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Inherently robust suboptimal nonlinear MPC: theory and application

Gabriele Pannocchia and James B. Rawlings and Stephen J. Wright

Abstract— **We discuss inherent robust stability properties of discrete-time nonlinear systems controlled by Model Predictive Control (MPC) algorithms that do not necessarily attain the global minimum of the optimization problem solved at each sample time. For these implementable suboptimal MPC algorithms, we prove nominal exponential stability of the origin of the closed-loop system. The stability property is robust with respect to (sufficiently small but otherwise arbitrary) process disturbances and state measurement/estimation errors. When (hard) state constraints appear in the control problem, our result requires a (local) continuity assumption of the feasible input space. If (hard) state constraints are not present, robustness of stability can be proved under standard assumptions. We show an example to illustrate the main ideas behind these results.**

I. INTRODUCTION

Nominal stability properties of MPC for discrete-time systems are well understood both for linear and nonlinear systems; in most cases nominal asymptotic or exponential stability can be established [1], [2]. These results tend to assume *exact* solution of the optimal control problem at each decision point. However, exact global solutions may not be attainable in practice, especially when dealing with nonlinear systems, which typically give rise to nonconvex problems. If a suboptimal solution is implemented, stability may not hold, or may be difficult to establish. In [3], however, it was shown that if the optimization yields a feasible, suboptimal solution whose cost is no worse than that of a well chosen warm start (and if some other technical assumptions hold), asymptotic stability of the equilibrium can be proved.

When considering systems subject to unknown disturbances and state measurement/estimation errors, so-called *robust* MPC formulations are usually proposed, in which a control law is required to satisfy the constraints for *all allowed* values of the unknown disturbances (see e.g. [4]–[7], [2, Ch. 3] and references therein). A major challenge in robust MPC design is in the handling of hard state constraints. To maintain feasibility of state constraints under disturbances, the authors in [8] propose modifying the nominal MPC problem by altering the state constraints to become progressively tighter with time. A different approach to the issue of robustness is to study stability properties of perturbed systems controlled by MPC algorithms that ignore such disturbances. Since most industrial (linear and nonlinear) MPC algorithms fall within this class, it is surprising that this approach has received much less attention in the literature [9], [10]. This observation is made by Grimm et al. [11], who present examples of nonlinear systems controlled by MPC in which the asymptotic stability of the equilibrium is destroyed by arbitrarily small perturbations. In a subsequent paper [12], these authors present sufficient conditions for robust stability of an MPC algorithm in which feasibility is maintained by means of time-varying tightening of the state constraints. Further results on robustness of discontinuous discrete-time systems and Lyapunov functions are discussed in [13]. The paper [11] also shows that for *linear* systems with a quadratic cost, the optimal MPC cost function is a continuous Lyapunov function for the closed-loop system (because the optimal state-feedback law is continuous), leading to inherent robust stability of the equilibrium. On the other hand, a suboptimal MPC law is not necessarily continuous, even for linear systems, and hence inherent robustness cannot be established even in such a simple case. Lazar and Heemels [14], in a significant paper, were the first to address robustness of suboptimal MPC explicitly. Their results require a *specified* degree of suboptimality to be satisfied, and employs the technique of time-varying tightening of state constraints (as in [8], [12]) to achieve recursive feasibility under disturbances.

Due to space limitations all proofs of the results of this paper are reported in [15].

Notation: The symbols $\mathbb{I}_{\geq 0}$ and $\mathbb{R}_{\geq 0}$ denote the sets of nonnegative integers and reals, respectively. The symbol $\mathbb{I}_{0:N-1}$ denotes the set $\{0, 1, \ldots, N-1\}$. The symbol $|\cdot|$ denotes the Euclidean norm and $\mathbb{B}$ denotes the closed ball of radius 1 centered at the origin. We denote the interior of a set X as $\text{int}(X)$. Given a nonnegative function $V : X \to \mathbb{R}_{\geq 0}$ and a positive scalar α, we define $\text{lev}_\alpha V = \{x \in X \mid V(x) \leq \alpha\}$.

II. SUBOPTIMAL NONLINEAR MPC

A. Basic definitions and assumptions

We consider discrete-time systems subject to state and input constraints in the following form:

$$x^+ = f(x, u), \qquad x \in \mathbb{X},\ u \in \mathbb{U} \tag{1}$$

in which $x \in \mathbb{R}^n$, $x^+ \in \mathbb{R}^n$ are the state at a given time and the successor state, respectively, while $u \in \mathbb{R}^m$ is the control input. Given an integer N, and an input sequence $\mathbf{u} \in \mathbb{U}^N$, $\mathbf{u} = \{u(0), u(1), \ldots, u(N-1)\}$, we denote with $\phi(k; x, \mathbf{u})$ the solution of (1) at time k for a given initial state $x(0) = x$. For any state $x \in \mathbb{R}^n$ and input sequence $\mathbf{u} \in \mathbb{U}^N$, we define a cost function over the finite horizon N:

$$V_N(x, \mathbf{u}) = \sum_{k=0}^{N-1} \ell(\phi(k; x, \mathbf{u}), u(k)) + V_f(\phi(N; x, \mathbf{u}))$$

