_{ij}). For any subset Λ of $\mathbf{S}$, we have $\frac{1}{2}Cap_K(\Lambda) \leq \mathbb{P}_{i_0}[Reach_\infty(\Lambda)] \leq Cap_K(\Lambda)$, where K is the Martin kernel (w.r.t. the initial state i_0) defined by $K(i,j) := \frac{U(i,j)}{U(i_0,j)}$.

The Case of the Brownian Motion: The results from the previous subsection can be easily extended to the case of the Wiener process (Brownian motion) (W_t) defined on the Euclidean space $\mathbb{R}^d$, $d \geq 3$. In this case, the Green operator is given as $U(x,y) = ||x-y||^{2-d}$; and the Martin kernel is $K(x,y) := \frac{||y||^{d-2}}{||x-y||^{d-2}}$, for $x \neq y$, and $K(x;x) = \infty$.

In [3], it is shown that replacing the Green kernel by the Martin kernel $U(x,y)/U(0,y)$ yields improved estimates, which are exact up a factor of 2.

Proposition 4: [3] Let Λ be any closed set in $\mathbb{R}^d$, $d \geq 3$. Then the reach set probability corresponding to (W_t) and Λ can be estimated as follows: $\frac{1}{2}Cap_K(\Lambda) \leq \mathbb{P}_0[Reach_\infty(\Lambda)] \leq Cap_K(\Lambda)$; where $\mathbb{P}_0$ is the law of the Brownian motion under $W_0 = 0$, and K is the associated Martin kernel. Here, the constants $1/2$ and 1 are sharp.

The Case of a Markov Process: Kai Lai Chung in [10] extended the formula (7) for temporally homogeneous transient Markov processes $\{x_t, t \geq 0\}$, taking values in a topological space $\mathbf{X}$, which is locally compact and has a countable base with its Borel σ-algebra $\mathcal{B}(\mathbf{X})$. The processes have also the càdlàg property. It is natural to put the problem of generalization of above Brownian motion result to more general Markov processes, using the Kai Lai Chung's result.

Throughout this section $\mathbf{M} = (x_t, \mathbb{P}_x)$ will be a Borel right Markov process on $(\mathbf{X}, \mathcal{B})$. In addition, we suppose that $\mathbf{M}$ has the càdlàg property and that $\mathbf{M}$ is *transient*. Let $p_t(x, B)$, $t > 0$, $x \in \mathbf{X}$, $B \in \mathcal{B}(\mathbf{X})$ be the transition function associated to the given Markov process. All the measures $p_t(x, \cdot)$ are supposed to be *absolutely continuous* w.r.t. a σ-finite measure μ on $(\mathbf{X}, \mathcal{B}(\mathbf{X}))$. We denote the Radon-Nycodim derivative of $p_t(x,\cdot)$ by $\rho_t(x,\cdot)$, i.e. $\rho_t(x,y) := p_t(x,dy)/\mu(dy)$. This can be chosen to be measurable in x, y and to satisfy $\int_{\mathbf{X}} \rho_s(x,y)\mu(dy)\rho_t(y,z) = \rho_{t+s}(x,z)$. A σ-finite measure μ on $(\mathbf{X}, \mathcal{B}(\mathbf{X}))$ is called *reference measure* if $\mu(B) = 0 \Leftrightarrow p_t(x, B) = 0$ for all t and x. Throughout this section we suppose that μ, in the absolutely continuity assumption, is a reference measure. We define the *Green kernel* as $u(x,y) := \int_0^\infty \rho_t(x,y)dt$ and the Martin kernel (w.r.t. to an initial state x_0)

$$K(x,y) = \frac{u(x,y)}{u(x_0,y)}. \tag{10}$$

It is clear that, if $(\mathbf{P}_t)$ is the transition semigroup and the kernel operator associated to the given Markov process defined in the standard way can expressed using the derivatives $\rho_t(x,y)$.

Assumption 1: i) $y \to u(x,y)^{-1}$ is finite continuous, for $y \in \mathbf{X}$; ii) $u(x,y) = +\infty$ if and only if $x = y$.

For a target set A we define a random variable $\gamma_A < \infty$ ($\mathbf{M}$ is transient), called the *last exit time* from A as follows: $\gamma_A(\omega) = \sup\{t > 0 | x_t(\omega) \in A\}$ if $\omega \in Reach_\infty(A)$, and 0 otherwise. Then, it follows that $x_{\gamma_A -} \in \overline{A}$ almost sure. The *distribution of the last exit position* $x_{\gamma_A -}$ is given by $L^A(x, B) = \mathbb{P}_x(\gamma_A > 0; x_{\gamma_A -} \in B)$, for all $x \in \mathbf{X}$, $B \in \mathcal{B}(\mathbf{X})$. Under these assumptions, there exists an equilibrium measure μ_A, which is $\sigma-$finite, concentrated in $\overline{A}$ such that $\mu_A(dy) := L^A(x,dy)u(x,y)^{-1}$ for all $x \in \mathbf{X}$. For a transient set A we have $\{0 \leq T_A < \infty\} = \{0 < \gamma_A < \infty\}$. The final result is that $\mathbb{P}_x(x_{\gamma_A -} \in B) = L^A(x,B) = \int_B u(x,y)\mu_A(d\beta)$, for each Borel set $A \subset \overline{E}$, and each $x \in \mathbf{X}$. In particular, $\mathbb{P}_x(T_A < \infty) = L^A(x, \overline{A}) = \int_{\overline{A}} u(x,y)\mu_A(dy)$.

