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Sufficient Conditions for Decentralized Navigation Functions Based Controllers using Canonical Vector Fields

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***Abstract*— A combination of dual Lyapunov analysis and properties of decentralized navigation function based controllers is used to check the stability properties of a certain class of decentralized controllers for navigation and collision avoidance in multi-agent systems. The derived results yield a less conservative condition from previous approaches, which relates to the negativity of the sum of the minimum eigenvalues of the Hessian matrices at the critical points, instead of requiring each of the eigenvalues to be negative itself. This provides an improved characterization of the reachable set of this class of decentralized navigation function based controllers, which is less conservative than the previous results for the same class of controllers.**

I. INTRODUCTION

Navigation of multi-agent systems is an area of increasing interest both from a research as well as an application viewpoint. When it comes to multi-robot/vehicle systems, collision avoidance and decentralization are two important specifications in the control design for guaranteeing safety and scalability. Thus there has been a growing demand for the development of decentralized navigation methodologies with guaranteed collision avoidance. In recent years the application of potential field based methods has been explored [7],[18] as a promising alternative for such algorithms.

A common problem with potential field based path planning algorithms in multi-agent systems is the existence of local minima [10],[12]. The seminal work of Koditschek and Rimon [11] involved navigation of a single robot in an environment of spherical obstacles with guaranteed convergence. In previous work, the closed loop single robot navigation methodology of [11] was extended to multi-agent systems. In [13],[9],[16],[4], [7],[8] this method was extended to take into account the volume of each robot while formation control for point agents using navigation functions was dealt with in [21], [3]. Decentralized navigation functions were also used for multiple UAV guidance in [2].

In previous work, analysis of potential field based controllers via density functions was considered in [14] for centralized and in [6], [5] for decentralized multi-agent navigation. In this work we extend the previous results by combining the canonical vector field formulation of [14] with the dual Lyapunov analysis of decentralized potential fields in [5]. We examine the convergence of the system using a combination of primal and dual [19] Lyapunov techniques. This combination has been used in [15],[1],[14],[22]. In particular in [14],[15] a density function is provided for a single robot driven by a navigation function in a static obstacle workspace. Primal analysis is used to show convergence to a neighborhood of the critical points and density functions are used to prove the instability of the undesirable critical points using the properties of the navigation functions. The difference in our case is that we consider a system of multiple moving agents driven by decentralized potential functions and the potential functions are not considered *a priori* navigation functions. On the contrary, the designed potentials are tuned properly to satisfy appropriate conditions to guarantee asymptotic stability from almost all initial conditions.

The outcome of the analysis is a less conservative sufficient condition for almost global navigation in the decentralized case. In [16],[4] the sufficient condition relied on Morse theory that required, among others, that the minimum eigenvalues of the Hessian matrices for each decentralized navigation function at the critical points is strictly negative. In [6], [5] we derived a similar conclusion with the use of density functions. Using the aforementioned tools, we show here that it is sufficient that only the *sum* of the minimum eigenvalues is strictly negative, and *not* the minimum eigenvalue of each decentralized navigation function itself. This yields an improved set of conditions for navigation.

More specifically, in [4],[7] the navigation functions were designed in such a way to allow agents that had already reached their desired destination to cooperate with the rest of the team in the case of a possible collision. In this paper, a construction similar to the initial navigation function construction in [11] is used. Hence each agent no longer participates in the collision avoidance procedure if its initial condition coincides with its desired goal. In essence, the agents might converge to critical points which are no longer guaranteed not to coincide with local minima. In [5] it was shown that in this case, each agent converges to a sphere around its target point and an estimate of this radius was given. It turns out that with the formulation of this paper a much less conservative bound can be derived that tends to zero in the case of point agents. Moreover, a broader set of initial conditions for successful navigation to the target set is derived for the case of non-point agents, establishing a direct correspondence between the convergence set radius and agents' maximum radii.

The rest of the paper is organized as follows: Section II presents the system and decentralized multi-agent navigation problem treated in this paper. The necessary mathematical preliminaries are provided in Section III, while Section IV

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provides the decentralized control design. Section V includes the convergence analysis and a simulated example is found in Section VI. Section VII summarizes the results of the paper and indicates further research directions.

II. Definitions and Problem Statement

Consider N agents operating in a planar spherical workspace $W \subset \mathbb{R}^2$, with radius R_W. Let $q_i \in \mathbb{R}^2$ denote the position of agent i, and let $q = [q_1^T, \ldots, q_N^T]^T$. We also denote $u = [u_1^T, \ldots, u_N^T]^T$. Agent motion is described by:

$$\dot{q}_i = u_i, i \in \mathcal{N} = \{1, \ldots, N\} \tag{1}$$

where u_i is the control input for each agent. We consider cyclic agents of specific radius $\varrho_i \geq 0, i \in \mathcal{N}$. We will also consider the particular cases when agents have common radii $\varrho_i = \varrho, \forall i \in \mathcal{N}$. For $\varrho = 0$, the problem is reduced to the case of point agents. Collision avoidance is meant in the sense that no intersections occur between the agents' discs. Each agent is assumed to have knowledge of the position of agents located in a cyclic neighborhood of specific radius d at each time instant, where $d > \max\limits_{i,j \in \mathcal{N}} (\varrho_i + \varrho_j)$. The function γ_{di} is agent's i goal function which is minimized once the desired objective with respect to this particular agent is fulfilled. In particular, let $q_{di} \in W$ denote the desired destination point of agent i. We then define $\gamma_{di} = \|q_i - q_{di}\|^2$.

