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On an Operator-Pencil Approach to Distributed Control of Heterogeneous Systems

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***Abstract*—In this paper we consider spatially distributed heterogeneous discrete-time systems which are interconnected over an infinite lattice. An operator-pencil approach is employed to develop analysis conditions, which are less conservative than those previously available. Synthesis conditions are also obtained and are in the form of operator inequalities. In general these are infinite dimensional but in the case of eventually invariant systems these reduce to a semidefinite program.**

I. Introduction

The recent past has seen a considerable push in the study of distributed control, for instance [1]–[7]. In particular the distributed control over infinite lattice systems have been studied in [1]–[4]. In this formulation distributed controllers are sought that inherit the interconnection topology of the plant. Typical systems that can fit this framework are lumped approximations of partial differential equations.

In the current paper we will build upon the tools and concepts that were developed in [2] for infinite heterogeneous lattice systems. We use an operator-pencil approach to represent the system dynamics and obtain stability conditions in the form of a generalized lyapunov inequality coupled with an inertia condition. A Kalman-Yakubovic-Popov(KYP)-type lemma is also presented to incorporate performance conditions in addition to stability. We approach the controller synthesis problem through the use of an elimination lemma presented in [8]. This lemma was also used in a similar context in [9], however in this paper we adapt the elimination lemma to be applicable to infinite systems. We also show that the inertia conditions obtained in the analysis step conform well with the hypothesis of the elimination lemma and no further restrictions are required. Based on the inertia conditions we develop the constraints required for the dimensions of the controller. In general these constraints were found to be tighter than [2].

Although the methods discussed in this paper can be extended to distributed lattice systems of higher dimensions, for simplicity we will restrict ourselves mainly to the case of a one dimensional lattice. Also using the ideas in [10], we can adapt this work to be applicable for infinite graphs. In [9] the authors have used dissipativity theory to develop analysis conditions for system distributed over a finite graph. We point here that in this work when we restrict the dimensions of the lattice to be finite, the analysis results thus obtained are equivalent to that of [9].

Lastly we discuss eventually invariant systems which are spatially invariant throughout, except over a finite set of indices. For such systems we argue that, given scaling matrices exist that satisfy the stability conditions, we can always find an eventually invariant version of the same. This would also lead to eventually invariant controllers which would require us to solve only a finite number of LMIs.

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II. Preliminaries

In this section we will familiarize ourselves with the mathematical notations used in the paper. We will denote the set of real numbers, integers, positive integers, non-negative integers by $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{N}$ and $\mathbb{N}_0$ respectively. For a symmetric matrix H, we define its inertia as the triplet $\mathrm{in}(H) = (\mathrm{in}_+(H), \mathrm{in}_0(H), \mathrm{in}_-(H))$ which respectively correspond to the number of positive, zero and negative eigenvalues of H. For an $n \times n$ symmetric matrix H and $n \times m$ matrix R following is a standard result,

$$\mathrm{in}_+(H) \geq \mathrm{in}_+(R^*HR),\ \mathrm{in}_-(H) \geq \mathrm{in}_-(R^*HR) \quad (1)$$

Given a Hilbert space $\mathcal{Y}$ we denote its associated norm by $\|\cdot\|_{\mathcal{Y}}$ and its inner product by $\langle\cdot,\cdot\rangle_{\mathcal{Y}}$. The set of all bounded linear operators mapping Hilbert spaces $\mathcal{Y}$ to $\mathcal{Z}$ is denoted by $\mathcal{L}(\mathcal{Y},\mathcal{Z})$. When the two spaces are same we abbreviate this as $\mathcal{L}(\mathcal{Y})$. The induced norm of an operator in $\mathcal{L}(\mathcal{Y},\mathcal{Z})$ is denoted by $\|\cdot\|_{\mathcal{Y}\to\mathcal{Z}}$. For convenience we will suppress the subscript when it is obvious. The adjoint of X is written as X^*. An operator X is *coercive* if there exists an $\alpha > 0$ such that $\|Xu\|_{\mathcal{Z}} \geq \alpha\|u\|_{\mathcal{Y}}$ holds for all u in $\mathcal{Y}$. A self adjoint operator $X \in \mathcal{L}(\mathcal{Y})$ is *negative definite* if there exists an $\alpha > 0$ such that $\langle u, Xu\rangle_{\mathcal{Y}} < -\alpha\|u\|^2_{\mathcal{Y}}$ holds for all non-zero $u \in \mathcal{Y}$. It is denoted by $X \prec 0$. The direct sum of two Hilbert spaces $\mathcal{Y}$ and $\mathcal{Z}$ is denoted by $\mathcal{Y} \oplus \mathcal{Z}$.

We will abbreviate $(t,k) \in \mathbb{Z}^2$ as $\bar{k}$. Suppose we have the sequence $\bar{n}(\bar{k})$ mapping $\mathbb{Z}^2$ to $\mathbb{N}_0$, we define $\ell(\mathbb{Z}^2, \{\mathbb{R}^{\bar{n}(\bar{k})}\})$ (or ℓ for short) to be the vector space of mappings w which satisfy $w : \bar{k} \in \mathbb{Z}^2 \mapsto w(\bar{k}) \in \mathbb{R}^{\bar{n}(\bar{k})}$. We will use $\ell_2(\mathbb{Z}^2, \{\mathbb{R}^{\bar{n}(\bar{k})}\})$ to be the subspace of ℓ which is a Hilbert space under the norm $\|w\|_2 = \left(\sum_{\bar{k}\in\mathbb{Z}^2} |w(\bar{k})|_2^2\right)^{1/2}$ where $|\cdot|_2$ is the Euclidean norm. Further, $\ell_{2e}(\mathbb{Z}^2, \{\mathbb{R}^{\bar{n}(\bar{k})}\})$ is used to denote the subspace of ℓ satisfying for each fixed $t \in \mathbb{Z}$ the inequality $\sum_{k\in\mathbb{Z}} |w(t,k)|_2^2 < \infty$.

We will now present some operator theoretic representations which will enable us to compactly represent the distributed systems. Let $\bar{n}$ and $\bar{v}$ be sequences mapping $\mathbb{Z}^2$ to $\mathbb{N}_0$. A linear operator Q mapping $\ell_2(\mathbb{Z}^2, \{\mathbb{R}^{\bar{n}(\bar{k})}\})$ to $\ell_2(\mathbb{Z}^2, \{\mathbb{R}^{\bar{v}(\bar{k})}\})$ is said to be a *hyperdiagonal* operator if there exists a uniformly bounded sequence of matrices $Q(\bar{k}) \in \mathbb{R}^{\bar{n}(\bar{k})\times\bar{v}(\bar{k})}$ such that the equality $(Qw)(\bar{k}) = Q(\bar{k})w(\bar{k})$ holds for each $\bar{k} \in \mathbb{Z}^2$. For a self-adjoint hyperdiagonal operator Q we define its inertia as the mapping, $\mathrm{In}(Q) : \mathbb{Z}^2 \to \mathbb{N}_0^3$ defined by $\mathrm{In}(Q)(\bar{k}) := \mathrm{in}(Q(\bar{k}))$. Similarly the positive and negative inertias of the operator are defined by $\mathrm{In}_+(Q)(\bar{k}) := \mathrm{in}_+(Q(\bar{k}))$ and $\mathrm{In}_-(Q)(\bar{k}) := \mathrm{in}_-(Q(\bar{k}))$ respectively. Following is a generalization of (1) to hyperdiagonal operators.