G. Pannocchia is with Dept. of Chem. Eng. (DICCISM), Univ. of Pisa, 56122 Pisa, Italy (g.pannocchia@diccism.unipi.it)

J.B. Rawlings is with Dept. of Chemical and Biological Engineering, Univ. of Wisconsin, Madison, WI 53706, USA (rawlings@engr.wisc.edu)

S.J. Wright is with Computer Sciences Dept., Univ. of Wisconsin, Madison, WI 53706, USA (swright@cs.wisc.edu)

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A pair $(x,\mathbf{u})$ is feasible if it belongs to the following set:

$$\mathbb{Z}_N = \{(x,\mathbf{u}) \mid u(k) \in \mathbb{U},\, \phi(k;x,\mathbf{u}) \in \mathbb{X} \text{ for all } k \in \mathbb{I}_{0:N-1}, \text{ and } \phi(N;x,\mathbf{u}) \in \mathbb{X}_f\}$$

in which $\mathbb{X}_f$ is the terminal constraint set. Consequently, the set of feasible states is:

$$\mathscr{X}_N = \{x \in \mathbb{R}^n \mid \exists \mathbf{u} \in \mathbb{U}^N \text{ such that } (x,\mathbf{u}) \in \mathbb{Z}_N\} \quad (2)$$

and for each $x \in \mathscr{X}_N$, the set of feasible input sequences is:

$$\mathscr{U}_N(x) = \{\mathbf{u} \mid (x,\mathbf{u}) \in \mathbb{Z}_N\}$$

Finally, for each $x \in \mathscr{X}_N$ we consider:

$$\mathbb{P}_N(x): \qquad \min_{\mathbf{u}} V_N(x,\mathbf{u}) \qquad \text{s.t. } \mathbf{u} \in \mathscr{U}_N(x)$$

We make the following standing assumptions.

Assumption 1: The functions $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, $\ell : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_{\geq 0}$ and $V_f : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ are continuous, $f(0,0) = 0$, $\ell(0,0) = 0$, and $V_f(0) = 0$.

Assumption 2: The set $\mathbb{U}$ is compact and contains the origin. The sets $\mathbb{X}$ and $\mathbb{X}_f$ are closed and contain the origin in their interiors, and $\mathbb{X}_f \subseteq \mathbb{X}$.

Assumption 3: For any $x \in \mathbb{X}_f$ there exists $u \in \mathbb{U}$ such that $f(x,u) \in \mathbb{X}_f$ and $V_f(f(x,u)) + \ell(x,u) \leq V_f(x)$.

Assumption 4: There exist positive constants a, a_1', a_2', a_f and $\bar{r}$, such that the cost function satisfies the inequalities

$$\ell(x,u) \geq a_1'|(x,u)|^a \qquad \text{for all } (x,u) \in \mathbb{X} \times \mathbb{U}$$
$$V_N(x,\mathbf{u}) \leq a_2'|(x,\mathbf{u})|^a \qquad \text{for all } (x,\mathbf{u}) \in \bar{r}\mathbb{B}$$
$$V_f(x) \leq a_f|x|^a \qquad \text{for all } x \in \mathbb{X}$$

B. Suboptimal solutions

In general $\mathbb{P}_N$ is a nonconvex optimization problem, and there is no guarantee that numerical solvers can actually achieve the global minimum, even within a pre-specified tolerance margin. Thus, instead of solving $\mathbb{P}_N$ exactly, we consider using any suboptimal algorithm having the following properties. Let $\mathbf{u} \in \mathscr{U}_N(x)$ denote the (suboptimal) control sequence for the initial state x, and let $\tilde{\mathbf{u}}$ denote a *warm start* for the successor initial state $x^+ = f(x,u(0;x))$, obtained from $(x,\mathbf{u})$ by setting

$$\tilde{\mathbf{u}} = \{u(1;x), u(2;x), \ldots, u(N-1;x), u_+\} \quad (3)$$

in which $u_+ \in \mathbb{U}$ is any input that satisfies the invariance conditions of Assumption 3 for $x = \phi(N;x,\mathbf{u}) \in \mathbb{X}_f$. We observe that the warm start is feasible for the successor state, i.e., $\tilde{\mathbf{u}} \in \mathscr{U}_N(x^+)$. Then, the suboptimal solution for the successor state is defined as any input sequence $\mathbf{u}^+ \in \mathscr{U}_N(x^+)$ that satisfies:

$$\mathbf{u}^+ \in \mathscr{U}_N(x^+) \quad (4a)$$
$$V_N(x^+,\mathbf{u}^+) \leq V_N(x^+,\tilde{\mathbf{u}}) \quad (4b)$$
$$V_N(x^+,\mathbf{u}^+) \leq V_f(x^+) \qquad \text{when } x^+ \in r\mathbb{B} \quad (4c)$$

in which r is a positive scalar sufficiently small that $r\mathbb{B} \subseteq \mathbb{X}_f$. We remark that condition (4b) ensures that the computed suboptimal cost is no larger than that of the warm start. In general, *all* numerical solvers can guarantee this kind of bound without requiring convergence to an optimal solution point. For instance, feasible sequential quadratic programming (fSQP) algorithms can be terminated at any finite number of iterations while respecting this bound.

Proposition 5: Any *optimal* solution $\mathbf{u}^0(x^+)$ to $\mathbb{P}_N(x^+)$, satisfies conditions (4a), (4b) for all $x^+ \in \mathscr{X}_N$. Moreover, condition (4c) is satisfied by $\mathbf{u}^0(x^+)$ for all $x^+ \in \mathbb{X}_f$.

