2011 50th IEEE Conference on Decision and Control and
European Control Conference (CDC-ECC)
Orlando, FL, USA, December 12-15, 2011

Identification of Nonlinear Systems with Stable Oscillations

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***Abstract*— We propose a convex optimization procedure for identification of nonlinear systems that exhibit stable limit cycles. It extends the "robust identification error" framework in which a convex upper bound on simulation error is optimized to fit rational polynomial models with a strong stability guarantee. In this work, we relax the stability constraint using the concepts of transverse dynamics and orbital stability, thus allowing systems with autonomous oscillations to be identified. The resulting optimization problem is convex. The method is illustrated by identifying a high-fidelity model from experimental recordings of a live rat hippocampal neuron in culture.**

I. Introduction

Black-box identification of highly nonlinear systems poses many challenges: flexibility of representation, efficient optimization of parameters, model stability, and accurate long-term simulation fits, to name but a few [1], [2]. It is especially challenging when the system exhibits autonomous oscillations: such a system is intrinsically nonlinear and lives on the "edge of stability", since periodic solutions must have at least one critically-stable Lyapunov exponent [3].

Recently, a new framework has been introduced for identifying a broad class of nonlinear systems along with certificates of model stability and accuracy of long-term predictions [4]. However this method necessarily fails if the system has autonomous oscillations. In this paper we extend the method of [4] to remove this restriction.

A. Identification of Oscillating Systems

In many scientific fields there is a need to capture oscillatory behaviour in the form of a compact mathematical model which can then be used for simulation, analysis, or control design. When the data comes from experimental recordings, this is known as *system identification*. It is also becoming more frequent to perform *model-order reduction* via system identification methods from solutions of a very high dimensional simulation, e.g. computational fluid dynamics [5] or a detailed electronic circuit model (see, e.g., [6], [7]).

In biology, systems that oscillate seem to be the rule rather than the exception: heartbeats, firefly synchronization, circadian rhythms, neuron spking, and many others [8], [9]. Nonlinear oscillator models have been used in speech analysis and synthesis, where stability of the identified model has been acknowledged as a major issue [10].

To the authors' knowledge, there is no generally applicable methods of system identification – or model-order reduction – for oscillating systems. One family of approaches popular for aerospace model reduction is harmonic balance (describing function) methods, in which the period of oscillation is assumed known and the model is reduced consider the problem in the fourier-series domain [11], [12], [5]. A similar approach has been taken to analyse phase-locked loops and oscillators, in which a local phase-offset system is of primary interest [7]. Neither of these approaches extend easily to situations in which the frequency of oscillation is input-dependent. Other papers assume a known decomposition into a stable linear part and a static nonlinear map, and consider it a problem of closed-loop linear system identification [13]. Applications have included identification of combustion instabilities [14], [15]. A mixed empirical/physics-based approach has been used to produce low-order models of periodic vortex shedding in fluid flows [16].

This was supported by National Science Foundation Grant No. 0835947.

B. Stability of Oscillations

No linear system can produce an asymptotically stable limit cycle. Identifying nonlinear models from data is a difficult problem, in particular because of the complex relationship between system parameters and long term behaviour of solutions. A recent approach, which this paper builds upon, works via convex optimization of a *robust identification error* which imposes an asymptotic stability constraint on the identified model [4].

However, if the system has a periodic solution, not driven by a periodic forcing term then this approach must fail: the stability constraint is too strong. To see this, suppose a system

$$\dot{x} = f(x) \in \mathbb{R}^n$$

has a non-trivial T-periodic solution $x^\star(t)$, then $x^\star(t+\tau), \tau \in (0,T)$ is also a solution which will never converge to $x^\star(t)$.

The natural notion of stability for oscillating systems is *orbital stability*. A T-periodic solution $x^\star$ is orbitally stable if nearby initial conditions coverge to the solution *set* in state space $\mathcal{X} = \{x(\tau) : \tau \in [0,T]\}$ and not necessarily to the particular time solution $x^\star(t)$. This is a weaker condition than standard (Lyapunov) asymptotic stability.

