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Observer-based LPV Control of a Nonlinear PDE

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Abstract— **In this paper, observer-based linear parameter-varying (LPV) control of the one-dimensional nonlinear Burgers' equation is presented. The partial differential equation is discretized using a finite difference scheme and the boundary conditions are taken as control inputs. A nonlinear high-order state space model is generated and proper orthogonal decomposition based Galerkin projection is used for model order reduction. A discrete-time quasi-LPV model that is affine in scheduling parameters is derived based on the reduced model and a polytopic dynamic output feedback LPV controller is synthesized. Since the scheduling parameters are linear combinations of system states, the synthesized output-feedback controller is converted to an observer-based state feedback controller which provides an estimate of the scheduling parameters. Simulation results demonstrate that the designed observer-based LPV controller has almost the same level of disturbance and measurement noise rejection capability compared to an output feedback LPV controller combined with a nonlinear observer. Moreover, both of the LPV controllers outperform an LQG controller based on a linearized model.**

I. INTRODUCTION

Modelling and control of distributed systems have been an active research topic in the recent years and flow control is an important subclass of such systems. Due to the complexity of governing partial differential equation (PDE) in flow control i.e. the Navier-Stokes equation, simpler analog models such as Burgers' and Euler's equations can be used to solve the control problem first and then it may be attempted to extend the solution to more general cases, see [1] and [2]. Burgers' equation includes the nonlinear convective term that is challenging to handle in flow-related problems. Moreover, supersonic flow about airfoils, shockwaves, some boundary layer problems and traffic flows can be modeled by Burgers' equation.

Application of proper orthogonal decomposition (POD) and Galerkin projection [3] to obtain a finite dimensional approximation of distributed parameter systems with a reasonable accuracy has become popular in the literature. This technique has been frequently used to obtain a suitable model of Burgers' equation for controller synthesis e.g. in [1], [2], [4] and [5]. However, linear controllers were designed in these reports based on locally linearized models. Stabilization of a family of stationary solutions of Burgers' equation using state feedback and output feedback was reported in [6] and [7].

Linear parameter-varying (LPV) gain-scheduling techniques have been developed into effective tools to control MIMO nonlinear plants. Their attractiveness lies in the extension of well known linear optimal control methods and the use of linear matrix inequalities (LMIs), to the solution of nonlinear control problems. Many nonlinear systems of practical interest can be represented as quasi-LPV systems, where the scheduling parameters include external signals and measured system outputs or states, see e.g. [8]. However, the application of these techniques has not been extensively explored in flow control problems.

An LPV controller design for transition control in a Poiseuille flow model was reported in [9], where the framework was limited for a particular spatially interconnected flow model. Control of a nonlinear Galerkin model using an adaptation-based LPV model was presented in [10]. The authors proposed an output feedback LPV controller for Burgers' equation in [11], where a nonlinear observer was implemented to estimate the scheduling parameters.

Conversion of an arbitrary linear time-invariant (LTI) controller to its equivalent observer-based form has been discussed in [12] and [13]. The motivation for such a conversion is to provide an estimate of plant states which can be used e.g. in gain scheduling.

In this paper, an observer-based state feedback LPV controller is designed for a low-order quasi-LPV model of the Burgers' equation. The one-dimensional Burgers' equation with Dirichlet boundary conditions is discretized using an explicit finite difference technique. A high order discrete-time nonlinear state space system is obtained, where boundary conditions are taken as control inputs and velocity of the two interior grids close to the boundaries are chosen as measured signals [11]. The model is simulated for a typical input trajectory and the method of snapshots [14] is used to obtain POD basis functions which form the model reduction transformation. Galerkin projection is then employed to map the original system to a lower-dimensional space.

A polytopic quasi-LPV model is derived and states of the reduced model are chosen as scheduling parameters. Since the derived quasi-LPV model has a small number of scheduling parameters and vertices, an output-feedback LPV controller with a fixed Lyapunov function [15] is designed which has a reasonable computation burden for synthesis and on-line implementation.