Corollary 6: For any $x^+ \in \mathscr{X}_N$, there exists a $\mathbf{u}^+ \in \mathscr{U}_N(x^+)$ satisfying all conditions (4) for all $\tilde{\mathbf{u}} \in \mathscr{U}_N(x^+)$.

It is important to notice that $\mathbf{u}$ is a set-valued map of the state x, and so too is the associated first component $u(0;x)$. If we denote the latter map as $\kappa_N(\cdot)$, we can write the evolution of the system (1) in closed-loop with suboptimal MPC as the following difference inclusion:

$$x^+ \in F(x) = \{f(x,u) \mid u \in \kappa_N(x)\} \quad (5)$$

Proposition 7: We have that $\kappa_N(0) = \{0\}$ and $F(0) = \{0\}$.

III. NOMINAL STABILITY

A. Supporting results for difference inclusions

Definition 8 (Exponential stability): The origin of the difference inclusion $z^+ \in H(z)$ is *exponentially stable* (ES) on $\mathscr{Z}$, $0 \in \mathscr{Z}$, if there exist scalars $b > 0$ and $0 < \lambda < 1$, such that for any $z \in \mathscr{Z}$, *all* solutions $\psi(k;z)$ satisfy:

$$\psi(k;z) \in \mathscr{Z}, \quad |\psi(k;z)| \leq b\lambda^k|z| \qquad \text{for all } k \in \mathbb{I}_{\geq 0}.$$

Definition 9 (Exponential Lyapunov function): V is an *exponential Lyapunov* function on the set $\mathscr{Z}$ for the difference inclusion $z^+ \in H(z)$ if there exist positive scalars a, a_1, a_2, a_3 such that the following holds for all $z \in \mathscr{Z}$:

$$a_1|z|^a \leq V(z) \leq a_2|z|^a, \qquad \max_{z^+ \in H(z)} V(z^+) \leq V(z) - a_3|z|^a.$$

We have the following results.

Proposition 10: If V is an *exponential Lyapunov* function on the set $\mathscr{Z}$ for the difference inclusion $z^+ \in H(z)$, there exists $0 < \gamma < 1$ such that:

$$\max_{z^+ \in H(z)} V(z^+) \leq \gamma V(z).$$

Lemma 11: If the set $\mathscr{Z}$, $0 \in \mathscr{Z}$, is positively invariant for the difference inclusion $z^+ \in H(z)$, $H(0) = \{0\}$, and there exists an exponential Lyapunov function V on $\mathscr{Z}$, the origin is ES on $\mathscr{Z}$.

B. Main results

We define an *extended* state $z = (x,\mathbf{u})$ and observe that it evolves according to the difference inclusion

$$z^+ \in H(z) = \{(x^+,\mathbf{u}^+) \mid x^+ = f(x,u(0;x)),\, \mathbf{u}^+ \in G(z)\} \quad (6)$$

in which (noting that both x^+ and $\tilde{\mathbf{u}}$ depend on z):

$$G(z) = \{\mathbf{u}^+ \mid \mathbf{u}^+ \in \mathscr{U}_N(x^+), V_N(x^+,\mathbf{u}^+) \leq V_N(x^+,\tilde{\mathbf{u}}), \text{ and } V_N(x^+,\mathbf{u}^+) \leq V_f(x^+) \text{ if } x^+ \in r\mathbb{B}\}.$$

We also define the following set (notice that $r\mathbb{B} \subseteq \mathbb{X}_f$):

$$\mathscr{Z}_r = \{(x,\mathbf{u}) \in \mathbb{Z}_N \mid V_N(x,\mathbf{u}) \leq V_f(x) \text{ if } x \in r\mathbb{B}\}.$$

Lemma 12: There exists a positive constant c such that $|\mathbf{u}| \leq c|x|$ for any $(x,\mathbf{u}) \in \mathscr{Z}_r$.

Lemma 13: $V_N(z)$ is an exponential Lyapunov function for the extended closed-loop system (6) in any compact subset of $\mathscr{Z}_r$.

Theorem 14: Under Assumptions 1, 2, 3, and 4, the origin of the closed-loop system (5) is ES on (arbitrarily large) compact subsets of $\mathscr{X}_N$.

Corollary 15: Under Assumptions 1, 2, 3, and 4, if $\mathscr{X}_N$ is compact, the origin of (5) is ES on $\mathscr{X}_N$.

IV. INHERENT ROBUSTNESS

A. Disturbances and robust stability definitions

For inherent robustness analysis, we consider the closed-loop evolution of the *perturbed* system

$$x^+ \in F_{ed}(x) = \{f(x,u)+d \mid u \in \kappa_N(x+e)\} \tag{7}$$

in which $d \in \mathbb{R}^n$ is an *unknown* process disturbance and $e \in \mathbb{R}^n$ represents an *unknown* state measurement/estimate error. It is important to remark that in the perturbed case, the control sequence $\mathbf{u}$ is computed as a suboptimal solution of $\mathbb{P}_N(x_m)$, with $x_m = x+e$, i.e., it is based on the evolution of the nominal system (1), for the initial measured state. We denote by $\phi_{ed}(k;x) = x(k)$ a solution to the perturbed closed-loop system (7) for the initial state $x(0) = x$ and given disturbance and measurement error sequences $\{d(k)\}$, $\{e(k)\}$. We now present the definition of *robust exponential stability* (RES), which resembles that of *robust asymptotic stability* (RAS) given in [11].