Orbital stability can be studied via the introduction of so-called *transverse coordinates*, also referred to as the *moving Poincaré section* [3], [17]. The basic idea is to contruct a new coordinate system at each point of the solution, decomposing the state into a scalar component tangential to the solution

curve, and a component of dimension $n-1$ transversal (often orthogonal) to the solution curve.

It is known that periodic solution of a nonlinear differential equation is orbitally stable if and only the dynamics in the transverse coordinates are stable [3, Chap. VI]. This framework has previously been used to design stabilizing controllers and analyze regions of attraction for oscillating systems [18], [19], [20], [21], [22]. It has also been used to analyze the convergence of prediction-error methods when identifying a linear/static-nonlinearity feedback interconnection that can oscillate [13]. In this chapter we extended the robust identification error method of [4] using a storage function in the transverse coordinates so as to robustly identify a broad class of nonlinear systems that may (or may not) admit autonomous oscillations.

II. Problem Statement

Given a data record[1] of states, inputs, and outputs $\{\tilde{x}(t), \tilde{u}(t), \tilde{y}(t)\}, t \in [0, T]$, the general problem is to construct a compact model in the form of a differential equation that, when simulated, faithfully reproduces the data. To pose the problem exactly we must specify both a model class to search over, and an optimization objective.

A. Model Class

The model class we will search over consists of continuous-time state-space models with state $x \in \mathbb{R}^n$, input $u \in \mathbb{R}^m$, output $y \in \mathbb{R}^p$, and dynamics defined in the following implicit form:

$$\frac{d}{dt}e(x) = f(x,u) \quad (1)$$

$$y = g(x,u) \quad (2)$$

where $e : \mathbb{R}^n \to \mathbb{R}^n, f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n, g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p$ are smooth functions. The Jacobians with respect to x of $e(x)$, $f(x,u)$, and $g(x,u)$ are denoted $E(x) = \frac{\partial}{\partial x}e(x)$, $F(x,u) = \frac{\partial}{\partial x}f(x,u), G(x,u) = \frac{\partial}{\partial x}g(x,u)$. We will enforce the constraint that $E(x)$ be nonsingular, so the above implicit model can equivalently be written in explicit form:

$$\dot{x} = E(x)^{-1}f(x,u).$$

Remark 1: To implement the methods described in this paper, $e(x), f(x,u)$, and $g(x,u)$ should come from a convex class of vector-valued functions for which we can efficiently check positivity. In practice, we use matrices of polynomials or trigonometric polynomials, so as to make use of sum-of-squares programming [23], [24].

B. Optimization Objective

The general problem we consider is to minimize, over choice of e, f, g, the value of the *simulation error:*

$$\mathfrak{E} = \int_0^T |y(t) - \tilde{y}(t)|^2 dt$$

[1]Here we assume that the data record consists of sufficiently smooth continuous-time signals on an interval, though in practice it will consist of a finite sequence of data points.

where $y(t)$ is the solution of (1), (2) with $x(0) = \tilde{x}(0)$. One may also wish to ensure that the dynamical system defined by (1), (2) is well-posed and has some sort of stability property. Note that we do not assume that the system from which data is recorded is in the model class.

Direct optimization of simulation error is not usually tractable: the relationship between system parameters and model simulation is highly nonlinear, and for black-box models we typically don't have good initial parameter guesses. We make the problem tractable (a convex program) through a series of approximations and relaxations.

III. Nonlinear System Identification via Robust Identification Error

In this section we briefly present some results from [4] and explain why they cannot be directly applied to systems with autonomous oscillations. The basic idea is to search jointly for system equations as well as a storage function with output reproduction error as a supply rate. Standard dissipation inequality arguments [25] then provide a bound on long-term simulation error.

A. Linearized Simulation Error

Suppose we have a model of the form (1), (2) and a data record $\{\tilde{x}(t), \tilde{u}(t), \tilde{y}(t)\}, t \in [0, T]$. We introduce the *linearized simulation error* as a local measure of the model's divergence from the data.