In order to encode inter-agent collision scenarios, we define a function γ_{ij}, for $j = 1, \ldots, N, j \neq i$, given by

$$\gamma_{ij}(\beta_{ij}) = \begin{cases} \frac{1}{2}\beta_{ij}, \ 0 \leq \beta_{ij} \leq c^2 \\ \phi(\beta_{ij}), \ c^2 \leq \beta_{ij} \leq d^2 - (\varrho_i + \varrho_j)^2 \\ 1, \ d^2 - (\varrho_i + \varrho_j)^2 \leq \beta_{ij} \end{cases} \tag{2}$$

where $\beta_{ij} = \|q_i - q_j\|^2 - (\varrho_i + \varrho_j)^2$. We also define the function γ_{i0} which refers to the workspace boundary (indexed by 0) and is used to maintain the agents within the workspace. We have $\beta_{i0} = (R_W - \varrho_i)^2 - \|q_i\|^2$. The function γ_{i0} is defined in the same way as $\gamma_{ij}, j > 0$. The positive scalar c and the function ϕ are chosen so that γ_{ij} is everywhere C^2. For example, we can chose ϕ to be a fifth degree polynomial whose coefficients are calculated so that γ_{ij} is everywhere twice continuously differentiable. In the sequel, we also use the notation $\nabla_i(\cdot) \triangleq \frac{\partial}{\partial q_i}(\cdot)$ for brevity.

III. Mathematical Preliminaries

A. Dual Lyapunov Theory

For functions $V : \mathbb{R}^n \to \mathbb{R}$ and $f : \mathbb{R}^n \to \mathbb{R}^n$ the notation $\nabla V = \begin{bmatrix} \frac{\partial V}{\partial x_1} & \cdots & \frac{\partial V}{\partial x_n} \end{bmatrix}^T$, $\nabla \cdot f = \frac{\partial f_1}{\partial x_1} + \ldots + \frac{\partial f_n}{\partial x_n}$ is used. The dual Lyapunov result of [19] is stated as follows:

Theorem 1: Given the equation $\dot{x}(t) = f(x(t))$, where $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ and $f(0) = 0$, suppose there exists a nonnegative density function $\rho \in C^1(\mathbb{R}^n \setminus \{0\}, \mathbb{R})$ such that $\rho(x) f(x) / \|x\|$ is integrable on $\{x \in \mathbb{R}^n : \|x\| \geq 1\}$ and

$$[\nabla \cdot (f\rho)](x) > 0 \text{ for almost all } x \tag{3}$$

Then, for almost all initial states $x(0)$ the trajectory $x(t)$ exists for $t \in [0, \infty)$ and tends to zero as $t \to \infty$. Moreover, if the equilibrium $x = 0$ is stable, then the conclusion remains valid even if ρ takes negative values.

Note that while Theorem 1 applies to the whole $\mathbb{R}^n$, we apply it here for the workspace W. The application of density functions to navigation function based systems was also used in [14]. A local version of Theorem 1 was used in [20], [17].

IV. Decentralized Navigation Functions

In this paper, we use a construction similar to the initial navigation function construction in [11]. An analysis of the proposed decentralized potentials was held in [5] using dual Lyapunov theory [19]. In this paper we extend the above results to non-point agents and derive less conservative sufficient conditions for convergence by decentralizing the canonical vector fields framework introduced in [14].

Specifically, a decentralized potential function $\varphi_i : \mathbb{R}^{2N} \to [0, 1]$ is defined as

$$\varphi_i = \frac{\gamma_{di}}{\left(\gamma_{di}^k + G_i\right)^{1/k}} \tag{4}$$

where $k > 0$ is a scalar positive parameter, and the function G_i is constructed in such a way to guarantee collision avoidance. A proposed control law is of the form

$$u_i = -K \nabla_i \varphi_i \tag{5}$$

where $K > 0$ is a positive scalar gain.

A. Construction of the G_i function

We now review briefly the construction of G_i, [4], [7] for the case of local sensing capabilities. The multi-agent team is associated with a graph whose vertices are indexed by the team members.

Definition 1: A *binary relation* with respect to an agent i is an edge between agent i and another agent.

Definition 2: A *relation* with respect to agent i is defined as a set of binary relations with respect to agent i.

Definition 3: The *relation level* is the number of binary relations in a relation with respect to agent i.