To schedule the designed output-feedback LPV controller, it is necessary to estimate the scheduling parameters which are the reduced states in this work. To remove the need to design and implement an additional observer [11], here it is proposed to convert the designed output feedback controllers at all vertices to their observer-based state feedback form. The technique developed in [12] and [13] for LTI systems is extended to the LPV case and the estimated states are

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used for state feedback as well as for scheduling the LPV controller.

Simulation results demonstrate that the designed observer-based LPV controller has almost the same level of disturbance and measurement noise rejection capability compared to an output feedback LPV controller combined with a nonlinear observer to estimate the scheduling parameters [11]. Moreover, both LPV controllers outperform an LQG controller based on a linearized model.

The contribution of this paper is to demonstrate the applicability of LPV gain-scheduling techniques to control a nonlinear PDE, where the scheduling parameters of the LPV controller are extracted from the states of a dynamic output feedback controller. Moreover, the number of measured and control signals are realistic. Another approach has been recently proposed by the authors [16] to include an observer dynamics in a generalized plant and then design an output feedback controller, which guarantees the closed-loop stability.

This paper is organized as follows. Discretization of the Burgers' equation and derivation of the nonlinear state space model are presented in Section 2. In Section 3 model reduction is discussed. LPV modelling is described in Section 4 and observer-based LPV controller synthesis is presented in Section 5. Simulation results are given in Section 6 and the last section gives the conclusions.

II. Nonlinear Modelling

Consider the one-dimensional nonlinear viscous Burgers' equation on the physical domain $\mathbb{S} = \{s|s \in [0, \mathcal{L}]\}$ and temporal domain $\mathbb{T} = \{t|t \in [0, \mathcal{T}]\}$

$$\frac{\partial w}{\partial t} + w\frac{\partial w}{\partial s} = \nu\frac{\partial^2 w}{\partial s^2}, \tag{1}$$

where $w(s,t) : \mathbb{S}\times\mathbb{T} \to \mathbb{W}$ is the space and time dependent velocity, s and t refer to space and time respectively and ν is a known constant representing viscosity.

The initial condition is given as

$$w(s, 0) = w_0(s), \tag{2}$$

and Dirichlet boundary conditions are applied at both ends

$$w(0, t) = u_1(t), \quad w(\mathcal{L}, t) = u_2(t), \tag{3}$$

where $u_1(t)$ and $u_2(t)$ are taken as control inputs.

An approximate discrete solution of (1) is represented by

$$w_i^k = w(\hat{s}, \hat{t}) : \hat{\mathbb{S}} \times \hat{\mathbb{T}} \to \mathbb{W}, \tag{4}$$

where i and k refer to time and space respectively, and the finite discrete sets $\hat{\mathbb{S}}$ and $\hat{\mathbb{T}}$ are defined as

$$\hat{\mathbb{S}} = \{s_1, \ldots, s_G\}, \quad \hat{\mathbb{T}} = \{t_1, \ldots, t_K\}, \tag{5}$$

where G is the number of grid points and K is the number of time samples.

Using the forward-time central-space (FTCS) method [17] which is an explicit finite difference scheme, (1) is discretized

$$\begin{aligned} w_i^{k+1} = & w_i^k - \frac{\lambda}{2}w_i^k(w_{i+1}^k - w_{i-1}^k) \\ & + r(w_{i+1}^k - 2w_i^k + w_{i-1}^k), \end{aligned} \tag{6}$$

where $\lambda = \dfrac{\Delta t}{\Delta s}$, $r = \dfrac{\nu\Delta t}{(\Delta s)^2}$, Δs is the spatial grid size and Δt is the time-stepping. As explained in [17], choosing some appropriate values for Δs and Δt leads to convergence of this scheme if ν is not very small.