Proposition 1: Suppose Q and M are hyperdiagonal operators, with Q self-adjoint. Then $\mathrm{In}_+(Q) \geq \mathrm{In}_+(M^*QM)$ and $\mathrm{In}_-(Q) \geq \mathrm{In}_-(M^*QM)$.

We now consider partitioned operators mapping the spaces $\ell_2(\mathbb{Z}^2, \{\mathbb{R}^{\bar{v}_1(\bar{k})}\}) \oplus \ell_2(\mathbb{Z}^2, \{\mathbb{R}^{\bar{v}_2(\bar{k})}\})$ to $\ell_2(\mathbb{Z}^2, \{\mathbb{R}^{\bar{q}_1(\bar{k})}\}) \oplus \ell_2(\mathbb{Z}^2, \{\mathbb{R}^{\bar{q}_2(\bar{k})}\})$. Let $W = \begin{bmatrix} H & P \\ G & J \end{bmatrix}$ be such an operator. We say that W is a *partitioned hyperdiagonal* operator if the constituent operators H, P, G, and J are hyperdiagonal. Given a hyperdiagonal operator W, we define its *hyperdiagonal representation* $[\![W]\!] : \ell_2(\mathbb{Z}^2, \{\mathbb{R}^{\bar{v}(\bar{k})}\}) \to \ell_2(\mathbb{Z}^2, \{\mathbb{R}^{\bar{q}(\bar{k})}\})$, (where $\bar{v} = \bar{v}_1 + \bar{v}_2$ and $\bar{q} = \bar{q}_1 + \bar{q}_2$) as the hyperdiagonal operator given by

$$\left([\![W]\!]x\right)(\bar{k}) := \begin{bmatrix} H(\bar{k}) & P(\bar{k}) \\ G(\bar{k}) & J(\bar{k}) \end{bmatrix} x(\bar{k})$$

Clearly these concepts generalize to arbitrary number of partitions. We will denote the set of all such partitioned hyperdiagonal operators as $\mathcal{P}(\bar{v}, \bar{q})$. When the partition dimensions are not important we will use the abbreviation $\mathcal{P}$.

III. System Model and Representation

We consider a spatially 1-dimensional lattice system $\mathcal{G}$, having the following representation (introduced in [1], [2], [11])

$$\begin{bmatrix} x_0(t+1,k) \\ x_+(t,k+1) \\ x_-(t,k-1) \end{bmatrix} = [\![A]\!](\bar{k})x(\bar{k}) + [\![B]\!](\bar{k}) \begin{bmatrix} w(\bar{k}) \\ u(\bar{k}) \end{bmatrix}$$
$$\begin{bmatrix} z(\bar{k}) \\ y(\bar{k}) \end{bmatrix} = [\![C]\!](\bar{k})x(\bar{k}) + [\![D]\!](\bar{k}) \begin{bmatrix} w(\bar{k}) \\ u(\bar{k}) \end{bmatrix}, \quad (2)$$

where

$$A = \begin{bmatrix} A_{00} & A_{0+} & A_{0-} \\ A_{+0} & A_{++} & A_{+-} \\ A_{-0} & A_{-+} & A_{--} \end{bmatrix},\ B = \begin{bmatrix} B_{w0} & B_{u0} \\ B_{w+} & B_{u+} \\ B_{w-} & B_{u-} \end{bmatrix},\ C = \begin{bmatrix} C_{z0} & C_{z+} & C_{z-} \\ C_{y0} & C_{y+} & C_{y-} \end{bmatrix},$$
$$D = \begin{bmatrix} D_{zw} & D_{zu} \\ D_{yw} & D_{yu} \end{bmatrix},\ x(\bar{k}) = \begin{bmatrix} x_0(\bar{k})^T & x_+(\bar{k})^T & x_-(\bar{k})^T \end{bmatrix}^T.$$

We combine the spatially shifted component of x as $x_1(\bar{k}) = \begin{bmatrix} x_+(\bar{k})^T & x_-(\bar{k})^T \end{bmatrix}^T$. Let us denote the sequences corresponding to the dimensions of $x(\bar{k})$, $x_0(\bar{k})$, $x_1(\bar{k})$, $x_+(\bar{k})$ and $x_-(\bar{k})$, as $\bar{n}(\bar{k})$, $\bar{n}_0(\bar{k})$, $\bar{n}_1(\bar{k})$, $\bar{n}_+(\bar{k})$ and $\bar{n}_-(\bar{k})$ respectively so that $\bar{n}_+ + \bar{n}_- = \bar{n}_1$ and $\bar{n}_0 + \bar{n}_1 = \bar{n}$. The dimensions of the inputs $w(\bar{k})$, $u(\bar{k})$ and outputs $z(\bar{k})$, $y(\bar{k})$ are given by sequences $\bar{n}_w(\bar{k})$, $\bar{n}_u(\bar{k})$, $\bar{n}_z(\bar{k})$ and $\bar{n}_y(\bar{k})$ respectively.

We define the following matrices which will be used later to define the pencil based system description

$$E = \left[\begin{array}{c|cc} I & 0 & 0 \\ \hline 0 & I & 0 \\ A_{-0} & A_{-+} & A_{--} \end{array}\right], B_E = \left[\begin{array}{cc} 0 & 0 \\ \hline 0 & 0 \\ B_{w-} & B_{u-} \end{array}\right] \quad (3)$$
$$F = \left[\begin{array}{c|cc} A_{00} & A_{0+} & A_{0-} \\ \hline A_{+0} & A_{++} & A_{+-} \\ 0 & 0 & I \end{array}\right], B_F = \left[\begin{array}{cc} B_{w0} & B_{u0} \\ \hline B_{w+} & B_{u+} \\ 0 & 0 \end{array}\right]$$

Let us also define the sequences $\bar{n}_E(\bar{k})$ and $\bar{n}_F(\bar{k})$ which correspond to the output dimensions of $[\![E]\!](\bar{k})$ and $[\![F]\!](\bar{k})$. It can be seen that $\bar{n}_E(\bar{k}) = \bar{n}_0(\bar{k}) + \bar{n}_+(\bar{k}) + \bar{n}_-(t, k-1)$ and $\bar{n}_F(\bar{k}) = \bar{n}_0(t+1, k) + \bar{n}_+(t, k+1) + \bar{n}_-(\bar{k})$.