The complementary set $(R_j^{i,C})_l$ of relation j with respect to agent i is the set that contains all the relations of the same level apart from relation j; γ_{ij} is called the "Proximity Function" between agents i and j. Let R_k^i denote the k^{th} relation of level l with respect to i. The "Relation Proximity Function" (RPF) is given by $(b_{R_k^i})_l = \sum_{j \in (R_k^i)_l} \gamma_{ij}$ where $j \in (R_k^i)_l$ denotes the agents that participate in the relation. We also use the simplified notation $b_r^i = \sum_{j \in P_r} \gamma_{ij}$ for the RPF, where r denotes a relation and P_r denotes the set of agents participating in the specific relation with respect to i. A "Relation Verification Function" (RVF) is defined by $(g_{R_k^i})_l = (b_{R_k^i})_l + \frac{\lambda (b_{R_k^i})_l}{(b_{R_k^i})_l + (B_{R_k^{i},C})_l^{1/h}}$, where $\lambda, h > 0$ and $(B_{R_k^i,C})_l = \prod_{m \in (R_k^C)_l} (b_m)_l$. Again for simplicity we also use the notation $(B_{R_k^{i},C})_l \equiv \tilde{b}_r^i = \prod\limits_{\substack{s \in S_r \\ s \neq r}} b_s^i$ for the term $(B_{R_k^{i},C})_l$ where S_r denotes the set of relations in the same level with relation r. The RVF is also written as

$g_r^i = b_r^i + \frac{\lambda b_r^i}{b_r^i + (\tilde{b}_r^i)^{1/h}}$. We have (a) $\lim_{b_r^i \to 0} \lim_{\tilde{b}_r^i \to 0} g_r^i \left(b_r^i, \tilde{b}_r^i \right) = \lambda$ (b) $\lim_{\substack{b_r^i \to 0 \\ \tilde{b}_r^i \neq 0}} g_r^i \left(b_r^i, \tilde{b}_r^i \right) = 0$. The function G_i is defined as $G_i = \prod_{l=1}^{n_L^i} \prod_{j=1}^{n_{R_l}^i} (g_{R_j^i})_l$ where n_L^i the number of levels and $n_{R_l}^i$ the number of relations in level-l with respect to i. Using the simplified notation, $G_i = \prod_{r=1}^{N_i} g_r^i$ where N_i is the number of all relations with respect to i.

We then have $\nabla_i \varphi_i = \frac{\left(\gamma_{di}^k + G_i\right)^{1/k} \nabla_i \gamma_{di} - \frac{\gamma_{di}}{k} \left(\gamma_{di}^k + G_i\right)^{1/k-1} \left(k \gamma_{di}^{k-1} \nabla_i \gamma_{di} + \nabla_i G_i\right)}{\left(\gamma_{di}^k + G_i\right)^{2/k}}$, so that

$$\nabla_i \varphi_i = \left(\gamma_{di}^k + G_i\right)^{-1/k-1} \left(G_i \nabla_i \gamma_{di} - \frac{\gamma_{di}}{k} \nabla_i G_i \right) \quad (6)$$

We can also compute

$$\nabla_i \varphi_j = \left(\gamma_{dj}^k + G_j\right)^{-1/k-1} \left(-\frac{\gamma_{dj}}{k} \nabla_i G_j \right) \quad (7)$$

A critical point of φ_i is defined by $\nabla_i \varphi_i = 0$. The following Proposition will be useful in the following analysis:

Proposition 1: For every $\epsilon > 0$ there exists a positive scalar $P(\epsilon) > 0$ such that if $k \geq P(\epsilon)$ then there are no critical points of φ_i in the set $F_i = \left\{ q \in W | g_r^i \geq \epsilon, \forall r = 1, \ldots, N_i \right\} \setminus \{\gamma_{di}\}$.

Proof: See [5]. $\diamondsuit$

V. Canonical Vector Fields for DNF's

In this section we redefine the canonical vector fields' framework defined in [14] for centralized navigation functions in the decentralized case.

Let q_{ki} be the k-th critical point for agent i, with $k = 1, \ldots, n_{si}$, where n_{si} the number of critical points. Similarly to [14], let d_{ki} denote the distance between the agent position and the corresponding critical point, i.e., $d_{ki} = ||q_i - q_{ki}||^2$. Let $\lambda_{\min_i}(q_{ki})$ be the minimum eigenvalue of the Hessian matrix $\frac{\partial^2 \varphi_i}{\partial q_i^2}$ at $q_i = q_{ki}$ and u_{ki} be the corresponding unit eigenvector. Define $U_{ki} = u_{ki} u_{ki}^T + \varepsilon_1 I$ where I is the two-dimensional unit matrix and $0 < \varepsilon_1 \leq 1$. Denote also $U_{n_{si}+1,i} = U_{n_{si}+2,i} = I$ and $d_{n_{si}+1,i} = \varphi_i$,$d_{n_{si}+2,i} = 1 - \varphi_i$. Define $\bar{d}_{ki} = \prod_{l=1, l \neq k}^{n_{si}+2} d_{li}$. Then for each agent i we define the matrix D_{φ_i} as $D_{\varphi_i} = \mu \sum_{k=1}^{n_{si}+2} \frac{\bar{d}_{ki}}{\bar{d}_{ki} + d_{ki}} U_{ki}$. It can be shown that D_{φ_i} fulfills similar properties to the matrix D_φ defined for a centralized navigation function in [14]. In particular, the following result holds:

Lemma 2: The matrix D_{φ_i} has the following properties: (i) $D_{\varphi_i} = \mu U_{ki} + \varepsilon_1 \mu I$, for $q_i = q_{ki}$ (ii) $D_{\varphi_i} = \mu I$, for $G_i = 0$ (iii) $D_{\varphi_i} = \mu I$, for $q_i = q_{di}$, (iv) $\nabla_i D_{\varphi_i} = 0$, for $q_i = q_{di}$ and $\nabla_i D_{\varphi_i} = 0$, for $q_i = q_{ki}$, and (v) $0 < x^T D_{\varphi_i} x \leq 2(n_{si}+2)\mu ||x||^2$, for all $x \in \mathbb{R}^2$.