The difference equation (6) is used to obtain the nonlinear discrete-time state space model

$$\begin{aligned} x(k+1) &= A_n x(k) + F_n(x(k), u(k)) \\ y(k) &= C_n x(k), \end{aligned} \tag{7}$$

where the state vector $x(k) \in \mathbb{R}^N$, the control input $u(k) \in \mathbb{R}^{n_i}$ and the control output $y(k) \in \mathbb{R}^{n_o}$ are defined as

$$\begin{aligned} x(k) &= [x_1(k), \ldots, x_N(k)]^\top = [w_2^k, \ldots, w_{G-1}^k]^\top \\ u(k) &= [w_1^k \quad w_G^k]^\top, \quad y(k) = [w_{N_1}^k \quad w_{N_2}^k]^\top, \end{aligned} \tag{8}$$

and $n_i = 2$, $n_o = 2$ and $N = G - 2$ are number of inputs, outputs and states respectively. The matrices A_n and C_n and the vector F_n are given as

$$A_n = \begin{bmatrix} 1-2r & r & 0 & \ldots & \ldots & \ldots & 0 \\ r & 1-2r & r & 0 & . & . & \vdots \\ 0 & r & \ddots & \ddots & 0 & . & \vdots \\ \vdots & 0 & \ddots & \ddots & \ddots & 0 & \vdots \\ \vdots & . & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & . & . & 0 & \ddots & \ddots & r \\ 0 & \ldots & \ldots & \ldots & 0 & r & 1-2r \end{bmatrix},$$

$$F_n = \begin{bmatrix} -\frac{\lambda}{2}x_2x_1 + \frac{\lambda}{2}x_1u_1 + ru_1 \\ \frac{\lambda}{2}x_2x_1 - \frac{\lambda}{2}x_3x_2 \\ \frac{\lambda}{2}x_3x_2 - \frac{\lambda}{2}x_4x_3 \\ \vdots \\ \frac{\lambda}{2}x_{G-3}x_{G-4} - \frac{\lambda}{2}x_Nx_{G-3} \\ \frac{\lambda}{2}x_Nx_{G-3} - \frac{\lambda}{2}x_Nu_2 + ru_2 \end{bmatrix},$$

$$C_n = \begin{bmatrix} \mathbf{0}_{1\times(N_1-1)} & 1 & \mathbf{0}_{1\times(N_2-1)} \\ \mathbf{0}_{1\times(N_2-1)} & 1 & \mathbf{0}_{1\times(N_1-1)} \end{bmatrix}. \tag{9}$$

The subscripts N_1 and N_2 specify the two grid points whose velocity are assumed to be measured. The nonlinearities in F_n come from the nonlinear convective term which is common in all flow problems.

The problem is to design a controller to compute the input signals $u(k)$ such that the control output $y(k)$ tracks a reference input and rejects process disturbance and measurement noise. The order of the nonlinear model (7) is too large to be used for controller synthesis and should be reduced.

III. Model Reduction

Using POD, orthonormal basis functions are extracted from an ensemble of experimental or simulation data which form a model reduction transformation. A solution of (1) can be approximated in terms of set of basis functions

$$w(\hat{s},\hat{t}) \approx \hat{w}(\hat{s},\hat{t}) = \sum_{j=1}^{M} \phi_j(\hat{s})\alpha_j(\hat{t}), \quad (10)$$

where ϕ_j define the set of orthonormal basis functions and α_j are coefficient functions.

The method of snapshots [14] is used to obtain the basis functions. The discretized model (6) is simulated for some typical boundary condition trajectories to form the matrix of snapshots $W_{snap} \in \mathbb{R}^{N\times K}$

$$W_{snap} = \begin{bmatrix} w_2^1 & \dots & w_2^K \\ \vdots & \ddots & \vdots \\ w_{G-1}^1 & \dots & w_{G-1}^K \end{bmatrix}. \quad (11)$$

Introduce the singular value decomposition of W_{snap}

$$W_{snap} = \Phi\Sigma\Psi^\top = [\Phi_s\ \Phi_n] \begin{bmatrix} \Sigma_s & 0 & 0 \\ 0 & \Sigma_n & 0 \end{bmatrix} \begin{bmatrix} \Psi_s^\top \\ \Psi_n^\top \end{bmatrix}, \quad (12)$$

where $\Phi \in \mathbb{R}^{N\times N}$ and $\Psi \in \mathbb{R}^{K\times K}$ and Φ_s, Σ_s and Ψ_s correspond to the M dominant singular values.

