

Nonlinear Estimation of Fluid Flow Velocity Fields

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Abstract—A proper orthogonal decomposition (POD)-based nonlinear estimator for fluid flow velocity fields is developed, which is capable of achieving finite-time convergence of the Galerkin coefficient estimates. Using Galerkin projection and POD-based model reduction, the incompressible Navier-Stokes equations are recast as a set of nonlinear ordinary differential equations in the Galerkin coefficients. A sliding-mode-based observer is designed to estimate the unknown time-varying Galerkin coefficients. Convergence of the estimates is proven to be achieved in finite time, and high-fidelity numerical simulation results are provided to complement the theoretical development.

I. INTRODUCTION

Control of air flow is of fundamental importance for aerospace or aeronautical applications, since the performance of aircraft systems, for example, is highly impacted by their aerodynamic characteristics. While numerous nonlinear flow control techniques have been presented in controls literature, nonlinear estimation techniques are not as often applied to fluid flow problems.

One of the challenges in designing controllers for fluid flow fields is that the dynamics are governed by nonlinear partial differential equations (i.e., the Navier-Stokes equations). Proper orthogonal decomposition (POD)-based model reduction is a well-established method for expressing the Navier-Stokes equations in a more tractable form [1]–[3]. In the POD-based method, the nonlinear Navier-Stokes equations are projected onto a subspace using the Galerkin projection. In the Galerkin method, the state variables (e.g., flow field velocity) can be expressed as linear combinations of the POD modes. By using this new representation of the state variables, the original Navier-Stokes equations can be rewritten as a set of nonlinear ordinary differential equations. The state variables in the transformed set of equations are the time-varying coefficients of the POD modes. By building an observer to estimate the coefficients, an observer of the fluid flow velocity field can thus be obtained [4], [5].

Utilization of POD-based observer design is well established in literature. In [4], an extended Kalman filter (EKF)-based observer is utilized to estimate the coefficients in the reduced-order model obtained from the Galerkin projection. An unscented Kalman Filter (UKF)-based observer is used in [6] to estimate velocity fields and contaminant concentration fields using the POD technique. It was found in [6] that the UKF did not significantly outperform the EKF for the given system. In [6], a higher-order Kalman filter is used to design

an observer for contaminant concentration fields. In [7], POD-based model reduction is used in conjunction with Quadratic Stochastic Estimation (QSE) to estimate the temporal behavior of the POD modes. The modified QSE technique presented in [7] allows estimation of the time-variation of the POD modes using wall-pressure measurements. In [8], a model for real-time estimation of velocity fields, called the “episodic POD model,” is presented. By utilizing measurements at the current time, an approximation of the entire 3D velocity fields can be obtained at past and future times using the episodic POD model.

While the aforementioned observer strategies perform well in their respective tasks, rigorous mathematical verification of estimator performance is not provided. By applying nonlinear estimation techniques, additional insights can be gained into the performance characteristics of fluid flow field observer designs.

The contribution in this paper is the development and rigorous mathematical validation of a POD-based nonlinear flow velocity field estimator design, which achieves finite-time estimation of the coefficients in the reduced-order model obtained from the Galerkin projection. Using POD and the Galerkin projection, the Navier-Stokes equations are recast as a set of nonlinear ordinary differential equations in terms of the unknown Galerkin coefficients. A sliding-mode observer is utilized to estimate the Galerkin coefficients, and convergence of the estimates is proven to be achieved in finite-time. High-fidelity numerical simulation results are provided to complement the theoretical development.

II. FLOW VELOCITY FIELD DYNAMICS

In this section, a reduced-order model for the fluid flow velocity field dynamics will be derived. To this end, POD-based model reduction will be utilized to express the incompressible Navier-Stokes equations as a set of nonlinear ordinary differential equations.

The (scaled) incompressible Navier-Stokes equations are given by

$$\operatorname{div}(u) = 0$$

$$\frac{\partial u}{\partial t} = -(u \cdot \nabla)u + \nu \nabla^2 u - \nabla p \quad (1)$$

where $u(z, t) : \Omega \times \mathbb{R} \rightarrow \mathbb{R}^3$ denotes the velocity field defined over a spatial domain, $z \in \Omega$; $p(z, t) \in \mathbb{R}^3$ is the pressure

field; and $\nu \triangleq 1/\text{Re}$ represents the (constant) kinematic viscosity, where Re denotes the Reynolds number.