The last property guarantees positive definiteness and boundedness of D_{φ_i}.

We will consider the modification of the control law in (5) by using D_{φ_i} as an additional gain matrix:

$$u_i = -K D_{\varphi_i} \nabla_i \varphi_i \quad (8)$$

The above system will be called the canonical system.

Note first that the two systems share the same critical points. Moreover, using the exact same arguments as in the proof of Proposition 3 in [14], the existence of an appropriate tuning of μ such that the trajectories of (5) are bounded by the trajectories of (8) can be established. This allows us to derive conclusions on the convergence of (5) by examining the convergence of (8).

VI. Convergence Analysis

A. Primal Lyapunov Analysis

The stability of system (1) under the control law (5) was analyzed in [5] and convergence to an arbitrarilly small neighborhood of the critical points was established. We now show that, as expected, the multiplication of the control with the positive definite matrix D_{φ_i} will yield the same behavior. Note that the closed loop kinematics of system (1) under the control law (8) are given by

$$\dot{q} = f(q) = \begin{bmatrix} -K D_{\varphi_1} \nabla_1 \varphi_1 \\ \vdots \\ -K D_{\varphi_N} \nabla_N \varphi_N \end{bmatrix}$$

Define $\varphi = \sum_i \varphi_i$. The derivative of φ can be computed by $\dot{\varphi} = (\nabla \varphi)^T \dot{q} = -K \sum_{i=1}^N (\nabla_i \varphi)^T (D_{\varphi_i} \nabla_i \varphi_i) = -K \sum_{i=1}^N \sum_{j=1}^N (D_{\varphi_i} \nabla_i \varphi_i)^T (\nabla_i \varphi_j)$ where φ_i is defined in (4). Consider $\varepsilon > 0$. Then we can further compute

$$\begin{aligned}
\dot{\varphi} &= -K \sum_{i=1}^N ((\nabla_i \varphi_i)^T D_{\varphi_i} \nabla_i \varphi_i + \sum_{j \neq i} (\nabla_i \varphi_j)^T (D_{\varphi_i} \nabla_i \varphi_i)) \\
&= -K \sum_{i: \|\nabla_i \varphi_i\| > \varepsilon} ((\nabla_i \varphi_i)^T D_{\varphi_i} \nabla_i \varphi_i \\
&\quad + \sum_{j \neq i} (\nabla_i \varphi_j)^T (D_{\varphi_i} \nabla_i \varphi_i)) \\
&\quad -K \sum_{i: \|\nabla_i \varphi_i\| \leq \varepsilon} ((\nabla_i \varphi_i)^T D_{\varphi_i} \nabla_i \varphi_i + \sum_{j \neq i} (\nabla_i \varphi_j)^T (D_{\varphi_i} \nabla_i \varphi_i)) \\
&\leq -K \sum_{i: \|\nabla_i \varphi_i\| > \varepsilon} (\lambda_{\min}(D_{\varphi_i}) \varepsilon^2 + \sum_{j \neq i} (\nabla_i \varphi_j)^T (D_{\varphi_i} \nabla_i \varphi_i)) \\
&\quad -K \sum_{i: \|\nabla_i \varphi_i\| \leq \varepsilon} (\nabla_i \varphi_j)^T (D_{\varphi_i} \nabla_i \varphi_i)
\end{aligned}$$

The terms in the first sum, where $\|\nabla_i \varphi_i\| > \varepsilon$, are lower bounded as follows: $\lambda_{\min}(D_{\varphi_i}) \varepsilon^2 + \sum_{j \neq i} (\nabla_i \varphi_j)^T (D_{\varphi_i} \nabla_i \varphi_i) \geq \lambda_{\min}(D_{\varphi_i}) \varepsilon^2 - \varepsilon ||D_{\varphi_i}||_{\max} \sum_{j \neq i} \|\nabla_i \varphi_j\|$. Using (7) we have $\|\nabla_i \varphi_j\| = \left(\gamma_{dj}^k + G_j\right)^{-1/k-1} \left(\frac{\gamma_{dj}}{k} \|\nabla_i G_j\|\right)$. For $\gamma_{dj} > \gamma_{\min}$, $k > 1$, the term $(\gamma_{dj}^k + G_j)^{1/k+1}$ in the above equation is minimized by $\gamma_{\min}^2$ so that

$$\lambda_{\min}(D_{\varphi_i})\varepsilon^2 + \sum_{j\neq i}(\nabla_i\varphi_j)^T(D_{\varphi_i}\nabla_i\varphi_i) \geq \lambda_{\min}(D_{\varphi_i})\varepsilon^2 - \varepsilon\frac{||D_{\varphi_i}||_{\max}}{k\gamma_{\min}^2}\sum_{j\neq i}\gamma_{dj}\|\nabla_i G_j\|$$