Definition 16 (RES): The origin of the closed-loop system (7) is *robustly exponentially stable* (RES) on $\text{int}(\mathscr{X}_N)$ if there exist scalars $b>0$ and $0<\lambda<1$ such that for all compact sets $\mathscr{C} \subset \mathscr{X}_N$, with $0 \in \text{int}(\mathscr{C})$, the following property holds: Given any $\varepsilon>0$, there exists $\delta>0$ such that for all sequences $\{d(k)\}$ and $\{e(k)\}$ with $x(0) = x \in \mathscr{C}$ satisfying

$$\max_{k\geq 0} |d(k)| \leq \delta,\ \max_{k\geq 0} |e(k)| \leq \delta,$$
$$x_m(k) = x(k)+e(k) \in \mathscr{X}_N,\ x(k) \in \mathscr{X}_N, \forall k \in \mathbb{I}_{\geq 0},$$

it follows that

$$|\phi_{ed}(k;x)| \leq b\lambda^k|x| + \varepsilon, \text{ for all } k \in \mathbb{I}_{\geq 0}. \tag{9}$$

We remark that in RES (or RAS given in [11]), the robust stability condition (9) is presented for those (if any) initial states, disturbance and measurement error sequences that a-priori ensure feasibility of the perturbed closed-loop trajectories. The next definition instead requires that feasibility is satisfied at all times, for *all sufficiently small* disturbance and measurement error sequences and all initial states in a given compact subset of $\text{int}(\mathscr{X}_N)$.

Definition 17 (SRES): The origin of the closed-loop system (7) is *strongly robustly exponentially stable* (SRES) on a compact set $\mathscr{C} \subset \mathscr{X}_N$, $0 \in \text{int}(\mathscr{C})$, if there exist scalars $b>0$ and $0<\lambda<1$ such that the following property holds: Given any $\varepsilon>0$, there exists $\delta>0$ such that for all sequences $\{d(k)\}$ and $\{e(k)\}$ satisfying

$$|d(k)| \leq \delta \text{ and } |e(k)| \leq \delta \quad \text{for all } k \in \mathbb{I}_{\geq 0},$$

and all $x \in \mathscr{C}$, we have that

$$x_m(k) = x(k)+e(k) \in \mathscr{X}_N,\ x(k) \in \mathscr{X}_N, \forall k \in \mathbb{I}_{\geq 0}, \tag{10a}$$
$$|\phi_{ed}(k;x)| \leq b\lambda^k|x| + \varepsilon, \forall k \in \mathbb{I}_{\geq 0}. \tag{10b}$$

B. Feasibility

Before presenting the robust stability results, we observe that although the warm start $\tilde{\mathbf{u}}$ is feasible for the *predicted* successor state $\tilde{x}^+ = f(x_m, u(0;x_m))$ (i.e., $(\tilde{x}^+,\tilde{\mathbf{u}}) \in \mathbb{Z}_N$), it may not be feasible for the *measured* successor state, i.e., $x_m^+ = f(x,u(0;x_m)) + d + e^+$. We remark that the *true* successor state, which is unknown in general, is $x^+ = f(x,u(0;x_m)) + d$. If $(x_m^+,\tilde{\mathbf{u}}) \notin \mathbb{Z}_N$, the right-hand side of the cost inequality (4b) is not meaningful. In such cases, we need to modify the warm start with a term $\mathbf{p}$ such that $(x_m^+, \tilde{\mathbf{u}}+\mathbf{p}) \in \mathbb{Z}_N$, and to this end we consider the following additional assumption.

Assumption 18: For any $x, x' \in \mathscr{X}_N$ and $\mathbf{u} \in \mathscr{U}_N(x)$, there exists $\mathbf{u}' \in \mathscr{U}_N(x')$ such that $|\mathbf{u}-\mathbf{u}'| \leq \sigma(|x-x'|)$ for some $\mathscr{K}$-function $\sigma(\cdot)$.

We remark that Assumption 18 has been shown to hold, e.g., for linear systems subject to polytopic constraints on $(x,\mathbf{u})$, and for nonlinear systems without state (or mixed) constraints. When the warm start $\tilde{\mathbf{u}}$ is not feasible, among various options for finding $\mathbf{p}$, we consider the following *feasibility* problem (recalling that $\tilde{x}^+$ is known):

$$\text{Find } \mathbf{p} \text{ s.t. } \tilde{\mathbf{u}}+\mathbf{p} \in \mathscr{U}_N(x_m^+) \text{ and } |\mathbf{p}| \leq \sigma(|x_m^+ - \tilde{x}^+|). \tag{11}$$

If $\tilde{\mathbf{u}} \in \mathscr{U}_N(x_m^+)$, then $\mathbf{p}=0$ satisfies the feasibility problem (11), and hence Assumption 18 is unnecessary. Furthermore, we do not require Assumption 18 when treating the case without state constraints in Section V.

Proposition 19: Under Assumption 18, for any $(\tilde{x}^+,\tilde{\mathbf{u}}) \in \mathbb{Z}_N$ and $x_m^+ \in \mathscr{X}_N$, the set of solutions to (11) is nonempty.