We define the temporal shift operator, $S_0 : \ell(\mathbb{Z}^2, \{\mathbb{R}^{\bar{q}(\bar{k})}\} \to \ell(\mathbb{Z}^2, \{\mathbb{R}^{\bar{q}_0(\bar{k})}\})$ by $(S_0 v)(\bar{k}) = v(t-1, k)$ and spatial shift operator, $S_1 : \ell(\mathbb{Z}^2, \{\mathbb{R}^{\bar{q}(\bar{k})}\}) \to \ell(\mathbb{Z}^2, \{\mathbb{R}^{\bar{q}_1(\bar{k})}\})$ by $(S_1 v)(\bar{k}) = v(t, k-1)$, where $\bar{q}_0(t+1, k) = \bar{q}(t, k)$ and $\bar{q}_1(t, k+1) = \bar{q}(t, k)$ are some predefined sequences. These shifts are also invertible, so we can similarly write $(S_0^{-1}v)(\bar{k}) = v(t+1, k)$ and $(S_1^{-1}v)(\bar{k}) = v(t, k+1)$. We also note that the shift operators are unitary. We can now compactly write (2) in an operator form as

$$Ex = \mathbf{\Lambda} F x + (\mathbf{\Lambda} B_F - B_E) \begin{bmatrix} w \\ u \end{bmatrix} \quad (4)$$
$$\begin{bmatrix} z \\ y \end{bmatrix} = Cx + D \begin{bmatrix} w \\ u \end{bmatrix},$$

where E, F, B_F, B_E, C and D are hyperdiagonal operators defined in (2)-(3) and $\mathbf{\Lambda} := \mathrm{diag}(S_0, S_1)$ is a composite shift operator with compatible partitioning. The operator equations in (4) can be interpreted by expanding them point-wise at $\bar{k}$ and using the properties of hyperdiagonal operators (for example $(Ex)(\bar{k}) = [\![E]\!](\bar{k})x(\bar{k})$) and the shift operator.

Assuming $(E - \mathbf{\Lambda} F)$ has an algebraic inverse, the input to output mapping is given by

$$G = -C(E - \mathbf{\Lambda} F)^{-1}(B_E - \mathbf{\Lambda} B_F) + D \quad (5)$$

The above description contains the operator $(E - \mathbf{\Lambda} F)$ which we call an operator pencil[1].

IV. Analysis

In this section we will discuss the conditions for well-posedness and stability for the system and develop a version of KYP lemma required for controller synthesis.

Definition 2: A system of the form (2) is said to be *well-posed* if, given inputs $w, u \in \ell_{2e}$, equations (2) admit unique solutions $x_0, x_1 \in \ell_{2e}$ and the corresponding mappings are causal.

Moreover from (4) it is clear that the system is well-posed only if the operator $E - \mathbf{\Lambda} F$ has an algebraic inverse on $\ell_{2e} \oplus \ell_{2e}$.

Definition 3: A system of the form (2) is said to be *stable* if it is well-posed and, given inputs $w, u \in \ell_2$, equations (2) admit unique solutions x_0, x_1 in ℓ_2 with corresponding mappings being causal.

If we define a partitioned hyperdiagonal operator $X = \mathrm{diag}(X_0, X_1)$, then for a compatibly partitioned $\mathbf{\Lambda}$, we have $\mathbf{\Lambda}^* X \mathbf{\Lambda}$ and $\mathbf{\Lambda} X \mathbf{\Lambda}^*$ in $\mathcal{P}$. Further $\mathbf{\Lambda}^* X \mathbf{\Lambda} = \mathrm{diag}(S_0^* X_0 S_0, S_1^* X_1 S_1)$. We define the set of invertible operators $\mathcal{X} \subset \mathcal{P}$ to be the self-adjoint, partitioned hyperdiagonal operators of the form

$$\mathcal{X} = \{X \in \mathcal{P}(\bar{n}_E) : X = \mathrm{diag}(X_0, X_1), X = X^*,$$
$$X^{-1} \in \mathcal{L}(\ell_2 \oplus \ell_2),\ X_0 \succ 0\} \quad (6)$$

Here the partitioning of $[\![X]\!](\bar{k})$ is done such that $X_0(\bar{k})$ and $[\![X_1]\!](\bar{k})$ have dimensions $\bar{n}_0(\bar{k})$ and $(\bar{n}_+(\bar{k}) + \bar{n}_-(t, k-1))$ respectively.

We now develop a stability result for systems described by equations of the form (4). It gives us sufficient conditions under which the operator $(E - \mathbf{\Lambda} F)$ is invertible on $\ell_2 \oplus \ell_2$. But first we present the following intermediate Lemmas.

[1]strictly speaking the operator pencil is the affine map $E - \lambda \Lambda F$ of the complex variable λ; our terminology is motivated by the fact that, by exploiting the properties of the shift and hyperdiagonal operators, it is possible to show that for each λ on the unit circle the spectrum of the image operator is equal to that of $E - \Lambda F$.

Lemma 4: Given sequences $\bar{n}_X$, $\bar{n}_Y$, $\bar{n}$ and hyperdiagonal operators $E \in \mathcal{P}(\bar{n}_X, \bar{n})$ and $F \in \mathcal{P}(\bar{n}_Y, \bar{n})$, suppose there exists self-adjoint hyperdiagonal operators, X and Y satisfying the inequality

$$E^*XE - F^*YF \succ 0 \tag{7}$$

and inertia condition $\mathrm{In}_+(X) + \mathrm{In}_-(Y) = \bar{n}$, then there exists a self-adjoint operator $Z \in \mathcal{P}(\bar{n})$ satisfying $E^*XE - Z \succ 0$ and $Z - F^*YF \succ 0$. Further Z satisfies $\mathrm{In}_+(Z) = \mathrm{In}_+(X)$ and $\mathrm{In}_-(Z) = \mathrm{In}_-(Y)$.

The proof is skipped and will appear in the journal version.

Lemma 5: Given operators E, F in $\mathcal{L}(\mathcal{Y}, \mathcal{Z})$, if there exists a self-adjoint $X \in \mathcal{L}(\mathcal{Z})$ satisfying $E^*XE - F^*XF \succ 0$, then the operator $(E - F)$ is coercive.

Proof follows from the equality $E^*XE - F^*XF = E^*X(E - F) + (E-F)^*XE - (E-F)^*X(E-F)$.

Lemma 6: Given hyperdiagonal operators $E \in \mathcal{P}(\bar{n}_E, \bar{n})$ and $F \in \mathcal{P}(\bar{n}_F, \bar{n})$, if there exists a $X \in \mathcal{X}$ satisfying the inequality

$$E^*XE - F^*\mathbf{\Lambda}^*X\mathbf{\Lambda}F \succ 0 \tag{8}$$

and the inertia condition $\mathrm{In}_+(X) + \mathrm{In}_-(\mathbf{\Lambda}^*X\mathbf{\Lambda}) = \bar{n}$ then $(E - \mathbf{\Lambda}F)$ is invertible on $\ell_2 \oplus \ell_2$.