We want to achieve a bound $\lambda_{\min}(D_{\varphi_i})\varepsilon^2 + \sum_{j\neq i}(\nabla_i\varphi_j)^T(D_{\varphi_i}\nabla_i\varphi_i) \geq \rho_1 > 0$, where $0 < \rho_1 < \lambda_{\min}(D_{\varphi_i})\varepsilon^2$. A sufficient condition for this to hold is $\frac{(N-1)||D_{\varphi_i}||_{\max}}{k\gamma_{\min}^2}\max_{j\neq i}\{\gamma_{dj}\|\nabla_i G_j\|\} \leq \frac{\lambda_{\min}(D_{\varphi_i})\varepsilon^2-\rho_1}{\varepsilon}$, or equivalently

$$k \geq \frac{\varepsilon}{\lambda_{\min}(D_{\varphi_i})\varepsilon^2-\rho_1}\frac{(N-1)||D_{\varphi_i}||_{\max}}{\gamma_{\min}^2}\max_{j\neq i}\{\gamma_{dj}\|\nabla_i G_j\|\} \tag{9}$$

We next compute a lower bound on the terms in the second sum, where $\|\nabla_i\varphi_i\| \leq \varepsilon$. Note first that

$$\begin{aligned}(\nabla_i\varphi_j)^T(D_{\varphi_i}\nabla_i\varphi_i) &= (D_{\varphi_i}\nabla_i\varphi_i)^T(\nabla_i\varphi_j)\\ &= \frac{\left(G_i\nabla_i\gamma_{di} - \frac{\gamma_{di}}{k}\nabla_i G_i\right)^T D_{\varphi_i}^T\left(-\frac{\gamma_{dj}}{k}\nabla_i G_j\right)}{\left(\gamma_{di}^k+G_i\right)^{1/k+1}\left(\gamma_{dj}^k+G_j\right)^{1/k+1}}\\ &= \frac{-\frac{\gamma_{dj}G_i}{k}\nabla_i\gamma_{di}^T D_{\varphi_i}^T\nabla_i G_j + \frac{\gamma_{dj}\gamma_{di}}{k^2}\nabla_i G_i^T D_{\varphi_i}^T\nabla_i G_j}{\left(\gamma_{di}^k+G_i\right)^{1/k+1}\left(\gamma_{dj}^k+G_j\right)^{1/k+1}}\end{aligned}$$

so that

$$\begin{aligned}(\nabla_i\varphi_j)^T(D_{\varphi_i}\nabla_i\varphi_i) \geq\\ \frac{1}{\gamma_{\min}^4}(-\frac{\gamma_{dj}G_i}{k}\|\nabla_i\gamma_{di}\|\|D_{\varphi_i}\|\|\nabla_i G_j\|\\ -\frac{\gamma_{dj}\gamma_{di}}{k^2}\|\nabla_i G_i\|\|D_{\varphi_i}\|\|\nabla_i G_j\|)\end{aligned}$$

We want to achieve a bound $\sum_{j\neq i}(\nabla_i\varphi_j)^T(D_{\varphi_i}\nabla_i\varphi_i) \geq -2\rho_2$, where $\rho_2 > 0$. A sufficient condition for this to hold is $\frac{1}{\gamma_{\min}^4}\frac{\gamma_{dj}G_i}{k}\|\nabla_i\gamma_{di}\|\|D_{\varphi_i}\|\|\nabla_i G_j\| \leq \rho_2$ and $\frac{1}{\gamma_{\min}^4}\frac{\gamma_{dj}\gamma_{di}}{k^2}\|\nabla_i G_i\|\|D_{\varphi_i}\|\|\nabla_i G_j\| \leq \rho_2$ or equivalently, that

$$k \geq \frac{\max_{j\neq i}\{\gamma_{dj}G_i\|\nabla_i\gamma_{di}\|\|D_{\varphi_i}\|\|\nabla_i G_j\|\}}{\rho_2\gamma_{\min}^4} \tag{10}$$

and

$$k \geq \sqrt{\frac{\max_{j\neq i}\{\gamma_{di}\gamma_{dj}\|\nabla_i G_i\|\|D_{\varphi_i}\|\|\nabla_i G_j\|\}}{\rho_2\gamma_{\min}^4}} \tag{11}$$

hold at the same time. Provided that k satisfies (9),(10),(11) we have $\dot{\varphi} \leq -K\rho_1 + K(N-1)^2\rho_2$, assuming there exists at least one agent such that $\|\nabla_i\varphi_i\| > \varepsilon$. The latter is strictly negative for $0 < (N-1)^2\rho_2 < \rho_1 < \lambda_{\min}(D_{\varphi_i})\varepsilon$.

In essence, $\dot{\varphi}$ can be rendered strictly negative as long as there exists at least one agent with $\|\nabla_i\varphi_i\| > \varepsilon$. Thus the system converges to an arbitrarily small region of the critical points, provided that $0 < (N-1)^2\rho_2 < \rho_1 < \lambda_{\min}(D_{\varphi_i})\varepsilon$ and the conditions on k hold. We have:

Proposition 3: Consider the system (1) with the control law (8). Assume that $\gamma_{di} \geq \gamma_{\min} > 0$. Pick $\varepsilon > 0, \rho_1, \rho_2 > 0$ satisfying $0 < (N-1)^2\rho_2 < \rho_1 < \lambda_{\min}(D_{\varphi_i})\varepsilon$ and assume that (9),(10),(11) hold. Then the system converges to the set $\|\nabla_i\varphi_i\| \leq \varepsilon$ for all i in finite time.