The columns of Φ form the set of basis functions $\{\phi_1,\dots,\phi_N\}$ which can be used to obtain an accurate low-order dynamic model via Galerkin projection. Such a projection captures the most energy and properties of the original system for a given number of modes or basis function [3]. The basis functions obtained by this method are also called POD modes.

For state space models like (7), the projection is simplified to multiplying both sides of (7) by the truncated orthonormal matrix $\Phi_s \in \mathbb{R}^{N\times M}$, see [18]

$$\Phi_s^\top x(k+1) = \Phi_s^\top A_n x(k) + \Phi_s^\top F_n(x(k),u(k)). \quad (13)$$

A low-dimensional state vector $x_r(k) \in \mathbb{R}^M$ is then defined as

$$x_r(k) = \Phi_s^\top x(k), \quad (14)$$

and the approximate state vector $\hat{x}(k) = \Phi_s\Phi_s^\top x(k) \in \mathbb{R}^N$ is given in the original dimension as

$$\hat{x}(k) = [\hat{x}_1(k),\dots\ \ ,\hat{x}_N(k)]^\top = [\hat{w}_2^k,\dots,\hat{w}_{G-1}^k]^\top. \quad (15)$$

Note that each element of x_r is a linear combination of x. The reduced order model and matrices are thus obtained as

$$\begin{aligned} x_r(k+1) &= A_r x_r(k) + F_r(\Phi_s x_r(k),u(k)) \\ y(k) &= C_r x_r(k), \end{aligned} \quad (16)$$

with

$$A_r = \Phi_s^\top A_n \Phi_s, \quad F_r = \Phi_s^\top F_n, \quad C_r = C_n\Phi_s. \quad (17)$$

A_r and C_r can be calculated offline, provided that basis functions have been determined. Since F_r is a function of x_r and u, it should be calculated on-line if needed.

Both boundary conditions for generating the matrix of snapshots are chosen as sinusoidal excitation trajectories covering frequencies up to 50 Hz, similar to the report by [2].

The parameters of the Burgers' equation and discretization are $\nu = 0.01$, $\Delta s = 0.02$, $\Delta T = 0.005$, $\mathcal{L} = 1$, $w_0(s) = 0$, $K = 10000$ and $G = 51$. Considering the accuracy and complexity of the reduced model, only three first dominant modes are selected, i.e. $M = 3$. More details about implementing POD and validation of the reduced model was given in [11].

The reliability of the resulting reduced-order model (16) strongly depends on the excitation signal; if the operating conditions become different from excitation trajectories, the accuracy of the model will degrade.

Although the resulting model (16) has a low order, it has a nonlinear term F_r which makes the controller synthesis challenging.

IV. LPV Modelling

Linearizion of the nonlinear reduced model of Burgers' equation around an operating point and linear controller design has been reported in [1], [2], [4] and [5]. The performance of such controllers can be limited depending on the operating range of the system. LPV gain-scheduling techniques offer a framework to use well-known linear optimal control methods for nonlinear plants.

The nonlinear discrete-time reduced-order model (16) can be converted to a parameter-dependent polytopic quasi-LPV model

$$\begin{aligned} x_r(k+1) &= A(\theta(k))x_r(k) + B(\theta(k))u(k) \\ y(k) &= Cx_r(k), \end{aligned} \quad (18)$$

where parameter dependent matrices $A(\theta) \in \mathbb{R}^{M\times M}$ and $B(\theta) \in \mathbb{R}^{M\times n_i}$ and constant matrix $C \in \mathbb{R}^{n_0\times M}$ are to be determined and θ is the scheduling parameter vector. Without loss of generality, assume that $\theta \in \mathbb{R}^M$. The LPV model can be represented by a linear input-output map

$$P(\theta) = \left[\begin{array}{c|c} A(\theta) & B(\theta) \\ \hline C & 0 \end{array}\right]. \quad (19)$$

Consider the compact set $\mathcal{P}_\theta \subset \mathbb{R}^M : \theta \in \mathcal{P}_\theta, \forall k > 0$. Here, it is assumed to be a polytope [15] defined by the convex hull

$$\mathcal{P}_\theta := \text{Co}\{\theta_{v_1},\theta_{v_2},\dots,\theta_{v_m}\}, \quad (20)$$

where $m = 2^M$ is the number of vertices.