Proof: Using Lemma 5, the inequality (8) directly gives us that $(E - \mathbf{\Lambda}F)$ is coercive. Now using Lemma 4 we know that there exists a $Z \in \mathcal{P}(\bar{n})$ satisfying

$$E^*XE - Z \succ 0, \quad Z - F^*\mathbf{\Lambda}^*X\mathbf{\Lambda}F \succ 0 \tag{9}$$

Now applying the Schur's complement formula to each of the above inequalities we obtain

$$\mathrm{In}(EZ^{-1}E^* - X^{-1}) = \mathrm{In}(E^*XE - Z) + \mathrm{In}(-X) - \mathrm{In}(-Z)$$

$$\mathrm{In}(\mathbf{\Lambda}^*X^{-1}\mathbf{\Lambda} - FZ^{-1}F^*) = \mathrm{In}(Z - F^*\mathbf{\Lambda}^*X\mathbf{\Lambda}F) + \mathrm{In}(\mathbf{\Lambda}^*X\mathbf{\Lambda}) - \mathrm{In}(Z)$$

Here we have used the fact that $\mathrm{In}(X) = \mathrm{In}(X^{-1})$ and $\mathrm{In}(\mathbf{\Lambda}^*X\mathbf{\Lambda}) = \mathrm{In}(\mathbf{\Lambda}^*X^{-1}\mathbf{\Lambda})$. Since Z satisfies $\mathrm{In}_+(Z) = \mathrm{In}_+(X)$ and $\mathrm{In}_-(Z) = \mathrm{In}_-(\mathbf{\Lambda}^*X\mathbf{\Lambda})$ we have both $\{\mathrm{In}(-X) - \mathrm{In}(-Z)\}$ and $\{\mathrm{In}(\mathbf{\Lambda}^*X\mathbf{\Lambda}) - \mathrm{In}(Z)\}$ with strictly positive inertias. Also using (9) we have the inertias of $(EZ^{-1}E^* - X^{-1})$ and $(X^{-1} - \mathbf{\Lambda}FZ^{-1}F^*\mathbf{\Lambda}^*)$ to be strictly positive and hence

$$EZ^{-1}E^* - X^{-1} \succ 0, \qquad X^{-1} - \mathbf{\Lambda}FZ^{-1}F^*\mathbf{\Lambda}^* \succ 0$$

Adding the above inequalities, we have

$$EZ^{-1}E^* - \mathbf{\Lambda}FZ^{-1}F^*\mathbf{\Lambda}^* \succ 0$$

From the above inequality we obtain $(E^* - F^*\mathbf{\Lambda}^*)$ to be coercive and hence $(E - \mathbf{\Lambda}F)$ is invertible. ■

We will now apply the above lemma to the system in (2) where the corresponding E and F operators (defined using (3)) are structured. The advantage of using an operator pencil approach lies in the fact that we can choose the structure of $[\![X_1]\!](\bar{k})$ to be full-block as opposed to that of [2] where $[\![X_1]\!](\bar{k})$ had to be chosen block-diagonal. This difference is made possible mainly due to the structure of the composite shift operator $\mathbf{\Lambda} = \mathrm{diag}(S_0, S_1)$ used here, compared to $\mathrm{diag}(S_0, S_1, S_1^{-1})$ in [2].

For a multidimensional system of the form in (2) we can find a permutation operator P (a partitioned hyperdiagonal operator which is a sequence of permutation matrices $[\![P]\!](\bar{k})$) so that, $[\![\begin{smallmatrix} E \\ F \end{smallmatrix}]\!](\bar{k}) = [\![P]\!](\bar{k}) \begin{bmatrix} I \\ [\![A]\!](\bar{k}) \end{bmatrix}$.

We can thus have the following alternative form of (8)

$$\begin{bmatrix} I \\ A \end{bmatrix}^* P^* \begin{bmatrix} X & 0 \\ 0 & -\mathbf{\Lambda}^*X\mathbf{\Lambda} \end{bmatrix} P \begin{bmatrix} I \\ A \end{bmatrix} \succ 0 \tag{10}$$

The point-wise inequalities thus obtained are similar to the Lyapunov inequalities developed in [9] where systems over finite dimensional graphs are considered. In this regard we have Remark 7 below. But first we obtain the following inequality by applying Proposition 1 to (10)

$$\mathrm{In}_+(X) + \mathrm{In}_-(\mathbf{\Lambda}^*X\mathbf{\Lambda}) \geq \bar{n} \tag{11}$$

So the inertia condition in Lemma 6 ensures that the above holds with equality.

Remark 7: In the case of distributed systems with finite indices, we can eliminate the need of the inertia condition in Lemma 6 because it is satisfied by default when inequality (8) holds. This can be proved by showing that the sum of the left and right hand sides of (11) over all spatial indices are infact equal.

Here is a version of KYP lemma which adds a performance criteria to the earlier discussed concept of stability.

Lemma 8: Suppose $X \in \mathcal{X}$ and the inertia condition $\mathrm{In}_+(X) + \mathrm{In}_-(\mathbf{\Lambda}^*X\mathbf{\Lambda}) = \bar{n}$ is satisfied then the following inequality implies that the system $\mathcal{G}$ is stable and the mapping G in (5) is causal and strictly contractive:

$$\begin{bmatrix} E & B_E \\ 0 & I \end{bmatrix}^* \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} E & B_E \\ 0 & I \end{bmatrix} - \begin{bmatrix} F & B_F \\ C & D \end{bmatrix}^* \begin{bmatrix} \mathbf{\Lambda}^*X\mathbf{\Lambda} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} F & B_F \\ C & D \end{bmatrix} \succ 0 \tag{12}$$

The proof utilizes the result in Lemma 6 and follows arguments in [2, Lemma 14], but is skipped in interest of space. Note that for the proof of causality we require E, F and B to be structured as in (3) at least for the temporal update (i.e. first block row).

When we restrict ourselves to structured matrices as in (3) we have the following alternative form for (12):

$$\begin{bmatrix} I \\ \hline A \; B \\ C \; D \end{bmatrix} P^* \left[\begin{array}{cc|cc} X & & & \\ & I & & \\ \hline & & -\mathbf{\Lambda}^*X\mathbf{\Lambda} & \\ & & & -I \end{array}\right] P \begin{bmatrix} I \\ \hline A \; B \\ C \; D \end{bmatrix} \succ 0 \tag{13}$$

where P is a permutation operator (similar to the one in (10))

V. Synthesis

This section deals with the synthesis of distributed linear controllers and the general technique follows [12]–[14]. For the system $\mathcal{G}$, we define an *admissible* controller to be one which ensures that the closed-loop system is stable and achieves the performance criteria of $\|w \mapsto z\| < 1$. The hyperdiagonal system operators for the controller will be denoted by (A_K, B_K, C_K, D_K) and that of the closed loop system by (A_{cl}, B_{cl}, C_{cl}, D_{cl}). Defining $Q = \begin{bmatrix} A_K & B_K \\ C_K & D_K \end{bmatrix}$, we have the following relation

$$\begin{bmatrix} A_{cl} & B_{cl} \\ C_{cl} & D_{cl} \end{bmatrix} = \begin{bmatrix} A + B_uD_KC_y & B_uC_K & B_w + B_uD_KD_{yw} \\ B_KC_y & A_K & B_KD_{yw} \\ C_z + D_{zu}D_KC_y & D_{zu}C_K & D_{zw} + D_{zu}D_KD_{yw} \end{bmatrix} = R + U^*QV,$$

where

$$R = \begin{bmatrix} A & 0 & B_w \\ 0 & 0 & 0 \\ C_z & 0 & D_{zw} \end{bmatrix}, \; U = \begin{bmatrix} 0 & I & 0 \\ B_u^* & 0 & D_{zu}^* \end{bmatrix}, \; V = \begin{bmatrix} 0 & I & 0 \\ C_y & 0 & D_{yw} \end{bmatrix}.$$

In the above equation we have assumed $D_{yu} = 0$ in order to have the above affine relation with respect to Q. We will denote the state dimensions of the controller and closed loop system by the sequences $\bar{n}_K$ and $\bar{n}_{cl} = \bar{n} + \bar{n}_K$.