First, we define the *equation error* signals associated with (1), (2) and the data:

$$\epsilon_x(t) = E(\tilde{x}(t))\dot{\tilde{x}}(t) - f(\tilde{x}(t), \tilde{u}(t)), \quad (3)$$

$$\epsilon_y(t) = \tilde{y}(t) - g(\tilde{x}(t), \tilde{u}(t)). \quad (4)$$

Now, consider the following family of systems parametrized by $\theta \in [0, 1]$:

$$E(x_\theta)\dot{x}_\theta = f(x_\theta, u) + f_\theta, \quad (5)$$

$$y_\theta = g(x_\theta, u) + g_\theta. \quad (6)$$

Let (x_θ, y_θ) be the solution of the above system with $f_\theta = (1-\theta)\epsilon_x$ and $g_\theta = (1-\theta)\epsilon_y$. Note that we have $(x_1, y_1) = (x, y)$, i.e. the results of simulation, and $(x_0, y_0) = (\tilde{x}, \tilde{y})$ i.e. the observed data. We can consider the following linearized simulation error about the recorded trajectory:

$$\mathcal{E} = \lim_{\theta \to 0} \frac{1}{\theta^2} \int_0^T |y_\theta(t) - \tilde{y}(t)|^2 dt$$

as local approximation of the true simulation error $\mathfrak{E}$.

B. Local Robust Identification Error

Note that $\mathcal{E}$ can alternately be represented as

$$\mathcal{E} = \int_0^T |G(\tilde{x}(t), \tilde{u}(t))\Delta + \epsilon_y(t)|^2 dt$$

where

$$\Delta(t) = \lim_{\theta \to 0} \frac{1}{\theta}[x_\theta(t) - \tilde{x}(t)] \quad (7)$$

which obeys the dynamics

$$\frac{d}{dt}E(\tilde{x}(t))\Delta(t) = F(\tilde{x}(t),\tilde{u}(t))\Delta(t) + \epsilon_x(t).$$

That is, $\Delta(t)$ is an estimate of the deviation of the model simulation $x(t)$ from the recorded data trajectory $\tilde{x}(t)$.

It was shown in [4] that

$$\mathcal{E} \leq \int_0^T \bar{\mathcal{E}}_Q(t)dt \tag{8}$$

for any $Q = Q' > 0$, where[2]

$$\bar{\mathcal{E}}_Q(t) = \sup_{\Delta\in\mathbb{R}^n} \{2\Delta' E' Q(F\Delta + \epsilon_x) + |G\Delta + \epsilon_y|^2\}. \tag{9}$$

The systems theory interpretation of (9) is that the first term in the supremum is the derivative of a positive-definite storage function with respect to linearized simulation error, and the second term is the output reproduction error. The storage function is an upper bound for "simulation-error to go" due to accumulated system deviations, and thus $\bar{\mathcal{E}}_Q(t)$ can be considered to be the worst-case increment to present *and future* output reproduction error due to the model mismatch at the current data point ϵ_x.

The bound (8) suggests searching over functions e, f, g and a matrix $Q = Q' > 0$ so as to minimize the right-hand-side of (8). This optimization is still non-convex, but a convex relaxation is given in [4] (we use a similar relaxation in Section V of the present paper).

Each of the supremums over Δ in (9) are finite if and only if the matrices

$$R = E'QF + F'QE + G'G$$

for each data point is negative semidefinite. If this property holds for all x, u, then it has been proven that the system is globally incrementally output stable. Roughly speaking: $\Delta' E' Q E \Delta$ is a contraction metric for the identified system [26] and $\Delta'(E'QF + F'QE)\Delta$ is its derivative. A formal proof of stability is given in [4].

For the purposes of the present paper, it is sufficient to note that enforcing global incremental stability is *too strong* to allow identification of systems exhibiting autonomous oscillations, since such systems *cannot* satisfy this property. The main purpose of this paper is to overcome this limitation via a reformulation of the local RIE in the transverse dynamics.

IV. Transverse Robust Identification Error

In oscillating systems, perturbations in phase cannot be stable and will therefore accumulate over time. The natural form of stability is *orbital* stability, which can be defined as stability to a solution set in state space, rather than a particular time solution. A standard framework for anlaysis of orbital stability is via transverse coordinates (see [3],[17]).