Theorem 5: Let $x_0 \in \mathbf{X}$ be the initial state. For any closed set A of $\mathbf{X}$ we have

$$\mathbb{P}_{x_0}(T_A < \infty) \leq Cap_K(A) \tag{11}$$

where Cap_K is the capacity defined, using (9), w.r.t. the Martin kernel K defined by (10).

Proof: To bound from above the probability of ever hitting A, consider the distribution

$$\nu_x(\Lambda) = L^A(x,\Lambda) = \mathbb{P}_x(0 < \gamma_A | x_{\gamma_A -} \in \Lambda);\ \Lambda \in \mathcal{B}(\mathbf{X}).$$

The Kai Lai Chung's result says that

$$L^A(x,\Lambda) = \int_\Lambda u(x,y)\mu_A(dy);\ \Lambda \in \mathcal{B}(\mathbf{X})$$

where μ_A is the equilibrium measure of A, given by

$$\forall x \in \mathbf{X} : \mu_A(dy) = L^A(x,dy)u(x,y)^{-1} = \nu_x(dy)u(x,y)^{-1}$$

in particular, for the initial state $x_0 \in \mathbf{X}$

$$\mu_A(dy) = L^A(x_0,dy)u(x_0,y)^{-1} = \nu_{x_0}(dy)u(x_0,y)^{-1}.$$

It follows that

$\int_A K(x,y)\nu_{x_0}(dy) = \int_A K(x,y)u(x_0,y)\mu_A(dy) =$
$= \mathbb{P}_x(T_A < \infty)$.

Therefore $K(\nu_{x_0}, \nu_{x_0}) \leq \nu_{x_0}(A)$ and thus

$$Cap_K(A) \geq [K(\nu_{x_0}/\nu_{x_0}(A))]^{-1} \geq \nu_{x_0}(A)$$

that yields the upper bound on the probability of hitting A. ∎

It is desired to obtain also a lower bound for the reach probabilities, as for Markov chains and Brownian motion, but the proof for these simple cases can not be modified directly to accommodate the general case of Markov processes.

Computational Issues: Suppose we have given an SHS H with the realization $\mathbf{M}$. For applying the results of the previous section, the main computational step that has to be accomplished is the computation of the kernel operator V. Using the connections between infinitesimal generator, operator semigroup/resolvent, we can state the following result.

Proposition 6: Let $\varphi : \mathbf{X} \rightarrow \mathbb{R}_+$ be a measurable function. Suppose for each $x \in \mathbf{X}$ the function φ is integrable on some interval $[0, \varepsilon]$, $\varepsilon > 0$ over the hybrid trajectories, and that the kernel operator $V\varphi$ is bounded and satisfies the following Poisson equation w.r.t. the infinitesimal generator $\mathcal{L}$

$$\begin{cases} \mathcal{L}V\varphi(x) + \varphi(x) = 0, & x \in \mathbf{X} \\ \int_{\mathbf{X}} V\varphi(y)R(x, dy) = V\varphi(x), & x \in \partial\mathbf{X}. \end{cases}$$

The Proposition 6 provides a quite insightful characterization for the kernel operator V. V is solution for a Poisson-type equation associated to $\mathcal{L}$. Due to the complexity of $\mathcal{L}$, this can lead to the use of viscosity solutions for such an equation. Then to simplify the approach, we may have to use Markov chain approximations/abstractions for the given stochastic hybrid process. These will provide us good lower/upper bounds (remember that for Markov chains we have also lower bounds) for the reach probabilities. However, all the probabilistic methods are, in the end, grounded on approximations. Then, it is difficult to evaluate if one is better than another one. Therefore, here, we consider another perspective on the problem. The kernel operator is, in fact, the expectation of a cost function over the trajectories. If $\varphi := 1_{dy}$, we can define $V(x, dy) := \int_0^\infty \mathbb{P}_x(x_t \in dy)dt$ that represents the mean of the *occupancy time of* dy. Then the Radon-Nycodim derivative w.r.t. μ provides the Green kernel $u(x, y)$. This kernel u can be thought of as a reward for the case when the process starts in x and arrives in y. Suppose that we have an 'approximation' of u by another payoff kernel $\pi : \mathbf{X} \times \mathbf{X} \rightarrow \mathbb{R}_+$ that is given a priori. Using this payoff kernel, one can construct both (Newtonian and Martin) capacities and derive the upper/lower bounds of the reach probabilities. Moreover, we can consider a controlled SHS, and take $\pi : \mathbf{X} \times \mathbf{U} \rightarrow \mathbb{R}_+$, where $\mathbf{U}$ is the control space. Then, the energy form has to be defined as $\sup\inf$, and the corresponding capacity would be the inverse of this energy. This development will constitute the subject for a follow-up paper.