We also refer the reader to the corresponding convergence result for the system (5) in [5], Proposition 2.

B. Dual Lyapunov Analysis

Having established convergence to an arbitrarily small neighborhood of the critical points, density functions are now used to pose sufficient conditions that the attractors of undesirable critical points are sets of measure zero.

For $\varphi = \sum_i \varphi_i$, define $\rho = \varphi^{-\alpha}, \alpha > 0$. In [5] it is shown that ρ fulfils the integrability condition of Theorem 1 and is a suitable density function for the equilibrium point $\gamma_{di} = 0, \forall i \in \mathcal{N}$. We have $\nabla\rho = -\alpha\varphi^{-\alpha-1}\nabla\varphi$ and $\nabla\cdot(f\rho) = \nabla\rho\cdot f + \rho\nabla\cdot f = -\alpha\varphi^{-\alpha-1}\nabla\varphi\cdot f + \varphi^{-\alpha}\nabla\cdot f$. Whenever $\nabla_i\varphi_i = 0$ for all $i \in \mathcal{N}$, we have $f = 0$. Moreover,

$$\nabla\cdot f = \nabla\cdot[-D_{\varphi_1}\nabla_1\varphi_1, \ldots, -D_{\varphi_N}\nabla_N\varphi_N]$$

For $\nabla_i\varphi_i = 0$ for all $i \in \mathcal{N}$, we can calculate

$$\begin{aligned}\nabla\cdot(f\rho) &= \varphi^{-\alpha}\nabla\cdot f\\ &= -\varphi^{-\alpha}\sum_i\{\mu\lambda_{\min_i} + \varepsilon_1\mu(\lambda_{\min_i} + \lambda_{\max_i})\}\end{aligned}$$

where $\lambda_{\min_i}, \lambda_{\max_i}$ denote the minimum and maximum eigenvalue, respectively, of the Hessian matrix $\frac{\partial^2\varphi}{\partial q_i^2}$ at the particular critical point of agent i. The following result is then straightforward:

Proposition 4: Assume that $\sum_i \lambda_{\min_i} < 0$. Then the right hand side of the last equation is rendered strictly positive by choosing

$$\varepsilon_1 < \frac{|\sum_i \lambda_{\min_i}|}{|\sum_i\{\lambda_{\min_i} + \lambda_{\max_i}\}|} \tag{12}$$

The above result implies that a sufficient condition for the fulfillment of the condition $\nabla\cdot(f\rho) > 0$ for $\nabla_i\varphi_i = 0$ for all $i \in \mathcal{N}$ is given by

$$\sum_i \lambda_{\min_i} < 0 \tag{13}$$

It thus turns out that negativity of the minimum Hessian eigenvalue of *all* φ_i is a sufficient but not a necessary condition for decentralized navigation. This condition was used in [4] using tools from Morse Theory. Using the combination of dual Lyapunov functions and canonical vector fields, we derived the sufficient condition (13), which is less conservative than the Morse condition $\lambda_{\min_i} < 0$ for all i. Moreover, the condition (13) is also less conservative than the condition $\sum_i \lambda_{\min_i} + \lambda_{\max_i} < 0$ that was derived in our previous work [5] using the same tools as in the current paper, apart from the canonical vector field formulation.

Let us now elaborate a little more on the condition (13). Using the notation $H_i(\varphi_i) \triangleq \dfrac{\partial^2\varphi_i}{\partial q_i^2}$ for the Hessian matrix of φ_i, it is true that $\sum_i \lambda_{\min_i} \leq \sum_i \hat{u}_i^T H_i\hat{u}_i$ holds for all vectors $\hat{u}_i$ with $||\hat{u}_i|| = 1$. Note also that the critical points of φ_i

and $\hat{\varphi}_i = \frac{\gamma_i^k}{G_i}$ coincide [11],[4]. So condition (13) is implied by the existence of a vector $\hat{u}_i$ with $||\hat{u}_i|| = 1$ such that

$$\sum_i \hat{u}_i^T H_i(\hat{\varphi}_i)\hat{u}_i < 0 \quad (14)$$

Note that the corresponding sufficient condition based on the Morse property in [4] had the form $\hat{u}_i^T H_i(\hat{\varphi}_i)\hat{u}_i < 0$ for *all* $i \in \mathcal{N}$. From Proposition 1 we know that at a critical point of $\hat{\varphi}_i$, we have $g_r^i \leq \epsilon$ for at least one relation of agent i. With a slight abuse of notation, we will denote $g_r^i = g_i$ in the sequel for brevity. Consider now $\hat{u} \triangleq \{\frac{\nabla_1 b_1(q_c)^\perp}{\|\nabla_1 b_1(q_c)^\perp\|}, \ldots, \frac{\nabla_N b_N(q_c)^\perp}{\|\nabla_N b_N(q_c)^\perp\|}\}$ and $\hat{u}_i \triangleq \frac{\nabla_i b_i(q_c)^\perp}{\|\nabla_i b_i(q_c)^\perp\|}$, where $q_c \in C_{\hat{\varphi}_i}$, and $C_{\hat{\varphi}_i}$ is the set of critical points of $\hat{\varphi}_i$. By its definition $\hat{u}_i$ is orthogonal to $\nabla_i b_i$ at a critical point q_c, and so $\hat{u}_i^T \cdot \nabla_i b_i = 0$ and $\nabla_i b_i^T \cdot \hat{u}_i = 0$.