We can thus write (13) for the closed loop system as

$$\begin{bmatrix} I \\ R + U^*QV \end{bmatrix}^* P^* W_{cl} P \begin{bmatrix} I \\ R + U^*QV \end{bmatrix} \succ 0 \qquad (14)$$

where, $W_{cl} = \left[\begin{array}{cc|cc} X_{cl} & & & \\ & I & & \\ \hline & & -\mathbf{\Lambda}^* X_{cl} \mathbf{\Lambda} & \\ & & & -I \end{array}\right]$ and $X_{cl} \in \mathcal{X}_{cl}$ corresponds to the closed loop version of the set (6), with appropriate dimensions.

Following is an infinite dimensional extension to the Elimination lemma developed in [8] and forms the basis of the synthesis step.

Lemma 9: For sequences $\bar{n}$, $\bar{m}$, $\bar{p}$ and $\bar{q}$ suppose we have operators $R \in \mathcal{P}(\bar{n}, \bar{m})$, $U \in \mathcal{P}(\bar{p}, \bar{n})$, $V \in \mathcal{P}(\bar{q}, \bar{m})$, operators $U_\perp$ and $V_\perp$ satisfying $\text{Im}\left(\llbracket U_\perp \rrbracket(\bar{k})\right) = \text{Ker}\left(\llbracket U \rrbracket(\bar{k})\right)$ and $\text{Im}\left(\llbracket V_\perp \rrbracket(\bar{k})\right) = \text{Ker}\left(\llbracket V \rrbracket(\bar{k})\right)$ are coercive and $W \in \mathcal{P}$ is self-adjoint and invertible with $\text{In}_+(W)(\bar{k}) = \bar{m}(\bar{k})$ and $\text{In}_-(W)(\bar{k}) = \bar{n}(\bar{k})$, then there exists a partitioned hyperdiagonal operator $Q \in \mathcal{P}(\bar{p}, \bar{q})$ satisfying

$$\begin{bmatrix} I \\ R + U^*QV \end{bmatrix}^* W \begin{bmatrix} I \\ R + U^*QV \end{bmatrix} \prec 0 \qquad (15)$$

if and only if the following operator inequalities hold

$$V_\perp^* \begin{bmatrix} I \\ R \end{bmatrix}^* W \begin{bmatrix} I \\ R \end{bmatrix} V_\perp \prec 0, \;\; U_\perp^* \begin{bmatrix} -R^* \\ I \end{bmatrix}^* W^{-1} \begin{bmatrix} -R^* \\ I \end{bmatrix} U_\perp \succ 0. \quad (16)$$

Since we are dealing with infinite dimensional operators here, apart from proving the pointwise matrix inequalities in lines of [8], we also have to prove the existence of uniform bounds for the same. However we skip the proof of this lemma due to lack of space.

Let us denote the dimension of the closed loop state vectors and its components in the the same way as defined for the plant but with an additional subscript cl (as $\bar{n}_{cl}$, $\bar{n}_{cl,0}$, $\bar{n}_{cl,1}$, $\bar{n}_{cl,+}$ and $\bar{n}_{cl,-}$). In order to apply Lemma 9 to the closed loop system we need to ensure that the inertia of W_{cl} (in (14)) and the dimensions of R comply with each other according to the hypothesis of the Lemma. The inertia of W_{cl} is

$$\begin{aligned} \text{in}_+(\llbracket W_{cl} \rrbracket(\bar{k})) &= \text{in}_+(\llbracket X_{cl} \rrbracket(\bar{k})) + \text{in}_-(\llbracket \mathbf{\Lambda}^* X_{cl} \mathbf{\Lambda} \rrbracket(\bar{k})) + \bar{n}_w(\bar{k}) \\ &= \bar{n}_{cl}(\bar{k}) + \bar{n}_w(\bar{k}) \\ \text{in}_-(\llbracket W_{cl} \rrbracket(\bar{k})) &= \text{in}_-(\llbracket X_{cl} \rrbracket(\bar{k})) + \text{in}_+(\llbracket \mathbf{\Lambda}^* X_{cl} \mathbf{\Lambda} \rrbracket(\bar{k})) + \bar{n}_z(\bar{k}) \\ &= \bar{n}_{cl,+}(t, k+1) + \bar{n}_{cl,-}(t, k-1) + \bar{n}_z(\bar{k}) \end{aligned}$$

which are respectively equal to the column and row dimensions of $\llbracket R \rrbracket(\bar{k})$ and hence we can apply Lemma 9.

Since $V_\perp = \begin{bmatrix} \text{Ker}(C_y) \\ 0 \\ \text{Ker}(D_{yw}) \end{bmatrix}$ and $U_\perp = \begin{bmatrix} \text{Ker}(B_u^*) \\ 0 \\ \text{Ker}(D_{zu}^*) \end{bmatrix}$, inequalities (16) reduce to

$$\begin{bmatrix} \text{Ker}(C_y) \\ \text{Ker}(D_{yw}) \end{bmatrix}^* \begin{bmatrix} I \\ \begin{bmatrix} A & B_w \\ C_z & D_{zw} \end{bmatrix} \end{bmatrix}^* P^* W P \begin{bmatrix} I \\ \begin{bmatrix} A & B_w \\ C_1 & D_{zw} \end{bmatrix} \end{bmatrix} \begin{bmatrix} \text{Ker}(C_y) \\ \text{Ker}(D_{yw}) \end{bmatrix} \prec 0 \qquad (17)$$

$$\begin{bmatrix} \text{Ker}(B_u^*) \\ \text{Ker}(D_{zu}^*) \end{bmatrix}^* \begin{bmatrix} \begin{bmatrix} A & B_w \\ C_z & D_{zw} \end{bmatrix} \\ -I \end{bmatrix}^* P^* W^{-1} P \begin{bmatrix} \begin{bmatrix} A & B_w \\ C_z & D_{zw} \end{bmatrix} \\ -I \end{bmatrix} \begin{bmatrix} \text{Ker}(B_u^*) \\ \text{Ker}(D_{zu}^*) \end{bmatrix} \succ 0 \qquad (18)$$

where

$$W = \left[\begin{array}{cc|cc} X & & & \\ & I & & \\ \hline & & -\mathbf{\Lambda}^* X \mathbf{\Lambda} & \\ & & & -I \end{array}\right], \; W^{-1} = \left[\begin{array}{cc|cc} Y & & & \\ & I & & \\ \hline & & -\mathbf{\Lambda}^* Y \mathbf{\Lambda} & \\ & & & -I \end{array}\right]$$

are the sub-matrices of W_{cl} and W_{cl}^{-1} which are constructed by retaining only the sub-matrix X and Y (corresponding exclusively to the plant) of X_{cl} and X_{cl}^{-1} as shown below:

$$X_{cl} = \begin{bmatrix} X & X_{GK} \\ X_{GK}^* & X_K \end{bmatrix} \qquad X_{cl}^{-1} = \begin{bmatrix} Y & Y_{GK} \\ Y_{GK}^* & Y_K \end{bmatrix} \qquad (19)$$

The operator inequalities (17) and (18) are in fact sequences of LMIs in variables $X(\bar{k})$ and $Y(\bar{k})$. Solving this system of inequalities for X and Y, the next step involves the completion of the operator X_{cl}. To meet this end we invoke the following lemma proved in [2, Lemma 21].