The LPV system is called parameter-affine, if the state space model depends affinely on the parameters

$$P(\theta) = \sum_{i=0}^{M} \theta_i P_i = P_0 + \theta_1 P_1 + \dots + \theta_M P_M. \quad (21)$$

Since θ can be expressed as a convex combination of m vertices θ_{v_i}, if (21) holds, it follows that the system can be

$$
A(\theta) = \Phi_s^\top \begin{bmatrix} 1-2r & -\frac{\lambda}{2}\hat{x}_1 + r & 0 & \dots & \dots & \dots & 0 \\ \frac{\lambda}{2}\hat{x}_2 + r & 1-2r & -\frac{\lambda}{2}\hat{x}_2 + r & 0 & . & . & \vdots \\ 0 & \frac{\lambda}{2}\hat{x}_3 + r & \ddots & \ddots & 0 & . & \vdots \\ \vdots & 0 & \ddots & \ddots & \ddots & 0 & \vdots \\ \vdots & . & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & . & . & 0 & \ddots & \ddots & -\frac{\lambda}{2}\hat{x}_{G-3} + r \\ 0 & \dots & \dots & \dots & 0 & \frac{\lambda}{2}\hat{x}_N + r & 1-2r \end{bmatrix} \Phi_s
$$

$$
B(\theta) = \Phi_s^\top \begin{bmatrix} \frac{\lambda}{2}\hat{x}_1 + r & 0 & \dots & \dots & 0 \\ 0 & \dots & \dots & 0 & -\frac{\lambda}{2}\hat{x}_N + r \end{bmatrix}^\top, \quad C = C_r = C_n \Phi_s. \tag{25}
$$

represented by a linear combination of LTI models at the vertices; this is called a polytopic LPV system

$$
P(\theta) \in \text{Co}\{P(\theta_{v_1}), P(\theta_{v_2}), \dots, P(\theta_{v_m})\} = \sum_{i=1}^{m} \alpha_i P(\theta_{v_i}), \tag{22}
$$

where $\sum_{i=1}^{m} \alpha_i = 1$, and $\alpha_i \geq 0$ are the convex coordinates.

The source of nonlinearity in (16) is the nonlinear convective term which is obvious in (6). By defining the scheduling parameter vector as

$$
\theta(k) = x_r(k), \tag{23}
$$

one can obtain state matrices $A(\theta)$, $B(\theta)$ and C as in (25) such that the model (18) is affine in θ. Note that these matrices are needed only for controller synthesis and during on-line implementation they do not have to be computed. Since the scheduling parameters are defined to be the states, the resulting model is a quasi-LPV model. The number of states is $M = 3$ and number of vertices is $m = 8$, which leads to low on-line computation and makes it possible to use standard synthesis tools.

V. Observer-based State Feedback Control

In [11] an output feedback LPV controller was designed for the quasi-LPV model (18). Since only w_{N_1} and w_{N_2} are assumed to be measurable, a nonlinear functional observer was designed to estimate the parameters (23) needed to schedule the LPV controller.

In this section an output feedback LPV controller is designed similar to [11], and to remove the need to design and implement an additional observer, the designed output feedback controller is converted to an observer-based state feedback form by extending the LTI method proposed in [13] to the LPV case.

A discrete-time output-feedback LPV controller with a fixed Lyapunov function is designed for the low-order quasi-LPV model of Burgers' equation (18) using the $\mathcal{H}_\infty$ loop-shaping approach [15]. This method has been proven to be an effective and practical tool for LPV synthesis due to the simplicity of the synthesis and implementation and low on-line computation effort.

The block diagram of this controller is shown in Fig. 1, where d and $n(k)$ are non-zero initial condition disturbance and measurement noise respectively. The nonlinear Burgers' system (7) is controlled by an LPV controller $\Omega(\theta)$.