We can now use similar calculations to the ones used in the proof of Lemma 5 in [4] to derive the following expression:

$$\sum_i \hat{u}_i^T H_i(\hat{\varphi}_i)\hat{u}_i = \sum_i \frac{\gamma_{di}^{k-1}}{G_i^2}\{\bar{g}_i c_i \mu_i + g_i(\gamma_{di}\eta_i - \gamma_{di}\xi_i + \frac{\nabla_i \bar{g}_i^T \nabla_i \gamma_{di}}{2} - \sigma_i)\} \quad (15)$$

where $\mu_i = \frac{1}{2}\nabla_i b_i^T \nabla_i \gamma_{di} - \upsilon_i \gamma_{di}$,$\upsilon_i = 2|P_r| > 2$,$c_i = 1 + \frac{\lambda}{b_i + \tilde{b}_i^{1/h}}$,$\xi_i = \hat{u}_i^T \cdot \nabla_i^2 \bar{g}_i \hat{u}_i + \frac{\bar{g}_i}{c_i} \cdot \hat{u}_i^T A_i \hat{u}_i - 2\frac{\lambda}{c_i\left(b_i + \tilde{b}_i^{1/h}\right)^2}\hat{u}_i^T \nabla_i \tilde{b}_i^{1/h} \nabla_i \bar{g}_i \hat{u}_i$,

$$\eta_i = \left(1 - \frac{1}{k}\right)\left[\begin{array}{c} \frac{\hat{u}_i^T \nabla_i \bar{g}_i \nabla_i \bar{g}_i^T \hat{u}_i}{\bar{g}_i} - 2\lambda \frac{\hat{u}_i^T \nabla_i \bar{g}_i \left(\nabla_i \tilde{b}_i^{1/h}\right)^T \hat{u}_i}{c_i\left(b_i + \tilde{b}_i^{1/h}\right)^2} \\ +\lambda^2 \bar{g}_i \frac{\hat{u}_i^T \nabla_i \tilde{b}_i^{1/h} \left(\nabla_i \tilde{b}_i^{1/h}\right)^T \hat{u}_i}{c_i^2\left(b_i + \tilde{b}_i^{1/h}\right)^4} \end{array}\right],$$

$$\sigma_i = \frac{\lambda \bar{g}_i}{2c_i\left(b_i + \tilde{b}_i^{1/h}\right)^2}\left(\nabla_i b_i + \nabla_i \tilde{b}_i^{1/h}\right)^T \nabla_i \gamma_{di},$$

and $A_i = \lambda\left[\begin{array}{c} 2\frac{\left(\nabla_i b_i + \nabla_i \tilde{b}_i^{1/h}\right)\left(\nabla_i b_i + \nabla_i \tilde{b}_i^{1/h}\right)^T}{\left(b_i + \tilde{b}_i^{1/h}\right)^3} - \\ -\frac{\left(\nabla_i^2 b_i + \nabla_i^2 \tilde{b}_i^{1/h}\right)}{\left(b_i + \tilde{b}_i^{1/h}\right)^2} \end{array}\right]$.

Note that the second term in the parenthesis in (15) can be made arbitrarily small by a small choice of ϵ but can still be positive, so the first term should be strictly negative. In particular, the condition

$$\sum_i \frac{\gamma_{di}^{k-1}}{G_i^2}\bar{g}_i c_i(\frac{1}{2}\nabla_i b_i^T \nabla_i \gamma_{di} - \upsilon_i \gamma_{di}) < 0 \quad (16)$$

is a sufficient condition for (14) to hold. Note that $\upsilon_i > 2$ and $\gamma_{di} > \gamma_{\min}$. Moreover, for $0 \leq g_i \leq \epsilon$ we have $0 \leq b_i = \sum_{j \in P_r} \beta_{ij} \leq \epsilon$ and thus $0 \leq \beta_{ij} \leq \epsilon$ for all $j \in P_r$, for the particular relation r with respect to agent i. We then have

$$\begin{aligned} ||\nabla_i b_i|| = ||2\sum_{j \in P_r}(q_i - q_j)|| &\leq 2\sum_{j \in P_r}||(q_i - q_j)|| \\ &\leq 2\sum_{j \in P_r}\sqrt{\epsilon + (\varrho_i + \varrho_j)^2} \end{aligned}$$

Moreover $||\nabla_i \gamma_{di}|| = 2\sqrt{\gamma_{di}}$ and we shall use the notation $M_i = \frac{\gamma_{di}^{k-1}}{G_i^2}\bar{g}_i c_i \sqrt{\gamma_{di}}$ in the sequel. It can easily be seen that the sufficient condition (16) is now implied by

$$\sum_i M_i \sum_{j \in P_r}\sqrt{\epsilon + (\varrho_i + \varrho_j)^2} < \sum_i M_i \sqrt{\gamma_{\min}} \quad (17)$$

which is in turn implied by

$$\max_i\{\sum_{j \in P_r}\sqrt{\epsilon + (\varrho_i + \varrho_j)^2}\} < \sqrt{\gamma_{\min}} \quad (18)$$