Such an analysis begins with knowledge of the vector field and a (not necessarily periodic) solution $x^\star(t)$. This curve is used to construct a time-varying change of coordinates locally about $x^\star(t)$. This change of coordinates transforms solutions nearby $x^\star(t)$ into a scalar component describing perturbations in the $\dot{x}^\star(t)$ direction (i.e. "phase" perturbations) and perturbations which are transversal (e.g. in directions orthogonal to $\dot{x}^\star(t)$).

In this work we consider a simplified situation wherein we take the data $\tilde{x}(t)$ in place of $x^\star(t)$. Define the following projection operators:

$$\pi(t) := \frac{\dot{\tilde{x}}(t)\dot{\tilde{x}}(t)'}{|\dot{\tilde{x}}(t)|^2},\ \Pi(t) := I - \pi(t), \tag{10}$$

i.e. $\pi(t)$ projects on to the one-dimeonsional subspace parallel to $\dot{\tilde{x}}(t)$ and $\Pi(t)$ projects on to the $(n-1)$-dimensional subspace orthogonal to this.

We examine a relaxed variation on the local RIE ((9) defined by:

$$\begin{aligned}\bar{\mathcal{E}}^\perp_Q(t) \quad = \quad & \sup_{\Delta\in\mathbb{R}^{n-1}} \{2\Delta\Pi^{r\prime} E' Q((F + E\dot{\Pi})\Pi^r\Delta + \epsilon_x) \\ & + |G\Pi^r\Delta + \epsilon_y|^2\}\end{aligned} \tag{11}$$

where $\Pi^r(t) \in \mathbb{R}^{n\times(n-1)}$ is a matrix with orthonormal rows spanning the subspace orthogonal to $\dot{\tilde{x}}$ and Q a symmetric positive-definite $n \times n$ matrix. It is a "reduced" form of the rank $n-1$ matrix $\Pi(t)$ containing only independent rows.

The motivating idea is to minimize deviations of the model from the recorded data in *phase space*, i.e. transversal to the curve $\tilde{x}$, allowing for time reparametrizations due to persistent perturbations in phase. Considering all deviations transversal to $\tilde{x}(t)$, the term $\bar{\mathcal{E}}^\perp_Q(t)$ is a bound on the increment to present and future output reproduction error induced by the modelling error ϵ_x. Precise statements about simulation-error bounds and model stability will appear in a future publication.

V. A Convex Upper Bound

We suggest minimizing:

$$\int_0^T \mathcal{E}^\perp_Q(t)dt$$

over choices of e, f, g and Q as an effective procedure for system identification. However, this is still a nonconvex optimization. In this section we propose a convex upper bound for which one can efficiently find the global minimum via semidefinite programming.

The basic idea is to decompose the each non-convex term into the sum of a convex and a concave part, and upper-bound the concave part with a linear relaxation.

Theorem 1: Define

$$\begin{aligned}\Delta^+_e &= E(\tilde{x})(I + \dot{\Pi})\Pi^r\Delta + F(\tilde{x},\tilde{u})\Pi^r\Delta + \epsilon_x \\ \Delta^-_e &= E(\tilde{x})(I - \dot{\Pi})\Pi^r\Delta - F(\tilde{x},\tilde{u})\Pi^r\Delta - \epsilon_x \\ \Delta_y &= G\Pi^r\Delta + \epsilon_y.\end{aligned}$$

Then $\bar{\mathcal{E}}^\perp_Q(t) \leq \hat{\mathcal{E}}^\perp_Q(t)$ where

$$\hat{\mathcal{E}}^\perp_Q = \sup_{\Delta\in\mathbb{R}^{n-1}} \left\{ \frac{|\Delta^+_e|^2_Q + |\Pi^r\Delta|^2_{Q^{-1}}}{2} - (\Pi^r\Delta)'\Delta^-_e + |\Delta_y|^2 \right\}$$

[2]Here, and frequently throughout the paper, we drop the arguments on $E(\tilde{x}(t)), F(\tilde{x}(t),\tilde{u}(t)), G(\tilde{x}(t),\tilde{u}(t)), \epsilon_x(t)$, and $\epsilon_y(t)$ for the sake of compactness of notation. It should be understood that these are always functions of time and the data.

which is convex in e, f, g, and $Q^{-1} > 0$. ■

The proof is omitted to meet space restrictions, but a similar statement was proved in [4, Section V].