Lemma 10: Given symmetric, nonsingular matrices X and Y with dimension η, and non-negative integers i_+, i_- and κ such that $i_+ + i_- = \eta + \kappa$, then there exists matrices $X_2, Y_2 \in \mathbb{R}^{\eta \times \kappa}$ and symmetric matrices X_3 , $Y_3 \in \mathbb{R}^{\kappa \times \kappa}$ satisfying $\begin{bmatrix} X & X_2 \\ X_2^* & X_3 \end{bmatrix}^{-1} = \begin{bmatrix} Y & Y_2 \\ Y_2^* & Y_3 \end{bmatrix}$ and $\text{in}\left(\begin{bmatrix} X & X_2 \\ X_2^* & X_3 \end{bmatrix}\right) = (i_+, 0, i_-)$ if and only if, $\text{in}_+\left(\begin{bmatrix} X & I \\ I & Y \end{bmatrix}\right) \le i_+$ and $\text{in}_-\left(\begin{bmatrix} X & I \\ I & Y \end{bmatrix}\right) \le i_-$.

The following lemma is a modified version of [2, Lemma 22], to serve the pencil setting. This lemma checks the feasibility of the controller dimensions along a spatial direction (with shift operator S and indexed by $k \in \mathbb{Z}$).

Lemma 11: Given sequences $\bar{n}_+$, $\bar{n}_-$, $\bar{n}$, $\bar{h}_+$, $\bar{h}_-$, $\bar{h}$, $\bar{\eta}$ and $\bar{\kappa}$ satisfying $\bar{n} = \bar{n}_+ + \bar{n}_-$, $\bar{h} = \bar{h}_+ + \bar{h}_-$, $\bar{\eta}(k) = n_+(k) + n_-(k-1)$ and $\bar{\kappa}(k) = h_+(k) + h_-(k-1)$, and hyperdiagonal operators X and Y in $\mathcal{P}(\bar{\eta})$, we can find hyperdiagonal operators X_2, Y_2 in $\mathcal{P}(\bar{\eta}, \bar{\kappa})$ and X_3, Y_3 in $\mathcal{P}(\bar{\kappa})$ satisfying $\begin{bmatrix} X & X_2 \\ X_2^* & X_3 \end{bmatrix}^{-1} = \begin{bmatrix} Y & Y_2 \\ Y_2^* & Y_3 \end{bmatrix}$ and

$$\text{In}_+\left(\left\llbracket \begin{matrix} X & X_2 \\ X_2^* & X_3 \end{matrix} \right\rrbracket\right) + \text{In}_-\left(S^* \left\llbracket \begin{matrix} X & X_2 \\ X_2^* & X_3 \end{matrix} \right\rrbracket S\right) = \bar{n} + \bar{h} \quad (20)$$

if and only if the following holds

$$\max_{j \in \mathbb{Z}} \left\{ \text{In}_+\left(\left\llbracket \begin{matrix} X & I \\ I & Y \end{matrix} \right\rrbracket\right)(j) - (\bar{n}_+ + \bar{h}_+)(j) \right\} + \max_{l \in \mathbb{Z}} \left\{ \text{In}_-\left(\left\llbracket \begin{matrix} X & I \\ I & Y \end{matrix} \right\rrbracket\right)(l) - (\bar{n}_- + \bar{h}_-)(l-1) \right\} \le 0. \quad (21)$$

Remark 12: In the earlier lemma, we can choose $\bar{h}_+$ and $\bar{h}_-$ sufficiently large so that (21) is always satisfied. One way to do so is to set the terms in the brackets in (21) to be independently less that or equal to zero. This yields the condition,

$$\bar{h}_+(k) \ge \bar{n}_+(k) + 2n_-(k-1), \quad \bar{h}_-(k) \ge \bar{n}_-(k) + 2n_+(k+1)$$

We now extend the earlier lemma to multi-dimensional setting.

Theorem 13: Given the system as in (2) and sequences $\bar{n}_{K0}, \bar{n}_{K+}, \bar{n}_{K-}$ corresponding to the controller dimensions, we can find an admissible controller with the specified dimensions if there exists $X = \text{diag}(X_0, X_1)$ and $Y = \text{diag}(Y_0, Y_1)$ in $\mathcal{X}$ satisfying inequalities (17), (18) and

$$\text{In}_+\left(\left\llbracket \begin{matrix} X_0 & I \\ I & Y_0 \end{matrix} \right\rrbracket\right) \le \bar{n}_0 + \bar{n}_{K0}, \;\; \text{In}_-\left(\left\llbracket \begin{matrix} X_0 & I \\ I & Y_0 \end{matrix} \right\rrbracket\right) = 0$$

$$\max_{j\in\mathbb{Z}}\left\{\mathrm{In}_+\left(\left[\!\left[\begin{matrix}X_1 & I\\ I & Y_1\end{matrix}\right]\!\right]\right)(j)-(\bar{n}_+ +\bar{n}_{K+})(j)\right\}+$$
$$\max_{l\in\mathbb{Z}}\left\{\mathrm{In}_-\left(\left[\!\left[\begin{matrix}X_1 & I\\ I & Y_1\end{matrix}\right]\!\right]\right)(l)-(\bar{n}_- +\bar{n}_{K-})(l-1)\right\}\leq 0$$

The proof uses the results in Lemma 11 applied to the spatial dimension but overall technique is similar to [2, Theorem 25].

VI. Eventually Invariant Systems

We define an eventually invariant operator as one which can be varying over a finite set of indices but eventually settles on to an invariant structure in both time and space. Eventually invariant systems which are defined by such operators are of particular interest because they lead to finitely representable controllers which can be solved by SDP. Particularly we would like to obtain eventually invariant scaling matrices that satisfy (12), given that some solution exists. However in contrast to [15] (where eventually time-periodic LTV systems are considered) doing so in the current work is difficult due to the additional inertia conditions which are not convex. For the sake of brevity we will prove the results only to incorporate stability, while noting that adding performance measure is a simple extension utilizing the Lemma 8.

Lemma 14: Suppose operators $E, F\in\mathcal{P}$ representing an eventually invariant 1-dimensional lattice system (structured as in (3)) have the spatially invariant components defined as

$$(E(\bar{k}),F(\bar{k}))=\begin{cases}(\bar{E}(t),\bar{F}(t)) \text{ for } k=-1,-2,\dots\\ (\hat{E}(t),\hat{F}(t)) \text{ for } k=N,N+1,\dots\end{cases}\quad(22)$$

and further for fixed k, $E(\bar{k})$ and $F(\bar{k})$ are invariant for $t\geq T$. If we have an $X\in\mathcal{X}$ satisfying the operator inequality $F^*\mathbf{\Lambda}^*X\mathbf{\Lambda}F-E^*XE\prec 0$, then there exists an eventually invariant $\tilde{X}\in\mathcal{X}$ which satisfies the inequality.