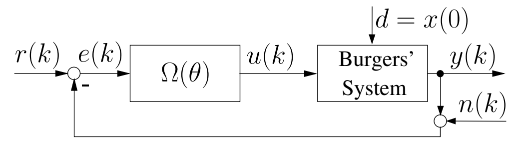

Fig. 1. Block diagram of the output feedback LPV controller

The design objective considered here is to stabilize (7) in the whole operating range with a high tracking capability, disturbance and measurement noise rejection and taking in consideration a limited bandwidth and amplitude for control signal. A mixed sensitivity loop-shaping approach is adopted to achieve the objectives. The generalized plant is shown in Fig. 2, where $W_S(z)$ and $W_K(z)$ are the weighting filters to shape sensitivity $S(\theta)$ and control sensitivity $\Omega(\theta)S(\theta)$, respectively.

For designs based on polytopic LPV models, the model must not have a parameter dependent input matrix $B(\theta)$; but this is not the case in (18). To solve this problem, the plant is augmented by a low-pass filter $W_B(z)$ with a sufficiently large bandwidth [15]. This removes the parameter dependency of the input matrix in the generalized plant state space realization.

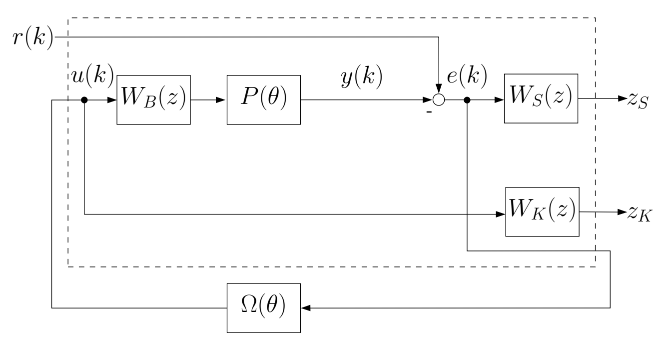

Fig. 2. Generalized plant for the synthesis of output feedback LPV controller

The mixed-sensitivity criterion to be minimized for controller design is the induced $\mathcal{L}_2$ gain of the closed-loop map between r and $[z_S^\top \quad z_K^\top]^\top$. A discrete-time output-feedback LPV controller $\Omega(\theta)$ is designed with the state space realization

$$\begin{aligned} x^c(k+1) &= A^c(\theta(k))x^c(k) + B^c(\theta(k))e(k) \\ u(k) &= C^c(\theta(k))x^c(k) + D^c(\theta(k))e(k), \end{aligned} \tag{25}$$

where $x^c \in \mathbb{R}^{n_c}$ and $n_c = 9$. The parameter dependent state matrices of the controller are determined by the convex coordinates and the controller vertices $A^c_{v_i}$, $B^c_{v_i}$, $C^c_{v_i}$ and $D^c_{v_i}$ with $i = 1, \ldots, 8$, which are computed using a modified discrete-time version of MATLAB robust control toolbox command `hinfgs`. More details bout the output feedback LPV controller design can be found in [11].

Conversion of an LTI output feedback compensator to its observer-based state feedback form has been discussed in [12] and [13] in detail. The motivation for such a conversion is to provide an estimate of plant states which can be used e.g. in gain scheduling. The technique is based on the solution of a generalized non-symmetric Riccati equation.

The conversion method proposed in [13] is applied to each vertex of the polytopic LPV system. First, Youla parameterization [19] of all stabilizing controllers built on an observer-based form for each vertex (18) is written as

$$\begin{aligned} \hat{x}_r(k+1) &= A_{v_i}\hat{x}_r(k) + B_{v_i}u(k) + L_{ob_i}(y(k) - C\hat{x}_r(k)) \\ x_Q(k+1) &= A_{Q_i}x_Q(k) + B_{Q_i}(y(k) - C\hat{x}_r(k)) \\ u(k) &= -K_{fb_i}\hat{x}_r(k) + C_{Q_i}x_Q(k) \\ &\quad + D_{Q_i}(y(k) - C\hat{x}_r(k)), \end{aligned} \tag{26}$$