Given any $\mathbf{p}$ satisfying (11), and for any given $x_m^+ \in \mathscr{X}_N$, we replace conditions (4) with the following:

$$\mathbf{u}^+ \in \mathscr{U}_N(x_m^+) \tag{12a}$$
$$V_N(x_m^+,\mathbf{u}^+) \leq V_N(x_m^+, \tilde{\mathbf{u}}+\mathbf{p}) \tag{12b}$$
$$V_N(x_m^+,\mathbf{u}^+) \leq V_f(x_m^+) \qquad \text{when } x_m^+ \in r\mathbb{B}. \tag{12c}$$

In the perturbed case, the extended state is $z=(x,\mathbf{u})$, with $\mathbf{u}$ a suboptimal solution to $\mathbb{P}_N(x_m)$ where $x_m = x+e$ is the measured state. The extended system evolves as follows:

$$z^+ \in H_{ed}(z) = \{(x^+,\mathbf{u}^+) \mid x^+ = f(x,u(0;x_m))+d,\ \mathbf{u}^+ \in G_{ed}(z)\}, \tag{13}$$

in which (note that both x_m^+ and $\tilde{\mathbf{u}}+\mathbf{p}$ depend on z):

$$G_{ed}(z) = \{\mathbf{u}^+ \mid \mathbf{u}^+ \in \mathscr{U}_N(x_m^+), V_N(x_m^+,\mathbf{u}^+) \leq V_N(x_m^+,\tilde{\mathbf{u}}+\mathbf{p}),\ V_N(x_m^+,\mathbf{u}^+) \leq V_f(x_m^+) \text{ if } x_m^+ \in r\mathbb{B}\}.$$

C. Main results

We define $z_m = (x_m, \mathbf{u}) = (x+e, \mathbf{u}) = z + (e, 0)$ and observe that $z_m \in \mathscr{Z}_r$. The following supporting result is fundamental.

Lemma 20: For every $\mu > 0$, there exists a $\delta > 0$ and $\gamma \in (0,1)$ such that, for all $(z_m, e, d, e^+) \in \mathscr{Z}_r \times \delta\mathbb{B} \times \delta\mathbb{B} \times \delta\mathbb{B}$ with $x_m^+ \in \mathscr{X}_N$, we have:

$$\max_{z^+ \in H_{ed}(z)} V_N(z^+) \leq \max\{\gamma V_N(z), \mu\},$$

where $z = z_m - (e, 0)$.

We now characterize the compact sets over which SRES is guaranteed to hold. Consider $\bar{V} > 0$ such that the set:

$$\mathscr{S} = \{z \in \mathbb{R}^n \times \mathbb{U}^N \mid V_N(z) \leq \bar{V}\}$$

satisfies $\mathscr{S} \subseteq \mathbb{Z}_N$, i.e., $\mathscr{S}$ is a sublevel set of $\mathbb{R}^n \times \mathbb{U}^N$ fully contained in $\mathbb{Z}_N$. Thus, by definition, for any $z = (x, \mathbf{u}) \in \mathscr{S}$, it follows that $x \in \mathscr{X}_N$. Next, given a scalar $\rho > 0$ and any $z_m \in \mathscr{Z}_r$, we define the following measure and associated set:

$$V_N^\rho(z_m) = \max_{e \in \rho\mathbb{B}} V_N(z) \quad \text{s.t. } z = z_m - (e, 0) \tag{14}$$

$$\mathscr{S}_\rho = \{z_m \in \mathscr{Z}_r \mid V_N^\rho(z_m) \leq \bar{V}\} \tag{15}$$

in which we assume that ρ is small enough that $\mathscr{S}_\rho$ is nonempty. Finally, we define the following compact set:

$$\mathscr{C}_\rho = \{x \in \mathbb{R}^n \mid x = x_m - e, e \in \rho\mathbb{B}, \exists \mathbf{u} : (x_m, \mathbf{u}) \in \mathscr{S}_\rho\}. \tag{16}$$

and we observe that $0 \in \text{int}(\mathscr{C}_\rho) \subset \mathscr{X}_N$ for ρ sufficiently small. The main SRES result of this paper is as follows.

Theorem 21: Under Assumptions 1, 2, 3, 4, and 18, the origin of the perturbed closed-loop system (7) is SRES on $\mathscr{C}_\rho$.

Corollary 22: Under Assumptions 1, 2, 3, 4, and 18, the origin of the perturbed closed-loop system (7) is RES on $\text{int}(\mathscr{X}_N)$.

V. CASE WITHOUT STATE CONSTRAINTS

A. Controller definition

We now specialize the results on inherent robustness for the case in which there are no state constraints. To this aim, we replace Assumption 2 with the following one.

Assumption 23: The set $\mathbb{U}$ is compact and contains the origin. The sets $\mathbb{X} = \mathbb{R}^n$ and $\mathbb{X}_f = \text{lev}_\alpha V_f = \{x \in \mathbb{R}^n \mid V_f(x) \leq \alpha\}$, with $\alpha > 0$.