To perform system identification via the optimization

$$\int_0^T \hat{\mathcal{E}}_Q^{\perp}(t)dt \to \min$$

over choices of e, f, g, and $Q^{-1} > 0$ subject to the constraint $E(x)' + E(x) > I$ for all x.

In practice, we will have a record of the true system at a finite number of times $t_i, i = 1, 2, ..., N$, and as a surrogate for the above we minimize the finite sum of the TRIE terms:

$$\sum_{i=1}^{N} \hat{\mathcal{E}}_Q^{\perp}(t_i) \to \min .$$

VI. Implementation

We now discuss practical considerations for data preparation and minimization of the upper bounds using semidefinite programming.

A. Extracting States from Input/Output Data

The RIE formulation assumes access to approximate state observations, $\tilde{x}(t)$. In most cases of interest, the full state of the system is not directly measurable and extraction of a state vector is a challenging problem in its own right. In practice, our solutions have been motivated by the assumption that future output can be approximated as a function of recent input-output history and future input. A common method for choosing a set of time-delays is to optimize mutual information [27]. To summarize recent history, we have had success applying linear filter banks, as is common in linear identification (e.g. Laguerre filters [28]).

Projection-based methods such as subspace identification [29] and proper-orthogonal-decomposition [30] are common methods for linear system identification and model reduction. However, even in fairly benign cases one expects the input-output histories to live near a nonlinear submanifold of the space of possible histories. Connections between nonlinear dimensionality reduction and system identification have been explored in some recent papers, e.g. [31] and [32].

For experimental recordings, derivatives can be estimated via differentiation filters or noncausal smoothing before numerical differentiation. Approximating additional states through filter banks allows the rates of these variables to be calculated analytically.

B. Semidefinite/Sum-of-Squares Programming Formulation

For each data point, the upper bound on the local TRIE is the supremum of a concave quadratic form in Δ. So long as e, f and g are chosen to be linear in the decision variables, this upper bound can be minimized by introducing an LMI for each data-point: we introduce a slack variable s_i for each data-point:

$$s_i \geq \hat{\mathcal{E}}_Q^{\perp}(t_i),$$

which can be transformed via Schur complement into an LMI constraint. Then we optimize for $\sum_i s_i \to \min$. We parametrize e, f, and g as polynomials, so that a sum-of-squares relaxation [23] is used to enforce the well-posedness constraint $E(x) + E(x)' \geq I \ \forall x \in \mathbb{R}^n$.

Note that the robust identification criterion typically has a relatively small number of decision variables, dependent on the order and degrees of the system model. However, to transform the problem into a standard semidefinite pogram, a very large number of slack variables, equality constraints, and LMI constraints are introduced, growing with the number of data points. The main reason for doing this is to make use of existing and well-tested semidefinite solvers, however it is likely that specialized solvers will be more computationally efficient. This will be a topic of future work.

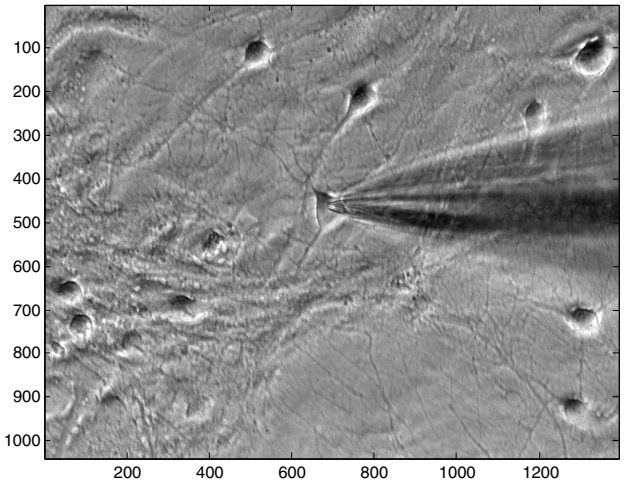

Fig. 1. The neuron in culture and the glass micropipette electrode used to interface to it. Imaging: phase contrast image at 20× magnification on an inverted Olympus IX-71 microscope. Scale: 100 pixels (marked on axes) = 43 microns.