V. CONCLUSIONS

This paper deals with stochastic reachability, only from the perspective of stochastic analysis developed for Markov processes using the operators derived from the transition probabilities. In this framework, upper/lower bounds for the reach probabilities can be obtained using some energy distribution measures called capacities. For Markov chains, and standard Wiener processes the results are at hand and quite intuitive. For general processes, like stochastic hybrid processes, the above capacities should correspond to some reward/cost functions. Then, the stochastic analysis tools need to be augmented with optimal control methods. Reachability methods based on other stochastic analysis tools have been developed elsewhere: e.g. martingales methods [5], [8]; Dirichlet forms methods [6]; approximation of the reduite function [9]. All of these methods (inclusive the one developed in this paper) provide only upper/lower bounds for the reach probabilities with a certain degree of accuracy. It is remarkable that such methods have quite intuitive standard solutions for simpler processes (like the Markov chains and Brownian motion), but these solutions can not be easily extended for complex processes. Then, the natural conclusion is that, in order to deal with the stochastic reachability problem for SHS, we need to enrich the stochastic analysis tools in a cross-fertilization manner with complementary tools from stochastic control (like dynamic programming techniques), and from numerical analysis (like model order reduction techniques).

REFERENCES

[1] Abate, A., Amin, A., Prandini, M., Lygeros, J., Sastry, S.: *Probabilistic Reachability for Discrete Time Stochastic Hybrid Systems*. Proc. IEEE CDC (2006): 258-263.

[2] Barles, C., Chasseigne, E., Imbert, C.: *On the Dirichlet Problem for Second-Order Elliptic Integro-Differential Equations*. Preprint (2007).

[3] Benjamini, I., Pemantle, R., Peres, Y:. *Martin Capacity for Markov Chains*. Ann. Probab., **23**, (1995): 1332-1346.

[4] Blom, H.A.P., Lygeros, J. (Eds.): "*Stochastic Hybrid Systems: Theory and Safety Critical Applications*". LNCIS **337** (2006).

[5] Bujorianu, M.L., Lygeros, J.: *Reachability Questions in Piecewise Deterministic Markov Processes*. Springer-Verlag, LNCS **2623** (2003): 126-140.

[6] Bujorianu, M.L.: *Extended Stochastic Hybrid Systems and their Reachability Problem*. Springer-Verlag, LNCS **2993** (2004): 234-249.

[7] Bujorianu, M.L., Lygeros, J.: *Towards Modelling of General Stochastic Hybrid Systems*. In [4]: 3-30.

[8] Bujorianu, M.L., Lygeros, J.: *New Insights on Stochastic Reachability*. Proc. 46th Conference in Decision and Control (2007).

[9] Bujorianu, M.L.: *Stochastic Reachability: From Markov Chains to Stochastic Hybrid Systems*. IFAC World Congress, (2011).

[10] Chung, K.L.: *Probabilistic Approach in Potential Theory to the Equilibrium Problem*. Annales de L'Institut Fourier, Grenoble, **23**, (1973): 313-322.

[11] Davis, M.H.A.: "*Markov Models and Optimization*" Chapman & Hall, (1993).

[12] Johnson, D.P.: *Diffusion approximations for optimal filtering of jump processes and for queueing networks*. Thesis (Ph. D.) (1983).

[13] Khinchine, A. Y.: *Asymptotische Gesetze der Wahrscheinlichkeitsrechnung. Ergebnisse*. d. Math, **77** H. 4, Berlin, Springer (1933).

[14] Kolmogoroff, A.: *Uber die analytischen Methoden in der Wahrscheinlichkeitsrechnung*. Math. Anal. **104**, 415-458 (1931).

[15] Koutsoukos, X.; Riley, D.: *Computational Methods for Verification of Stochastic Hybrid Systems*. IEEE Trans. on Systems, Man and Cybernetics. Part A. **38**(2), 385-396 (2008).

[16] Koutsoukos, X.: *Optimal Control of Stochastic Hybrid Systems Based on Locally Consistent Markov Decision Processes*. IEEE International Symposium on Intelligent Control ISIC (2005)

[17] Krystul, J., Bagchi, A.: *Approximation of first passage times of switching diffusion*. MTNS (2004).

[18] Lygeros, J.: *On reachability and minimum cost optimal control*. Automatica **40** (6) (2004).

[19] Petrowsky, I.: *Uber das Irrfahrtproblem*. Math. Ann., Bd. **109**, (1934): 425-444.

[20] Syski, R.: "*Passage Times for Markov Chains*". Studies in Probability, Optimization, and Statistics, Vol. 1 (1991): 556p.