As discussed later on, Assumption 18 will not be necessary, whereas Assumptions 1, 3 and 4 (with $V_N(\cdot)$ replaced by $V_N^\beta(\cdot)$ later defined) are required. We modify the cost function as follows:

$$V_N^\beta(x, \mathbf{u}) = \sum_{k=0}^{N-1} \ell(\phi(k; x, \mathbf{u}), u(k)) + \beta V_f(\phi(N; x, \mathbf{u}))$$

in which $\beta \geq 1$ is a parameter that will be chosen in way that the terminal constraint, $\phi(N; x, \mathbf{u}) \in \mathbb{X}_f$, is unnecessary as it will be satisfied inherently for any suboptimal input sequence with appropriately bounded cost. Given the warm start $\tilde{\mathbf{u}}$ for the successor state $x^+ = f(x, u(0; x))$, defined as in (3), we modify the requirements to the suboptimal MPC algorithm as follows:

$$\mathbf{u}^+ \in \mathbb{U}^N \tag{17a}$$

$$V_N^\beta(x^+, \mathbf{u}^+) \leq V_N^\beta(x^+, \tilde{\mathbf{u}}) \tag{17b}$$

$$V_N^\beta(x^+, \mathbf{u}^+) \leq \beta V_f(x^+) \qquad \text{when } x^+ \in r\mathbb{B} \tag{17c}$$

We observe that the main difference between the above requirements and those in (4) is that in (17a) we allow any input $\mathbf{u}^+ \in \mathbb{U}^N$, whereas in (4) the terminal constraint, $\phi(N; x, \mathbf{u}) \in \mathbb{X}_f$, is explicitly enforced by (4a). Condition (17c) is also slightly different and follows from the modification of the terminal penalty. To avoid unnecessary repetition, we again use (5) (or (6) when referring to the extended state) to describe the evolution of the nominal closed-loop system under suboptimal MPC with modified terminal penalty. We choose scalar (maximal cost) $\bar{V} \geq \alpha > 0$ and define the following compact sets:

$$\bar{\mathscr{Z}}_r = \{(x, \mathbf{u}) \in \mathbb{R}^n \times \mathbb{U}^N \mid V_N^\beta(x, \mathbf{u}) \leq \bar{V}, \text{ and } V_N^\beta(x, \mathbf{u}) \leq \beta V_f(x) \text{ if } x \in r\mathbb{B}\}$$

$$\mathbb{X}_0 = \{x \in \mathbb{R}^n | \exists \mathbf{u} \in \mathbb{U}^N \text{ such that } (x, \mathbf{u}) \in \bar{\mathscr{Z}}_r\} \tag{18}$$

For the remainder of the paper we choose $\beta = \bar{\beta} := \bar{V}/\alpha$, in which $\bar{V}$ is the maximal cost in the previous set definitions and $\alpha > 0$ is the terminal region sublevel set parameter of Assumption 23. Note that all the results to follow also hold if we choose any β satisfying $\beta \geq \bar{\beta}$. We point out that the choice $\beta \geq \bar{\beta}$ also implies that $\bar{\mathscr{Z}}_r$ does *not* contain any trajectories terminating on the boundary of $\mathbb{X}_f$. For such trajectories, $V_f(x(N)) = \alpha$, and thus $\sum_{i=0}^{N-1} \ell(x(i), u(i)) \leq 0$, which is satisfied only by $(x(i), u(i)) = (0, 0)$ for $i \in \mathbb{I}_{0:N-1}$, which implies that $x(N) = 0$, which is a contradiction. Hence, $\mathbb{X}_0$ contains only states that can be steered to $\text{int}(\mathbb{X}_f)$.

B. Nominal stability

Lemma 24: $V_N^\beta(z)$ is an exponential Lyapunov function for the extended closed-loop system (6) in any compact subset of $\bar{\mathscr{Z}}_r$.

Theorem 25: Under Assumptions 1, 23, 3, and 4, the origin of the closed-loop system (5) is ES on $\mathbb{X}_0$.

We now characterize the set $\mathbb{X}_0$ and its limit for large $\bar{V}$. To this end we define a (slightly) restricted feasible set of initial states that can be taken by an admissible input sequence to the *interior* of $\mathbb{X}_f$, rather than all of $\mathbb{X}_f$ (note that the interior of $\mathbb{X}_f$ is not empty because $\alpha > 0$):

$$\bar{\mathscr{X}}_N := \{x \in \mathbb{R}^n \mid \exists \mathbf{u} \in \mathbb{U}^N \text{ s.t. } \phi(N; x, \mathbf{u}) \in \text{int}(\mathbb{X}_f)\} \tag{19}$$

Proposition 26: The admissible set $\mathbb{X}_0$ and restricted feasible set $\bar{\mathscr{X}}_N$ satisfy the following:

$$\mathbb{X}_0(\bar{V}) \subseteq \bar{\mathscr{X}}_N \text{ for all } \bar{V} \geq 0, \text{ and } \bar{\mathscr{X}}_N \subseteq \bigcup_{\bar{V} \geq 0} \mathbb{X}_0(\bar{V}) \tag{20}$$

C. Inherent robustness

For inherent robustness analysis of the case without state constraints, we again consider that the closed-loop system evolves according to (7). We observe that having removed the terminal constraint has the immediate benefit that the warm start $\tilde{\mathbf{u}}$ is "feasible" for the measured successor state $x_m^+ = x^+ + e^+$, because $\tilde{\mathbf{u}} \in \mathbb{U}^N$. Hence, there is no need to solve the feasibility problem (11). Therefore, we can write the evolution of the extended closed-loop system as $z^+ \in H_{ed}(z)$ in which $H_{ed}(z)$ is still defined in (13) with $G_{ed}(z)$ modified as follows:

$$G_{ed}(z) = \{\mathbf{u}^+ \mid \mathbf{u}^+ \in \mathbb{U}^N, V_N^\beta(x_m^+, \mathbf{u}^+) \le V_N^\beta(x_m^+, \tilde{\mathbf{u}}), \quad V_N^\beta(x_m^+, \mathbf{u}^+) \le \beta V_f(x_m^+) \text{ if } x_m^+ \in r\mathbb{B}\}$$

We also observe that the fundamental result of Lemma 20 still holds for the modified cost V_N^β, with $\mathscr{Z}_r$ replaced by $\bar{\mathscr{Z}}_r$.