Although we are considering incompressible flow in this paper, the following model reduction approach can be directly applied to compressible flows as well [9]–[11]. The subsequent nonlinear estimator design and analysis can also be applied to the compressible flow case with minor modifications, provided one can develop the appropriate POD basis.

A. POD-based Model Reduction

In POD-based model reduction, the velocity field $u(z, t)$ is expanded in terms of a finite sum of POD modes $\phi(z)$ defined on the spatial domain Ω .¹ The POD expansion can be expressed in terms of n modes as

$$u(z, t) = \sum_{j=1}^n a_j(t) \phi_j(z). \quad (2)$$

By rewriting the velocity field $u(z, t)$ in terms of the modal decomposition in (2), the Navier-Stokes equation in (1) can be recast in a nonlinear ordinary differential equation in terms of the unknown time-varying coefficients $a_i(t)$, $i = 1, \dots, n$. To this end, the decomposition in (2) is substituted into (1) to yield the following:

$$\begin{aligned} \sum_{i=1}^n \dot{a}_i(t) \phi_i(z) = & \quad (3) \\ - \left(\sum_{j=1}^n a_j(t) \phi_j(z) \cdot \nabla \right) \sum_{k=1}^n a_k(t) \phi_k(z) & \\ + \nu \sum_{i=1}^n a_i(t) \nabla^2 \phi_i(z) - \nabla p. & \end{aligned}$$

Where $\text{div}(\phi_i) = 0$ was utilized. By orthogonality of modes,

$$\langle \phi_i(z), \phi_j(z) \rangle = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}. \quad (4)$$

After projecting the terms in (3) onto the space of POD modes, and utilizing (4), (3) can be rewritten and simplified as

$$\begin{aligned} \dot{a}_k(t) = & \quad (5) \\ - \sum_{i=1}^n \sum_{j=1}^n a_i(t) a_j(t) \langle (\phi_j(z) \cdot \nabla) \phi_i(z), \phi_k(z) \rangle & \\ + \nu \sum_{i=1}^n a_i(t) \langle \nabla^2 \phi_i(z), \phi_k(z) \rangle - \langle \nabla p, \phi_k(z) \rangle. & \end{aligned}$$

In (5), the notation $\langle \cdot, \cdot \rangle$ represents the standard inner product given by

$$\langle \alpha(z), \beta(z) \rangle = \int_{\Omega} \alpha(z) \cdot \beta(z) dV \quad (6)$$

where $\alpha(z) \cdot \beta(z)$ denotes the standard dot product between the vectors $\alpha(z)$ and $\beta(z)$ in Euclidean space, and dV is a

¹In the current analysis, the POD expansion is expressed in terms of the spatial modes $\phi(z)$, but for some applications temporal modes $\psi(t)$ might be more appropriate.

volume element. The expression in (5) constitutes a reduced-order system of ordinary nonlinear (quadratic) differential equations. Given a set of representative POD modes, the system in (5) provides a description of the flow dynamics. The expression in (5) can be rewritten in the more compact form

$$\dot{a}_k(t) = L_k a(t) + a^T(t) Q_k a(t) \quad (7)$$

where $L_k(z) \in \mathbb{R}^{1 \times n}$ is a row vector with elements $L_{k_i}(z)$ given by

$$L_{k_i}(z) = \nu \langle \nabla^2 \phi_i(z), \phi_k(z) \rangle \quad (8)$$

and $Q_k(z) \in \mathbb{R}^{n \times n}$ is a matrix with elements $Q_{k_{ij}}(z)$ given by

$$Q_{k_{ij}}(z) = \langle (\phi_j(z) \cdot \nabla) \phi_i(z), \phi_k(z) \rangle, \quad k = 1, \dots, n. \quad (9)$$

The pressure term is ignored in (7). Since $\text{div}(\phi) = 0$, it can be shown that $\phi_k(z) = 0$ on the boundary of Ω . Thus, the pressure term in (5) vanishes over a closed domain [5].