For the case of common equal radii ϱ for all agents, the above is simplified to $\max_i\{\sum_{j \in P_r}\sqrt{\epsilon + 4\varrho^2}\} < \sqrt{\gamma_{\min}}$. Since the maximum number of binary relations in a relation can be equal to 6 in the case of decentralized navigation functions, the above is implied by

$$\gamma_{\min} > 36(\epsilon + 4\varrho^2) \quad (19)$$

We now use the argument of [14] mentioning that since (3) is satisfied exactly at the critical points, it is satisfied also in an arbitrary small neighborhood around them. From the primal Lyapunov analysis, we know that indeed the system converges to an arbitrarily small neighborhood of the critical points. The dual Lyapunov analysis guarantees that the attractors of the undesirable critical points are sets of measure zero. The following then holds:

Proposition 5: Consider the system (1) with the control law (8). Assume that the assumptions of Propositions 1,3 and (19) or (18) hold. Then for almost all initial conditions the closed loop system (1), (8) converges to the set $\gamma_{di} \leq \gamma_{\min}$ for all $i \in \mathcal{N}$.

The latter along with the fact that the trajectories of (5) are bounded by the trajectories of (8) for appropriate tuning of μ, guarantees that the above Proposition holds also for the closed loop system (1), (5).

Note that for point agents, (19) becomes

$$\gamma_{\min} > 36\epsilon \quad (20)$$

This establishes that for point agents, convergence to an arbitrarily small neighborhood around the destination points is guaranteed. The radius of this neighborhood becomes larger as the radii of the agents increase, as per (19),(18).

VII. SIMULATIONS

The derived results are now supported through a computer simulation using different common radii for four agents in the same scenario. In particular, in both scenarios the initial and desired destinations of the four agents are identical.

In particular, for the first simulation we use the scenario for four agents in [6]. In the first case of Fig. 1, the initial position and desired destination of agent $i, i = 1,2,3,4$ are denoted by $A - i$,$T - i$ respectively. We can see that the final configuration of the agents is not close enough to their desired trajectory, i.e., $\gamma_{\min}$ is quite large. From the previous derivations, we expect that $\gamma_{\min}$ will be smaller for smaller agent radii. Indeed, this case is depicted in Fig. 2, where we have decreased the radii of the agents in half. We can see

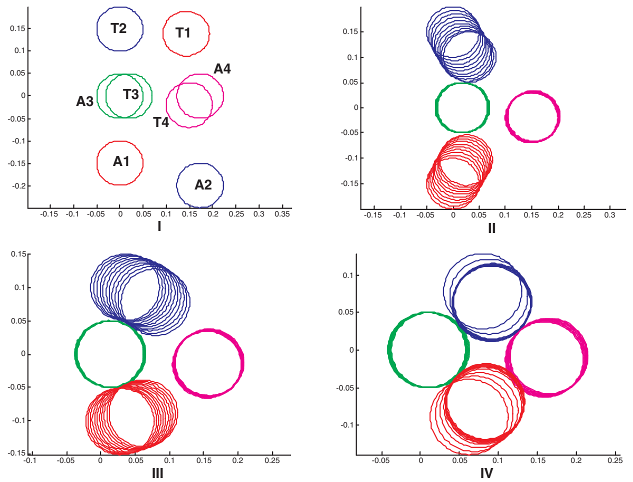

Fig. 1. Simulation 1, depicted from [6]. Agents fail to converge to their desired destinations.

now that the agents converge to a smaller region around the target points, i.e., $\gamma_{\min}$ is smaller, as expected by (19).

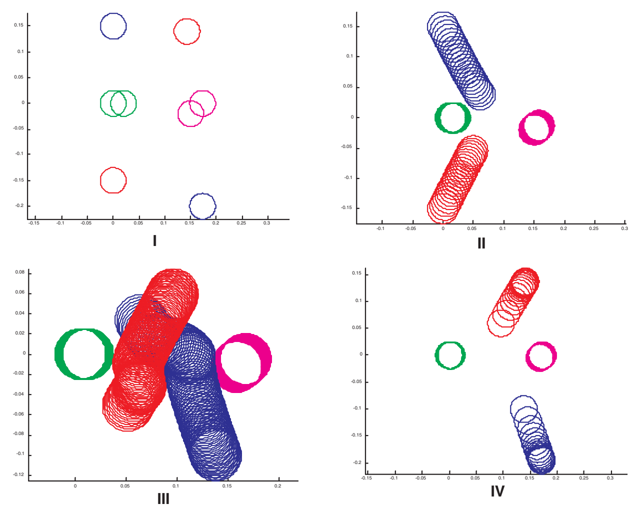

Fig. 2. The scenario of Simulation 1, [6] is recapped for smaller radii. Agents converge to a smaller region around their target points.

VIII. Conclusions

A combination of dual Lyapunov analysis and properties of decentralized navigation function based controllers were used to check the stability of a class of decentralized controllers for navigation and collision avoidance in multi-agent systems. The derived results yield a less conservative condition which relates to the negativity of the sum of the minimum eigenvalues of the Hessian matrices at the critical points, instead of requiring each of the eigenvalue to be negative itself. This provides an improved characterization of the reachable set of this certain class of decentralized navigation function based controllers, which is less conservative than the previous results for the same class of controllers.

Future research involves applying the decentralized navigation functions' framework to the design of [18].

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