We now present a set over which we prove SRES. Given a scalar $\rho > 0$ and any $z_m \in \bar{\mathscr{Z}}_r$, we define:

$$\bar{V}_N^\rho(z_m) = \max_{e \in \rho\mathbb{B}} V_N^\beta(z) \quad \text{s.t. } z = z_m - (e, 0) \tag{21a}$$

$$\bar{\mathscr{S}}_\rho = \{z_m \in \bar{\mathscr{Z}}_r \mid \bar{V}_N^\rho(z_m) \le \bar{V}\} \tag{21b}$$

in which we assume that ρ is small enough that $\bar{\mathscr{S}}_\rho$ is nonempty. Finally, the candidate set for SRES is defined as:

$$\bar{\mathscr{C}}_\rho = \{x \in \mathbb{R}^n \mid x = x_m - e, e \in \rho\mathbb{B}, \exists \mathbf{u} : (x_m, \mathbf{u}) \in \bar{\mathscr{S}}_\rho\} \tag{22}$$

Theorem 27: Under Assumptions 1, 23, 3 and 4, the origin of the closed-loop system (7) is SRES on $\bar{\mathscr{C}}_\rho$.

When $|d|, |e| \to 0$, it follows directly from (21) and (22) that $\bar{\mathscr{C}}_\rho \to \mathbb{X}_0$ and SRES holds over a set approaching the admissible set of initial conditions. This observation, coupled with (20) gives the desired result: in the limit of small disturbances and large parameter $\bar{V}$, the robust region of attraction for the case without state constraints converges to (the closure of) the restricted feasible set.

VI. ILLUSTRATIVE EXAMPLE

A. System and controllers

We consider the following system:

$$x_1^+ = x_1 + u$$
$$x_2^+ = bx_2 + u^3$$

with $0 < |b| < 1$. The horizon is $N = 3$, $\mathbb{U} = [-1, 1]$, and the stage cost function is given by: $\ell(x, u) = |x|^2 + u^2$. Three different nonlinear MPC formulations are considered.

C1. No state constraints are enforced, $\mathbb{X} = \mathbb{R}^2$, and the terminal constraint set is the origin, $\mathbb{X}_f = \{0\}$.

C2. No state constraints are enforced, $\mathbb{X} = \mathbb{R}^2$, and the terminal constraint set is $\mathbb{X}_f = \{x \in \mathbb{R}^2 \mid x'Px \le \alpha\}$ with $\alpha > 0$ and P later defined.

C3. State constraints are enforced, $\mathbb{X} = [-2, 2]^2$, and the terminal constraint set is the same of C2.

We remark that controller C1 *does not* satisfy Assumption 2 because $\mathbb{X}_f$ does not contain the origin in its interior. In the definition of controllers C2 and C3, we note that the linearization of system around the origin can be written as:

$$x^+ = Ax + Bu \qquad \text{with } A = \begin{bmatrix} 1 & 0 \\ 0 & b \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and we observe that the pair (A, B) is stabilizable. Therefore, we follow the procedure described in [2, Par. 2.5.3.2], and we define a linear control law $\kappa_f(x) = Kx = [k,\, 0]x$. We note that such control law which is stabilizing for the linearized system if and only if $|1 + k| < 1$. Assuming that k satisfies the previous condition, let $Q_K = I + K'K$, $A_K = A + BK$, and solve the following Lyapunov equation (notice the factor 2 multiplying Q_K): $A_K' P A_K + 2Q_K = P$. Consequently, we define the terminal cost $V_f(x) = x'Px$, while the terminal constraint set is given by $\mathbb{X}_f = \{x \in \mathbb{R}^2 \mid V_f(x) \le \alpha\}$, in which $\alpha > 0$; we notice that P is positive definite because Q_K is positive definite. It can be shown [2, Par. 2.5.3.2] that there exists $\alpha > 0$ such that Assumption 3 holds for $u = \kappa_f(x)$. Furthermore, it can be verified that Assumptions 1 and 2 hold. Similarly, Assumption 4 holds for $a = 2$.