Introducing the state vector $x(t) = [a_1(t), \dots, a_n(t)]^T \in \mathbb{R}^n$ we can represent the system (7) in the following form:

$$\dot{x} = A(x)x, \quad (10)$$

where

$$A(x) = L + \sum_{i=1}^n x_i P_i, \quad (11)$$

where $L \in \mathbb{R}^{k \times n}$ is a matrix with rows L_k , and P_i is the matrix comprised of the i th rows of the matrices Q_k .

Based on the standard assumption that multiple velocity field measurements are available, the measurable system outputs can be expressed as

$$y = Cx, \quad (12)$$

where $y \in \mathbb{R}^1$, and $C \in \mathbb{R}^{1 \times n}$ is a known constant vector.

III. SLIDING MODE OBSERVER FOR GENERAL NONLINEAR SYSTEM

In this section, we will describe the general approach that we utilize to design an observer for the nonlinear system (10) - (12). The observer design is based on the sliding mode technique presented in [12].

Consider a nonlinear system given by

$$\dot{x} = f(x), \quad (13)$$

where $x(t) \in \mathbb{R}^n$, with output vector measurements

$$y = h(x) \in \mathbb{R}^m. \quad (14)$$

The system (10) - (12) is a particular case of the system (13) - (14).

Let us introduce the following vector function:

$$H(x) = [h_1(x), \dots, h_n(x)]^T, \quad (15)$$

where

$$h_1(x) = h(x),$$

and

$$h_i(x) = \frac{\partial h_{i-1}(x)}{\partial x} f(x) = L_f^{i-1} h(x), \quad i = 2, \dots, n. \quad (16)$$

are repeated Lie derivatives of $h(x)$ in the direction of the vector field $f(x)$.

From (13) - (14) and (16), we have that if $x(t)$ is a solution of (13), then

$$\frac{d}{dt} h_i(x(t)) = h_{i+1}(x(t)), \quad i = 1, \dots, n-1. \quad (17)$$

As can be seen, the functions $f(x)$ and $h(x)$ in (10) - (12) are sufficiently smooth so that all of the partial derivatives given in (16) exist and are continuous.

In order to ensure that the state $x(t)$ can be found through observation of the output $y(t)$, a nonlinear observability condition is needed. Namely, that for a given domain $X_0 \subset \mathbb{R}^n$ of initial conditions, which is assumed to be bounded, all the solutions of the system (13) belong to the open one-component domain $X \subset \mathbb{R}^n$ for all $0 \leq t < \infty$, and the Jacobian of the diffeomorphism

$$H : \mathbb{R}^n \rightarrow \mathbb{R}^n \quad (18)$$

where $H(x) = [h_1(x), \dots, h_n(x)]^T$, is nonsingular in X , i.e.:

$$\left| \det \frac{\partial H(x)}{\partial x} \right| \geq \delta > 0 \quad (19)$$

for some $\delta > 0$ and every $x \in X$. From here, it follows that the map H is an injection.

To estimate the state variables of system (13) using measurements (14) we can use Drakunov observer suggested in [12] of the form:

$$\dot{\hat{x}} = \left[\frac{\partial H(\hat{x})}{\partial x} \right]^{-1} M(\hat{x}) \text{sign}[V(t) - H(\hat{x})] \quad (20)$$

where $V(t) = [v_1(t), \dots, v_n(t)]^T$ and

$$v_1(t) = y(t) \quad (21)$$

$$v_{i+1}(t) = \{m_i(\hat{x}) \text{sign}[v_i(t) - h_i(\hat{x}(t))]\}_{eq}, \quad i = 1, \dots, n-1 \quad (22)$$

where $\{\dots\}_{eq}$ denotes an ‘‘equivalent value operator’’ of a discontinuous function in sliding mode. The basic notion of the equivalent value in sliding mode was introduced by Utkin (see, for example, [13] for original development). It is consistent with the Filippov definition [14], [15]. The equivalent value in sliding mode can be calculated using different means. For example, if the sliding mode is implemented with chattering, then the equivalent value can be approximated via low-pass filter of the *signum* function. By using higher-order sliding modes, the exact value of the equivalent value can be obtained in finite time without the use of filters.