Proof: The proof is by construction. The earlier inequality implies the existence of an $\epsilon>0$ such that the following component-wise inequality is satisfied:

$$F(\bar{k})^*\begin{bmatrix}X_0(t+1,k) & \\ & X_1(t,k+1)\end{bmatrix}F(\bar{k})- E(\bar{k})^*\begin{bmatrix}X_0(\bar{k}) & \\ & X_1(\bar{k})\end{bmatrix}E(\bar{k})\prec-\epsilon I\quad(23)$$

By averaging the scaling matrices $X(\bar{k})$ for $k\geq N$ and $k<0$ separately, we can argue that there exists corresponding operators $\hat{X}$ and $\bar{X}$ which are invariant in k and satisfy

$$\hat{F}(t)^*\begin{bmatrix}\hat{X}_0(t+1) & \\ & \hat{X}_1(t)\end{bmatrix}\hat{F}(t)-\hat{E}(t)^*\begin{bmatrix}\hat{X}_0(t) & \\ & \hat{X}_1(t)\end{bmatrix}\hat{E}(t)\prec-\epsilon I$$
$$\bar{F}(t)^*\begin{bmatrix}\bar{X}_0(t+1) & \\ & \bar{X}_1(t)\end{bmatrix}\bar{F}(t)-\bar{E}(t)^*\begin{bmatrix}\bar{X}_0(t) & \\ & \bar{X}_1(t)\end{bmatrix}\bar{E}(t)\prec-\epsilon I\quad(24)$$

For any $\mu\geq 0$ we can take a weighted sum of (23) and (24) to have

$$\hat{F}(t)^*\begin{bmatrix}\hat{X}_{0\mu}(t+1,k) & \\ & \hat{X}_{1\mu}(t,k+1)\end{bmatrix}\hat{F}(t)- \hat{E}(t)^*\begin{bmatrix}\hat{X}_{0\mu}(\bar{k}) & \\ & \hat{X}_{1\mu}(\bar{k})\end{bmatrix}\hat{E}(t)\prec-\epsilon I\quad\text{for } k\geq N$$
$$\bar{F}(t)^*\begin{bmatrix}\bar{X}_{0\mu}(t+1,k) & \\ & \bar{X}_{1\mu}(t,k+1)\end{bmatrix}\bar{F}(t)- \bar{E}(t)^*\begin{bmatrix}\bar{X}_{0\mu}(\bar{k}) & \\ & \bar{X}_{1\mu}(\bar{k})\end{bmatrix}\bar{E}(t)\prec-\epsilon I\quad\text{for } k\leq -1\quad(25)$$

where $\hat{X}_{i\mu}(\bar{k})=\frac{1}{1+\mu}\left(X_i(\bar{k})+\mu\hat{X}_i(t)\right)$ and $\bar{X}_{i\mu}(\bar{k})=\frac{1}{1+\mu}\left(X_i(\bar{k})+\mu\bar{X}_i(t)\right)$ for $i=0,1$.

The inequalities (25) suggest that we can find $\xi>0$ and $0<\epsilon'\leq\epsilon$ such that for all $\mu\geq 0$ the following holds

$$\hat{F}(t)^*\begin{bmatrix}\hat{X}_{0\mu}(t+1,k) & \\ & \hat{X}_{1(\mu+\xi)}(t,k+1)\end{bmatrix}\hat{F}(t)- \hat{E}(t)^*\begin{bmatrix}\hat{X}_{0\mu}(\bar{k}) & \\ & \hat{X}_{1\mu}(\bar{k})\end{bmatrix}\hat{E}(t)\prec-\epsilon' I\quad\text{for } k\geq N\quad(26)$$
$$\bar{F}(t)^*\begin{bmatrix}\bar{X}_{0\mu}(t+1,k) & \\ & \bar{X}_{1(\mu-\xi)}(t,k+1)\end{bmatrix}\bar{F}(t)- \bar{E}(t)^*\begin{bmatrix}\bar{X}_{0\mu}(\bar{k}) & \\ & \bar{X}_{1\mu}(\bar{k})\end{bmatrix}\bar{E}(t)\prec-\epsilon' I\quad\text{for } k\leq -1$$

Here $\hat{X}_{1(\mu+\xi)}$ and $\bar{X}_{1(\mu+\xi)}$ are defined in the same way as $\hat{X}_{1\mu}$ and $\bar{X}_{1\mu}$ by replacing μ with $\mu+\xi$.

We define $\zeta(\bar{k})=\begin{cases}(k-N-1)\xi & \text{for } k>N\\ (-k+1)\xi & \text{for } k<-1\end{cases}$

using which we can define, $X_\zeta(\bar{k})=$

$$\begin{cases}\frac{1}{1+\zeta(\bar{k})}\left(X(\bar{k})+\zeta(\bar{k})\hat{X}(t)\right) & \text{for } k>N\\ \frac{1}{1+\zeta(\bar{k})}\left(X(\bar{k})+\zeta(\bar{k})\bar{X}(t)\right) & \text{for } k<-1\\ X(\bar{k}) & \text{otherwise}\end{cases}$$

In inequalities (26) if we substitute $\mu=\zeta(\bar{k})$ then we obtain the inequality $F^*\mathbf{\Lambda}^*X_\zeta\mathbf{\Lambda}F-E^*X_\zeta E\prec 0$. Clearly as $|k|$ tends to infinity $\hat{X}_{i\zeta}(\bar{k})$ and $\bar{X}_{i\zeta}(\bar{k})$ respectively tend to $\hat{X}_i(t)$ and $\bar{X}_i(t)$ for $i=1,2$. We can thus choose integers $N_1\geq N$ and $N_2\leq 0$ such that replacing $X_\zeta(\bar{k})$ with $\hat{X}(t)$ for $k\geq N_1$ and with $\bar{X}(t)$ for $k\leq N_2$, X_ζ still satisfies the preceding inequality. The operator X_ζ thus constructed is eventually invariant only in the spatial dimension. Since the system is also eventually invariant in time, by a similar argument we can also replace $X_\zeta(\bar{k})$ by time invariant matrices as t is sufficiently large. The resulting operator (heterogeneous over finite indices) which we constructed is the desired $\tilde{X}$. ■

Following lemma shows that under the given inertia conditions for scaling matrices, their inertias are uniquely determined by the system matrices in the spatially invariant region.

Lemma 15: Given operators E, F as defined in Lemma 14, if there exists an operator $X\in\mathcal{X}$ satisfying the inequality $F^*\mathbf{\Lambda}^*X\mathbf{\Lambda}F-E^*XE\prec 0$ and the inertia condition $\mathrm{In}_+(X)+\mathrm{In}_-(\mathbf{\Lambda}^*X\mathbf{\Lambda})=\bar{n}$, then at time t the inertia of $X_1(\bar{k})$ (and hence $X(\bar{k})$) is invariant over the indices $k\geq N$ and $k\leq -1$ and is completely determined by $\hat{E}(t),\hat{F}(t)$ and $\bar{E}(t),\bar{F}(t)$ respectively.