where A_{v_i} and B_{v_i} are the state and input matrices of the LPV plant (19) at each vertex, $x_Q \in \mathbb{R}^{n_q}$ is the state vector of the Youla parameter with $n_Q = n_c - M = 6$, $A_{Q_i} \in \mathbb{R}^{n_Q \times n_Q}$, $B_{Q_i} \in \mathbb{R}^{n_Q \times n_o}$, $C_{Q_i} \in \mathbb{R}^{n_i \times n_Q}$ and $D_{Q_i} \in \mathbb{R}^{n_i \times n_o}$ are state space matrices at each vertex of the Youla parameter $Q(\cdot)$, and $K_{fb_i} \in \mathbb{R}^{n_i \times M}$ and $L_{ob_i} \in \mathbb{R}^{M \times n_o}$ are static state feedback and state observer gains at each vertex. Following the procedure given in [13], all of these matrices are obtained for $i = 1, \ldots, m$ using the state space matrices of the closed-loop system, plant and output feedback controller vertices.

Since the plant, the designed output feedback controller and consequently the closed-loops system are convex combination of vertices of some polytopes, the LPV polytopic representation of Youla parameter, state feedback and state observer gains can be obtained as

$$Q(\theta) = \left[\begin{array}{c|c} A_Q(\theta) & B_Q(\theta) \\ \hline C_Q(\theta) & D_Q(\theta) \end{array}\right] = \sum_{i=1}^{m} \alpha_i \left[\begin{array}{c|c} A_{Q_i} & B_{Q_i} \\ \hline C_{Q_i} & D_{Q_i} \end{array}\right] \tag{27}$$

$$K_{fb}(\theta) = \sum_{i=1}^{m} \alpha_i K_{fb_i} \tag{28}$$

$$L_{ob}(\theta) = \sum_{i=1}^{m} \alpha_i L_{ob_i}, \tag{29}$$

and the resulting observer-based state feedback LPV controller is given as

$$\begin{aligned} \hat{x}_r(k+1) &= A(\theta)\hat{x}_r(k) + B(\theta)u(k) \\ &\quad + L_{ob}(\theta)(y(k) - C\hat{x}_r(k)) \\ x_Q(k+1) &= A_Q(\theta)x_Q(k) + B_Q(\theta)(y(k) - C\hat{x}_r(k)) \\ u(k) &= -K_{fb}(\theta)\hat{x}_r(k) + C_Q(\theta)x_Q(k) \\ &\quad + D_Q(\theta)(y(k) - C\hat{x}_r(k)). \end{aligned} \tag{30}$$

The block diagram of the closed -loop system with the control scheme (30) is shown in Fig. 3, where δ is a delay operator. The estimated scheduling parameters $\hat{\theta}(k) = \hat{x}_r(k)$ are calculated and provided in the loop, but the connecting signal lines are not shown for compactness. Moreover, these connections are not taken into account in proving the stability of the closed loop, i.e. the quasi-LPV system is treated as a pure LPV system [8]; thus, the results are local. Note that the local stability has already been proved in designing the output feedback controller. Since the observer-based controller is input-output equivalent to the original output feedback controller, stability has been preserved after conversion to the observer-based form (30).

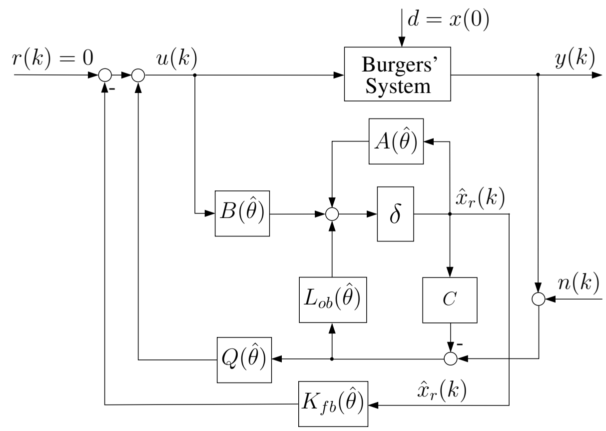

Fig. 3. Block diagram of the observer-based LPV controller

VI. Simulation Results

The system shown in Fig. 3 is simulated in closed loop in MATLAB/SIMULINK using the full-order finite difference model (6) and the designed observer-based state feedback LPV controller (30). The simulation sampling frequency is set to 200 Hz. The control outputs $y_1 = w_{N_1}$ and $y_2 = w_{N_2}$ are corrupted using the band-limited white noise block of SIMULINK with power $P_n = 10^{-6}$. Control outputs in flow control problems are typically close to the boundaries, thus $N_1 = 6$ and $N_2 = 44$ are chosen.