VII. Experimental Results on Live Neurons

We now demonstrate the method by identifying the membrane-potential dynamics of a live, *in-vitro* hippocampal neuron. A micropipette is used to establish an interface with the cell such that current can be injected into the soma, and the membrane potential response can be recorded. A microscopic photograph of the neuron and patch is shown in Figure 1. The preparation of the culture is described in the appendix.

The membrane dynamics of a single neuron are highly complex: at low currents the system responds like a low-order passive linear system. However, when certain ion channels are activated rapid *spiking* can occur. After spiking there is a *refractory* period in which sensitivity is reduced, and with sufficient current input spiking can repeat at an input-dependent frequency.

There is a spectrum of models of neuron dynamics, ranging from simple "integrate and fire" models to highly complex biophysical models of ion channels and conductances [33]. Threshold based models generally have a very small number of parameters, but do not provide high fidelity reproduction of the membrane potential dynamics.

By contrast biophysical models can be very accurate, but are highly nonlinear and are very difficult to identify [34] – they can have many locally optimal fits in disconnected regions of parameter space [35]. In this section we use the proposed method to identify a black-box nonlinear model with comparatively few states (three) which reproduces the experimentally observed spiking and subthreshold behavior with very high fidelity.

Three increasing step currents are applied to the neuron resulting in increasing firing rate and a characteristic change in the spike amplitude and shape.

As discussed in Section VI, we must find a good proxy for the internal state of the system. Here we used two Laguerre filters with identical pole locations to summarize the recent history voltage history.

Figure 2 presents a comparison of fit performance using three methods. The first is equation error minimization, i.e. simply optimizing

$$\sum_i |E(\tilde{x}(t_i))\dot{\tilde{x}}(t_i) - f(\tilde{x}(t_i), \tilde{u}(t_i))|^2 \rightarrow \min$$

subject to the well-posedness constraint $E(x) + E(x)' \geq I$ but without constraints on stability or long-term simulation error (this is similar in principle to NARX and prediction error methods). The second method is the comparison is the original RIE minimization from [4], and the third is the proposed Transverse RIE method.

We see that while equation error minimization (top) leads to initially good performance, the model goes unstable quite quickly. Fitting with the RIE (middle) leads to the anticipated overly stable model dynamics (see Section III). The final plot presents the Transverse RIE identification, which matches the experimentally observed spike patterns very well.

We have also had success identifying behavior which covers both the subthreshold and spiking regime of a neuron. The applied stimulus was a variety of multisine signals. Figure 3 presents validation of a Transverse RIE fit on held-out data. The lower plot is the multisine input in pico-Amperes. The upper plot presents the original data and fit. Both the subthreshold regime and spikes are generally well reproduced.

VIII. Conclusions

This paper has introduced a new technique for identification of nonlinear systems which may produce autonomous oscillations, i.e. system oscillations which are produced internally by the dynamics rather than as a response to a periodic input.

The proposed method worked well on the challenging problem of accurately modeling the membrane dynamics of a live neuron from experimental current-voltage recordings. The input-dependent absence or presence, and frequency of repitition, of spiking events was well captured in the model.

Future work will include theoretical investigations into the properties of identified models, further software development, and testing on a wider range of systems exhibiting autonomous oscillations.