Let us note here, that sliding mode should not be necessarily associated with chattering phenomena, which is a common misconception. Chattering is a result of nonideal implementation of the sliding mode, but not sliding mode itself. For example, a mass resting on an inclined surface due to Coulomb friction is, in fact, a system with sliding

mode, since idealization of the Coulomb friction model is $-N \text{sign}(\dot{x})$, and for sufficiently large N we have $\dot{x} = 0$. Zero velocity of the mass is classical sliding mode, since $\dot{x} = 0$ is a stable invariant manifold, where the ‘‘control’’ $-N \text{sign}(\dot{x})$ is discontinuous. As a rule, there is no chattering here.

In digital implementation, the finite-time convergence and close to sliding motion without any chattering can be achieved by using continuous approximation of the discontinuous function in small ε -vicinity of the sliding manifold. The correct choice of ε will result in convergence to the vicinity of the sliding manifold that may be sufficient for practical purposes. The other possibility is to use higher-order numerical algorithms. In this case the *sign*-function in a numerical scheme is replaced by a variable v_k calculated on the basis of the previous steps of the algorithm, and once the sliding manifold is reached $v_k = u_{eq}$.

The matrix $M(\hat{x})$ in (20) is an $n \times n$ diagonal matrix with positive diagonal entries $m_i(\hat{x})$, $i = 1, \dots, n$:

$$M(\hat{x}) = \text{diag}[m_1(\hat{x}), \dots, m_n(\hat{x})]. \quad (23)$$

In [12] it is shown that by a suitable choice of $M(\hat{x})$, the observer (20) converges in *any* prescribed finite time interval. The choice of this matrix M is defined by the region of initial conditions X_0 for the system (13) and the upper estimates of h_i .

In the observers shown above we used the *sign*(σ) to provide convergence to the manifold $\sigma = 0$ and then to keep the extended system (system+observer) state on this manifold in spite of the ‘‘disturbance,’’ but in fact, any other type of control with such properties can be used for this purpose (for example with internal dynamics). This issue is computational and depends on the numerical algorithms used to implement continuous-time observer systems.

IV. SLIDING MODE OBSERVER FOR FLUID FLOW

In this section, the sliding mode observer design approach described in the previous section will be applied to the fluid flow dynamic system given in (10) and (12). In this case $f(x) = A(x)x$ is a quadratic function, but the observation equation is linear. This fact can be expressed mathematically as

$$h_1(x) = Cx \quad (24)$$

where the state $x(t) \in \mathbb{R}^n$ contains the unknown Galerkin coefficients, as defined previously. Based on (24), the Lie derivatives of $h(x)$ can be calculated as follows:

$$\begin{aligned} h_2(x) &= \frac{\partial h_1(x)}{\partial x} f(x) = C f(x) \\ h_3(x) &= \frac{\partial h_2(x)}{\partial x} f(x) = C \frac{\partial f(x)}{\partial x} f(x) \\ &\vdots \\ h_m(x) &= C F_m(x) \end{aligned} \quad (25)$$

where the vector fields $F_m(x)$ are defined via the recurrent relation

$$F_m(x) = \frac{\partial F_{m-1}(x)}{\partial x} f(x) \quad (26)$$

and

$$F_1(x) = x.$$

We can use the specific form of $f(x)$ to calculate $F_2(x)$. This will allow us to use the third order sliding mode observer to identify the Galerkin coefficients $a_1(t)$, $a_2(t)$ and $a_3(t)$.

By comparing (10) and (11) with (13), we have

$$F_2(x) = f(x) = \left(L + \sum_{i=1}^n x_i P_i \right) x. \quad (27)$$

The function F_3 is obtained as

$$F_3(x) = \frac{\partial f(x)}{\partial x} f(x)$$

where due to the quadratic form of $f(x)$, the derivative can be calculated as

$$\frac{\partial f(x)}{\partial x} = L + 2 \sum_{i=1}^n x_i P_i. \quad (28)$$

So $F_3(x)$ can be expressed as

$$F_3(x) = \left(L + 2 \sum_{i=1}^n x_i P_i \right) \left(L + \sum_{i=1}^n x_i P_i \right) x. \quad (29)$$