For comparison, the reduced model (16) is linearized around an equilibrium $(x_r, u) = (0, 0)$ and an LQG controller similar to the report in [4] is designed and tuned for the best disturbance and measurement noise rejection. In the mentioned report a distributed measurement was assumed and no measurement noise was injected. To have a better comparison, the reference input is set to $r = 0$. The task is to reject a non-zero initial condition disturbance

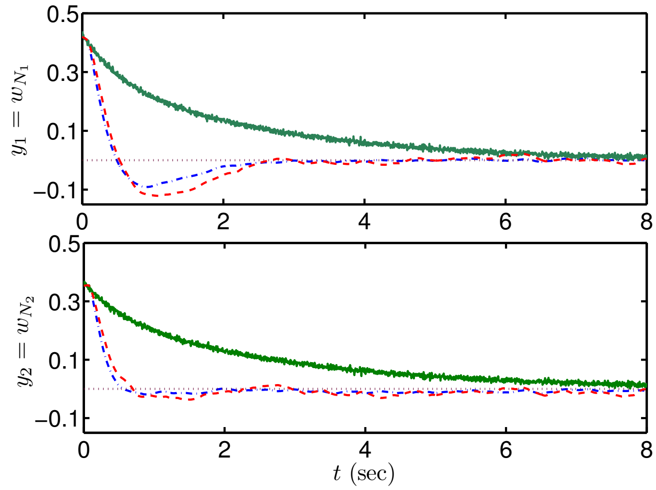

Fig. 4. Rejection of a non-zero initial condition disturbance by observer-based state feedback LPV (dashed), output feedback LPV with a nonlinear observer (dashed-dotted) and LQG (solid) controllers

and suppress the injected measurement noise. The output feedback controller with a nonlinear observer proposed in [11] is also implemented.

Fig. 4 illustrates the output of the three simulated controllers for the mentioned task. It is obvious that both LPV controllers suppress the measurement noise better, and reject the disturbance much faster than the LQG controller. This indicates that the loop-shaping objectives in both low and high frequency ranges have been met. Faster disturbance rejection was possible by LQG controller, but measurement noise was amplified significantly in that case. The mean square error of the observer-based state feedback LPV controller is only 5.3% worse than the output-feedback LPV controller which needs an additional observer [11].

VII. Conclusion

Observer-based state feedback LPV controller synthesis for the nonlinear viscous Burgers' equation has been presented in this paper. The one-dimensional nonlinear Burgers' equation was discretized using a finite difference scheme and the boundary conditions were taken as control inputs. A discrete-time nonlinear state space model with a high order was obtained and proper orthogonal decomposition together with Galerkin projection was used for model order reduction.

A discrete-time polytopic quasi-LPV model which is affine in scheduling parameters was derived based on the reduced model, where the scheduling parameters are the reduced states. An output-feedback LPV controller with a fixed Lyapunov function was designed which has a reasonable synthesis and on-line computation cost.

To schedule the designed output-feedback LPV controller, it is necessary to estimate the scheduling parameters. To remove the need to design and implement an additional observer, it has been proposed to convert the designed output feedback controllers at all vertices to their observer-based state feedback form.

Simulation results demonstrate the high process disturbance and measurement noise rejection capabilities of the designed observer-based LPV controller compared with an LQG controller based on a linearized model. An additional nonlinear observer to estimate the scheduling parameters does not improve the performance significantly.

The reliability of the derived reduced-order model and the designed controller strongly depends on the excitation signal which is used to generate the POD modes; if the operating conditions become much different from excitation trajectories, accuracy of the model and performance of the controller will degrade. Since the scheduling parameters are linear combination of states of the original system, the guaranteed stability and performance results are local and valid only if the parameters do not leave the parameter set.

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