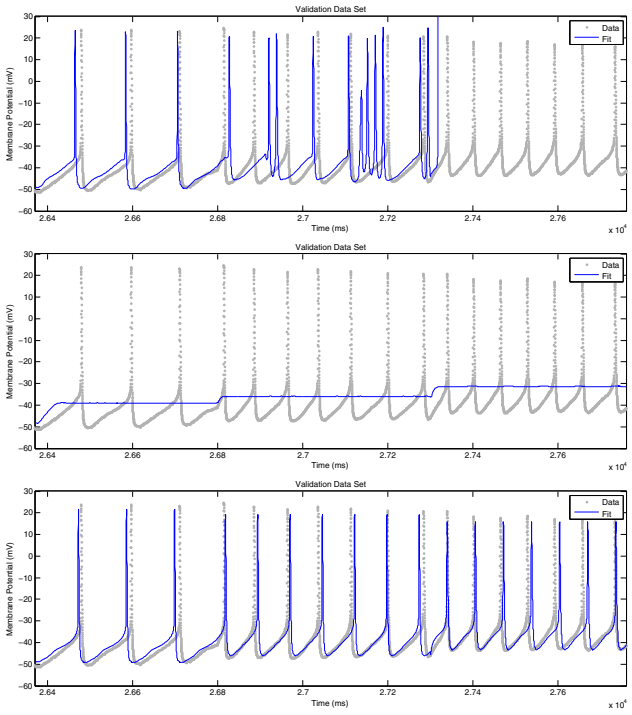

Fig. 2. A neuron was subjected to increasing step-currents, and spiked with increasing frequency. Long-term simulation of an equation error fit (top) is unstable. RIE (middle) minimization provides an overly stable fit. The proposed TRIE method (bottom) reproduces the spikes accurately.

Appendix

A. Live Neuron Experimental Procedure

Primary rat hippocampal cultures were prepared from P1 rat pups, in accordance with the MIT Committee on Animal Care policies for the humane treatment of animals. Dissection and dissociation of rat hippocampi were performed in a similar fashion to [36]. Dissociated neurons were plated at a density of 200K cells/mL on 12 mm round glass coverslips coated with 0.5 mg/mL rat tail collagen I (BD Biosciences) and 4 μg/mL poly-D-lysine (Sigma) in 24-well plates. After 2 days, 20 μM Ara-C (Sigma) was added to prevent further growth of glia.

Cultures were used for patch clamp recording after 10 days in vitro. Patch recording solutions were previously described in [37]. Glass pipette electrode resistance ranged from 2-4 MΩ. Recordings were established by forming a GΩ seal between the tip of the pipette and the neuron membrane. Perforation of the neuron membrane by amphotericin-B (300 μg/mL) typically occurred within 5 minutes, with resulting access resistance in the range of 10-20 MΩ. Recordings with leak currents smaller than -100 pA were selected for analysis. Leak current was measured as the current required to voltage clamp the neuron at -70 mV. Synaptic activity was blocked with the addition of 10 μM CNQX, 100 μM APV, and 10 μM bicuculline to the bath saline. Holding current was applied as necessary to compensate for leak current.

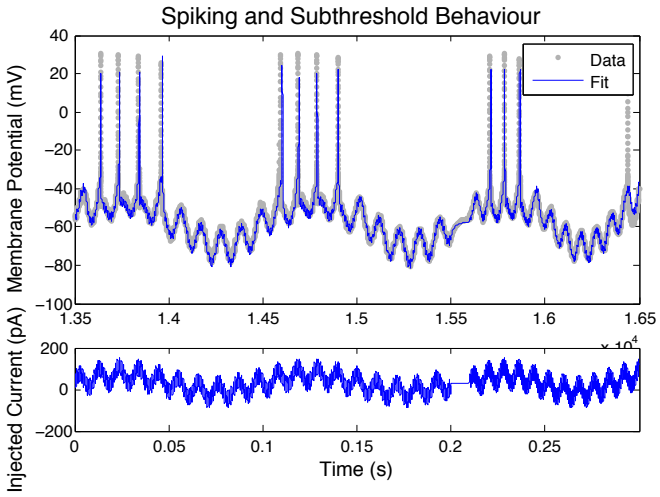

Fig. 3. Response of real neuron and TRIE model simulation, showing both stable sub-threshold behavior and spiking.

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