Considering the case $n = 2$, we have

$$\begin{aligned} H(x) &= [h_1(x), h_2(x)]^T \\ &= \left[Cx, C \left(L + \sum_{i=1}^2 x_i P_i \right) x \right]^T. \end{aligned} \quad (30)$$

Based on (30), the Jacobian $\frac{\partial H(x)}{\partial x}$ can be calculated as

$$\frac{\partial H(x)}{\partial x} = \begin{bmatrix} c_1 & c_2 \\ C(L + 2 \sum_{i=1}^2 x_i P_i)_1 & C(L + 2 \sum_{i=1}^2 x_i P_i)_2 \end{bmatrix} \quad (31)$$

where $c_1, c_2 \in \mathbb{R}$ are constants, and the notation $(\cdot)_i$ represents the i^{th} column of the matrix in parentheses.

For the case $n = 3$ we have

$$\begin{aligned} H(x) &= [h_1(x), h_2(x), h_3(x)]^T \\ &= \begin{bmatrix} (Cx)^T \\ \left(C(L + \sum_{i=1}^3 x_i P_i) x \right)^T \\ \left(C(L + 2 \sum_{i=1}^3 x_i P_i) (L + \sum_{i=1}^3 x_i P_i) x \right)^T \end{bmatrix}. \end{aligned} \quad (32)$$

Theorem 1: For any $t_1 > 0$, there exists a diagonal matrix $M(\hat{x})$ such that $\hat{x}(t) \equiv x(t)$ for $t \geq t_1$.

Proof: Proof of Theorem 1 can be found in [12]. ■

It is assumed that the locations of flow velocity sensors can be judiciously selected such that the Jacobian in (31) satisfies observability condition (19). Preliminary results show that this is a mild assumption.

V. SIMULATION RESULTS

The numerical example presented in this section serves only the purpose of numerical verification of implementability of the described method to estimation of Galerkin coefficients.

We used the following system parameters found in the literature (see (7) - (9)): $c_1 = -5; c_2 = 32.7; l_{11} = 30; l_{12} = 36.5; l_{21} = -111.7; l_{22} = -120.5; q_{112} = -45.4; q_{122} = 47.2; q_{211} = -361.1; q_{212} = 304$.

The remaining coefficients q_{kij} were assumed to be zero. Our observer (20) was implemented in Matlab using saturation functions sat_ε with small linear ε -zone around zero, rather than discontinuous (i.e., *sign*) functions. It enabled us to completely avoid using equivalent filters and to just directly use the value of v in (22): $v_{i+1}(t) = m_i(\hat{x}) sat_\varepsilon[v_i(t) - h_i(\hat{x}(t))]$.

The simulation results are shown in Fig. 1 - Fig. 3. In Fig. 1, we demonstrate convergence of the observer for several initial conditions of the Galerkin system. The simulation time interval is 0.5 sec. As can be seen, the observer has a certain overshoot due to the fact that during the reaching phase to the first sliding surface $y - h(\hat{x}) = 0$, the equivalent value of the second sat_ε function is ± 1 . However, once sliding occurs the equivalent value provides enough information for convergence to the second sliding surface $v_2 - L_f h(\hat{x}) = 0$, resulting in convergence of the estimates to the actual values of the Galerkin parameters as shown in Fig. 2 and Fig. 3. Fig. 2 shows the time plots of $a_1(t)$, $a_2(t)$, and their estimates for the entire duration of the simulation. Fig. 3 highlights the first 0.1 sec of the simulation interval to more clearly illustrate that sliding mode starts within 0.01 sec. The initial conditions for the estimates are zero. In Fig. 3, the estimates appear to start from 0.2 and -0.2 , but zooming the simulation plots at the start shows that the estimates almost instantly (within 0.001 sec) jump to 0.2 and -0.2 and then converge within 0.01 sec to the vicinity of the actual $a_i(t)$.

Small differences between $a_i(t)$ and their estimates that are evident in the plot of Fig. 3 are due to numerical error in the algorithm implementation. This is because the sat_ε function was utilized in the Matlab program, instead of the *sign*-function. By reducing ε and the simulation step size, this error can be rendered negligible.