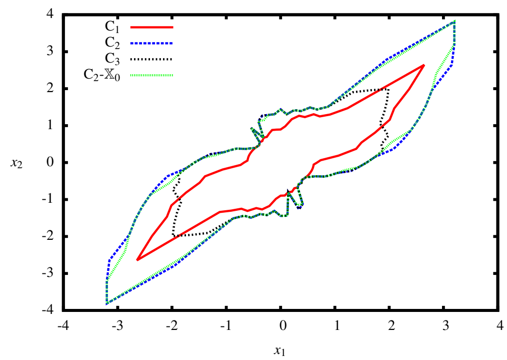

Fig. 1. Approximate feasibility sets of the three controllers

B. Results and discussion

We now present some numerical results, considering in the system dynamics $b = 0.9$. In the definition of V_f and P discussed in the previous paragraph, we use $k = -1$. It follows that $P = \left[\begin{smallmatrix} 4 & 0 \\ 0 & 10.53 \end{smallmatrix}\right]$, and it can be verified that $\alpha = 1.1$ is such that Assumption 3 holds for $u = [-1,\, 0]x$. We report in Fig. 1 the approximate feasibility sets, $\mathscr{X}_N$, for the three controllers. For controller C2, we also show the restricted feasibility set $\mathbb{X}_0$ obtained for $\bar{V} = 100$. As expected, we notice for C2 that $\mathbb{X}_0 \subseteq \mathscr{X}_N$, although we can notice that $\mathbb{X}_0$ is covering almost all $\mathscr{X}_N$. Furthermore, the feasibility set $\mathscr{X}_N$ for C2 contains both the feasibility sets for C1 and C3. We report in Fig. 2 the first component of $\mathscr{U}_N(x)$, with $x = 0.5\,[\cos(\theta), \sin(\theta)]$, as a function of θ for the controllers C1 and C2 (the plot for C3 is not reported as it identical to that of C2). For C1, we can notice that Assumption 18 does not hold at the point indicated by the arrow. On the other hand, no such points can be noticed in this plot for C2.

VII. CONCLUSIONS

This paper analyzes the nominal and robust stability properties of discrete-time nonlinear systems in closed-loop with *general* and *implementable* suboptimal MPC algorithms. The

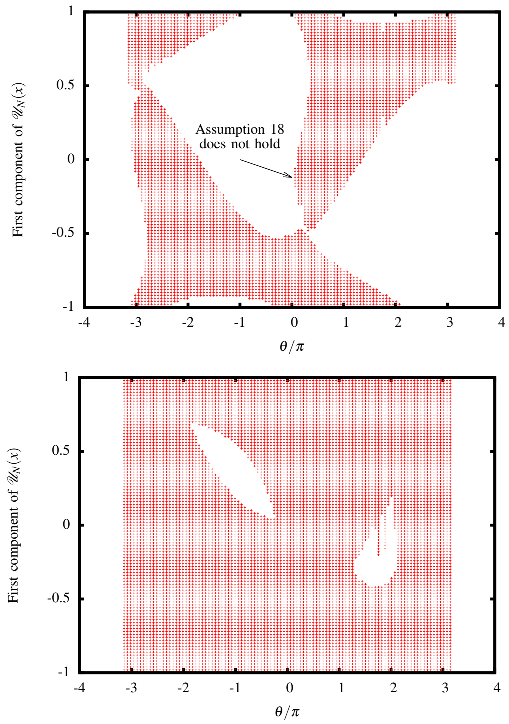

Fig. 2. First component of the input feasibility set $\mathscr{U}_N(x)$ as a function of the initial state $x = 0.5\,[\cos(\theta), \sin(\theta)]$. C1 (top), C2 (bottom)

class of suboptimal algorithms analyzed in this paper simply require computing a control sequence that improves the cost of a *warm start* sequence. In the nominal case, such warm start is immediately available from the previous decision time, while in the perturbed case it may happen that the warm start available from the previous decision time is infeasible, and a feasibility recovery step is required. No preassigned tolerance with respect to the optimal cost is required [14] to prove the results of this paper.

The paper [3] proved nominal asymptotic stability in a neighborhood of the origin. We went several steps further, and proved nominal exponential stability in arbitrarily large compact subsets of the feasible region. However, the most relevant contribution of this paper was to establish *inherent robust* exponential stability of the origin, with respect to sufficiently small but otherwise arbitrary unknown process disturbances and state measurement/estimation errors. Inherent robustness, in the spirit of ideas and results proved by Teel and coworkers [11], [12], means that the controller is based on the nominal system model and ignores such unknown perturbations. To prove the robust stability properties we require an continuity assumption on input feasible set, with respect to the initial state. That assumption is used to show that in the perturbed case, the feasibility recovery step size of the warm start is bounded by some $\mathscr{K}-$function of the perturbation size. Such an assumption holds, e.g., (i) when no state constraints are enforced and (ii) for linear systems subject to polytopic constraints (on input and state).

When state constraints are excluded in the controller formulation, e.g. when state constraints are softened, a variant of the controller can be used, in which the terminal constraint is also excluded, while its satisfaction required for stability analysis is ensured implicitly by using an inflated terminal penalty. The major benefit of this formulation is that the warm start from the previous decision time is always feasible, and hence no recovery step is required in the perturbed case.

All the results proved in this paper apply to optimal MPC as well, and thus *suboptimal* and *optimal* nonlinear MPC have the same (qualitative) robust stability properties, although we can expect that the size of perturbations that can be tolerated by optimal MPC may be larger. The nonlinear MPC formulation considered in this paper is *as simple as possible*, e.g., we did not use any state constraint tightening approach [8], [12] to ensure recursive robust feasibility of the optimization problem. Essentially most industrial (linear and nonlinear) MPC algorithms fall within this class, and the results of this paper are expected to provide further confidence in the use of MPC for nonlinear systems where global optimization is *usually* out of reach.

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