Proof: Since E, F is structured as in (3), we can have the following partitioning to separate the temporal and spatial components (i.e. $\bar{n}_0(\bar{k})$ and $\bar{n}_1(\bar{k})$)[2]

$$E(\bar{k})=\begin{bmatrix}I & 0\\ E_{10}(\bar{k}) & E_{11}(\bar{k})\end{bmatrix},\ F(\bar{k})=\begin{bmatrix}F_{00}(\bar{k}) & F_{01}(\bar{k})\\ F_{10}(\bar{k}) & F_{11}(\bar{k})\end{bmatrix}\quad(27)$$

Substituting the above into inequality (23) and considering the $(2,2)$ block

[2]Note that the second block row/column of the matrices $E(\bar{k})$ and $F(\bar{k})$ are further structured as in (3), but are not explicitly shown as it is not required for the proof

$$F_{01}(\bar{k})^* X_0(t+1,k) F_{01}(\bar{k}) + F_{11}(\bar{k})^* X_1(t,k+1) F_{11}(\bar{k}) - E_{11}(\bar{k})^* X_1(\bar{k}) E_{11}(\bar{k}) \prec -\epsilon I$$
$$\Rightarrow F_{11}(\bar{k})^* X_1(t,k+1) F_{11}(\bar{k}) - E_{11}(\bar{k})^* X_1(\bar{k}) E_{11}(\bar{k}) \prec -\epsilon I \quad (28)$$

The last inequality holds because X_0 is positive-definite. Let us partition $\hat{E}(t), \hat{F}(t)$ and $\bar{E}(t), \bar{F}(t)$ as in (27). Now starting with inequalities in (24) and following the steps above, we can arrive at the following matrix inequalities

$$\hat{F}_{11}(t)^* \hat{X}_1(t) \hat{F}_{11}(t) - \hat{E}_{11}(t)^* \hat{X}_1(t) \hat{E}_{11}(t) \prec -\epsilon I$$
$$\bar{F}_{11}(t)^* \bar{X}_1(t) \bar{F}_{11}(t) - \bar{E}_{11}(t)^* \bar{X}_1(t) \bar{E}_{11}(t) \prec -\epsilon I \quad (29)$$

We know that the solutions of the above inequalities exist, as the solution $\hat{X}$ and $\bar{X}$ can be constructed from operator X. It is also known that the inertia of $\hat{X}(t)$ (and similarly for $\bar{X}(t)$) is determined by the number of eigenvalues of the matrix pencil $(\lambda \hat{E}(t) - \hat{F}(t))$ inside and outside of the unit circle[3]. Now to relate these inertias to that of X_1, we take a convex combination of (28) and (29) for $k \geq N$ as

$$\hat{F}_{11}(t)^* X_{1\theta}(t,k+1) \hat{F}_{11}(t) - \hat{E}_{11}(t)^* X_{1\theta}(\bar{k}) \hat{E}_{11}(t) \prec -\epsilon I$$

Here $X_{1\theta}(\bar{k}) = \theta X_1(\bar{k}) + (1-\theta)\hat{X}_1(t)$, for $\theta \in [0,1]$. Now if inertias of $X_1(\bar{k})$ and $\hat{X}_1(t)$ are not the same, we can increase θ from 0 until an eigenvalue of $X_{1\theta}(\bar{k})$ is 0. In such a scenario we will have $\text{in}_+(X_{1\theta}(\bar{k})) + \text{in}_-(X_{1\theta}(t,k+1)) < \bar{n}_1(\bar{k})$ which would violate the above inequality. Hence we conclude that the inertia of $X_1(\bar{k})$ should be same as that of $\hat{X}_1(t)$ for $k \geq N$ which is uniquely determined by $\hat{E}(t), \hat{F}(t)$. A similar argument can be given for $k < 0$. ■

Following result incorporates the inertia condition into Lemma 14 and completes the construction of the eventually invariant scaling operator (equivalently the existence of eventually invariant controller) which guarantees stability of the system.

Theorem 16: Suppose E, F are eventually invariant operators as defined in Lemma 14 and suppose there exists $X \in \mathcal{X}$ satisfying $F^* \mathbf{\Lambda}^* X \mathbf{\Lambda} F - E^* X E \prec 0$ and the inertia condition $\text{In}_+(X) + \text{In}_-(\mathbf{\Lambda}^* X \mathbf{\Lambda}) = \bar{n}$ then there exists eventually invariant $\tilde{X} \in \mathcal{X}$ which satisfies both the inequality $F^* \mathbf{\Lambda}^* \tilde{X} \mathbf{\Lambda} F - E^* \tilde{X} E \prec 0$ and the inertia condition $\text{In}_+(\tilde{X}) + \text{In}_-(\mathbf{\Lambda}^* \tilde{X} \mathbf{\Lambda}) = \bar{n}$.

Proof: The construction of eventually invariant $\tilde{X}$ is exactly same as that in Lemma 14 and as a result we get $F^* \mathbf{\Lambda}^* \tilde{X} \mathbf{\Lambda} F - E^* \tilde{X} E \prec 0$. As in Lemma 14 we first construct an eventually spatially invariant X_ζ and prove the inertia condition for the same. Since $X_{1\zeta}(\bar{k}) = X_1(\bar{k})$ for $k = -1, \ldots, N+1$, we have

$$\text{in}(X_{1\zeta}(\bar{k})) = \text{in}(X_1(\bar{k})) \text{ for } k = -1, \ldots, N+1 \quad (30)$$

This leads to the following for all t

$$\text{in}(X_{1\zeta}(t,-1)) = \text{in}(X_1(t,-1)) = \text{in}(\bar{X}_1(t)),$$
$$\text{in}(X_{1\zeta}(t,N)) = \text{in}(X_1(t,N)) = \text{in}(\hat{X}_1(t)) \quad (31)$$

[3]Matrix inequalities of the form $\tilde{F}^* Y \tilde{F} - \tilde{E}^* Y \tilde{E} \prec 0$ appear in literature as generalized lyapunov inequalities. For this and background on matrix/operator pencils and their eigenvalues readers are directed to [16], [17] and references therein

The second equalities in the above lines result from Lemma 15. Using (11) and the fact that $X_{0\zeta}(\bar{k})$ is positive definite we have $\text{in}_-(X_{1\zeta}(t,k+1)) + \text{in}_+(X_{1\zeta}(\bar{k})) \geq \bar{n}_1(\bar{k})$. Further for indices where the system is spatially invariant the preceding inequality is same as

$$\text{in}_-(X_{1\zeta}(t,k+1)) \geq \text{in}_-(X_{1\zeta}(\bar{k})) \text{ for } k \geq N, k \leq -1 \quad (32)$$

By construction we have

$$X_{1\zeta}(\bar{k}) = \hat{X}_1(t) \text{ for } k \gg N \text{ and}$$
$$X_{1\zeta}(\bar{k}) = \bar{X}_1(t) \text{ for } k \ll -1 \quad (33)$$

Combining (30)-(33), we have $\text{In}(X_\zeta) = \text{In}(X)$. To complete the proof, we can follow a similar procedure in the temporal dimension to prove $\text{In}(\tilde{X}) = \text{In}(X_\zeta) = \text{In}(X)$. ■

VII. Conclusions

This paper discusses an operator-pencil approach for analysis of systems distributed over an infinite lattice. The corresponding synthesis conditions thus obtained are less conservative than earlier work on heterogeneous systems. The special case of eventually invariant lattice systems is also analyzed.

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