VI. CONCLUSIONS

A POD-based nonlinear observer is presented, which achieves finite-time estimation of the Galerkin coefficients. Utilizing Galerkin projection in conjunction with POD-based model reduction, the incompressible Navier-Stokes equations are recast as a set of nonlinear ordinary differential equations in terms of the unknown coefficients obtained via Galerkin projection. By using POD-based model reduction, the problem of estimating the fluid flow velocity field is achieved by estimating the time-varying Galerkin coefficients. A sliding-mode-based observer is designed to estimate the Galerkin coefficients. Finite-time convergence of the estimates is achieved, and high-fidelity simulation results are provided to illustrate the performance of the observer design. Future work will

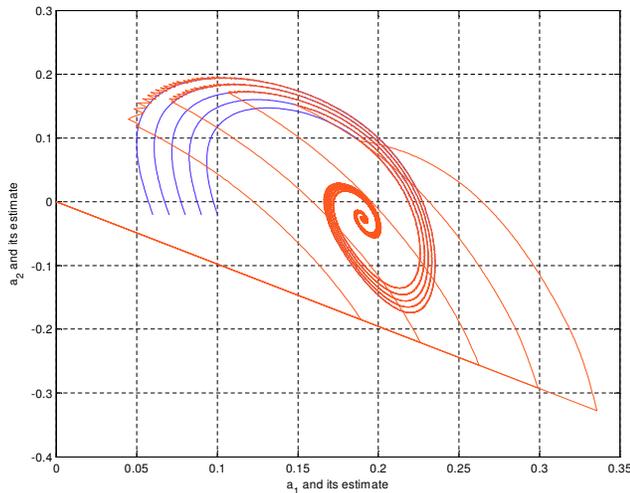


Fig. 1. State space plots of $a_1(t)$, $a_2(t)$ (blue) and their estimates (red).

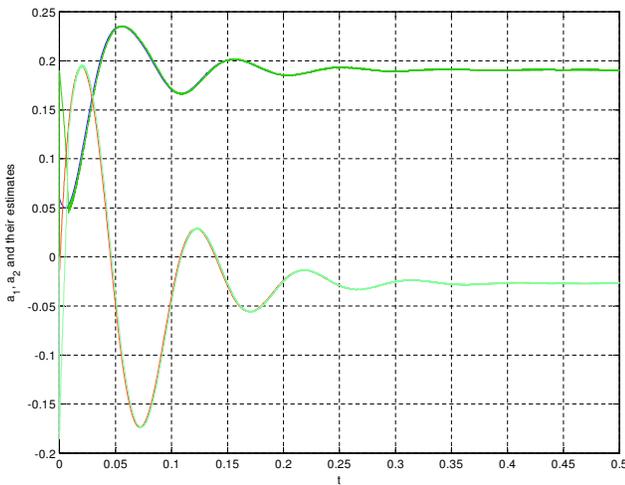


Fig. 2. Time plots of $a_1(t)$, $a_2(t)$, and their estimates for the entire simulation interval.

address the challenges involved in dealing with an unknown output mapping (i.e., where the matrix C is unknown).

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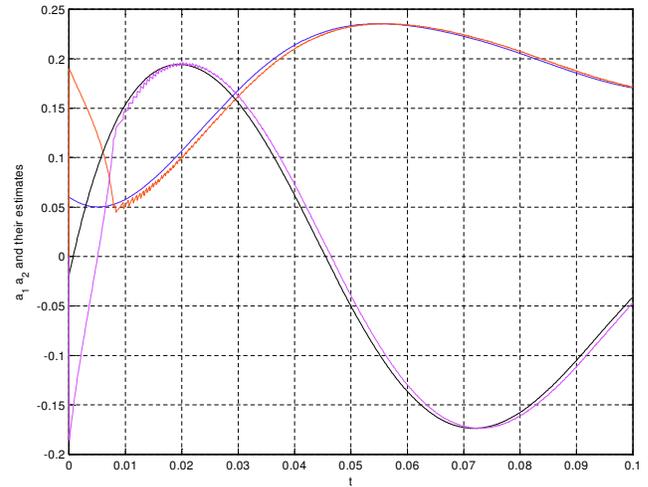


Fig. 3. Initial convergence phase of time plots of $a_1(t)$, $a_2(t)$ (blue), and their estimates (red), highlighting that sliding mode starts within 0.01 sec.

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