

On the Geometric Aspects of the Invariant Ellipsoid Method: Application to the Robust Control Design

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Abstract—This paper deals with a robust control design for a class of nonlinear affine control systems. The dynamical models under consideration are described by ordinary differential equations in the presence of some additive bounded uncertainties. The design procedure for the robust linear feedback control associated with the linearized dynamic model is based on an extended version of the classical invariant ellipsoid method. The stability/robustness analysis of the resulting closed-loop system involves the celebrated Clarke stability theorem that represents a theoretic extension of the celebrated Lyapunov-type methodology. The obtained analytic results are illustrated by a simple computational example.

I. INTRODUCTION

Our paper is devoted to a geometric interpretation of the extended invariant (attractive) ellipsoid techniques. Recall that the conventional method of the invariant ellipsoid (see e.g., [14], [10], [11]) constitutes a powerful theoretical and numerical approach to the problem of control design in some dynamic systems with uncertainties (see [9], [13], [17], [18], [19]).

Recently, the problem of effective methods for the robust design of controllers has attracted a lot of attention, thus both theoretical results and applications were developed, (see e.g., [1], [3], [4], [7], [8], [13], [18], [19]). Various control design strategies are based on modifications of the classic invariant ellipsoid approach. Let us recall that a set in the state space of a dynamic system is said to be positively invariant if any trajectory initiated in this set remains inside the set at all future time instants. Note that some authors also define so called flow-invariant set associated with a dynamic system under consideration (see [6]). If we replace the above invariance by an attractive property related to the trajectories of the given dynamic system, we obtain a concept of an attractive ellipsoid that corresponds to this system (see e.g., [13], [19] for the corresponding results. The general question of existence of an invariant set for an arbitrary dynamical system constitutes a very sophisticated mathematical problem. We discuss this existence problem under some restrictive assumptions related to the structure of the control system under consideration. The next challenging problem (assuming the existence) is related to a constructive characterization of the invariant set. In some cases it is possible to specify an invariant set constructively and, namely, in the form of an ellipsoidal set (see [9], [14], [10], [11], [17], [18], [19]). Evidently, an ellipsoidal

invariant set in the state space of the given closed-loop control system represents a simple suitable region that also can be called as a region of practical stability of the given system. It is well-known that a concrete construction of an invariant or attractive ellipsoid and the corresponding problem of synthesis of a robust state feedback controller usually involves an auxiliary minimization procedure that is related to the size of the ellipsoid. In the framework of linear or quasi-linear systems with bounded uncertainties the size minimization problem mentioned above can usually be reduced to an auxiliary LMI-constrained optimization problem [5], [13], [15], [16], [17], [19]. This LMI-based approach is a direct consequence of the Lyapunov analysis based on quadratic Lyapunov-type functions.

In this paper we use a geometric characterization of an ellipsoidal invariant set. The consideration is based on a general approach proposed in [6]. Our analysis involves a class of control systems with affine structure and the corresponding linearization techniques. Our aim is to give an alternative interpretation of the implementable results obtained in the recent works (for example, in [9], [13], [17], [19]) which are devoted to the attractive ellipsoid method. A possible additional characterization of the developed analytic techniques extends a new viewpoint on the elaborated theoretical and computational techniques. Moreover, the conventional invariant/attractive ellipsoid method was developed for stationary quasi-linear systems (see [9], [15], [17], [19]). Note that the stationarity assumption here is a significant hypothesis that has a restrictive nature. As a direct consequence of this assumption we obtain a possibility to reformulate the above size optimization problem and to deal with an equivalent LMI-constrained optimization. The geometrical approach to the robust control design studied in this paper makes it possible to extend the above techniques to a non-stationary case and to eliminate the above-mentioned restrictive stationarity hypothesis from the consideration.

The remainder of our paper is organized as follows. Section II contains the problem formulation and some necessary mathematical concepts and facts. Section III is devoted to a geometrical characterization of the invariant ellipsoid method. We also discuss here the design procedure for the associated robust feedback control design. The robust control synthesis is obtained as a formal consequence of the invariance requirement. Section IV contains a numerical example. In that section we also propose a new variational method for a possible numerical treatment of the standard LMIs. Section V summarizes the paper.

II. PROBLEM FORMULATION AND SOME PRELIMINARIES

Let us start introducing the following basic initial value problem of affine structure:

$$\begin{aligned} \dot{x}(t) &= f(x(t)) + B(t)u(t) + \xi(t) \text{ a.e. on } \mathbb{R}_+, \\ x(0) &= x_0, \end{aligned} \quad (1)$$

where $x_0 \in \mathbb{R}^n$ is a fixed initial state, The given function $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is assumed to be uniformly Lipschitz continuous on an open bounded set $\mathcal{R} \subseteq \mathbb{R}^n$. By $B(t) \in \mathbb{R}^{n \times m}$, $t \in \mathbb{R}_+$ we denote here a control matrix of the non-stationary affine system. The initial dynamic system contains the uncertainties $\xi(\cdot)$ that are assumed to be uniformly bounded

$$\sup_{t \in \mathbb{R}_+} \|\xi(t)\| \leq M.$$

By $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$ we denote here the state and the control vector, respectively. Motivated by numerous engineering applications, let us firstly consider a system (1) over a control set $\mathcal{U}$ of essentially bounded measurable control inputs.

In parallel with (1) we examine the corresponding linearized control system

$$\begin{aligned} \dot{y}(t) &= f_x(x^u(t))y(t) + B(t)v(t) + \xi(t), \\ y(0) &= 0, \end{aligned} \quad (2)$$

where $u(\cdot) \in \mathcal{U}$ and $x^u(\cdot)$ is the absolutely continuous solution to the initial system (1) generated by the admissible $u(\cdot)$. Note that the auxiliary linear system (1) is studied for a fixed admissible control function $u(\cdot)$. This preselected control strategy is usually called the reference control. We also make the following additional technical assumptions related to (2), namely, we suppose that the pair $(A(t), B(t))$, where $A(t) := f_x(x^u(t))$ is controllable for every $t \in \mathbb{R}_+$.

Let us now determine a class of admissible control functions for the linearized system (2). In this paper we deal with a class of locally Lipschitz feedback type control strategies $w(\cdot)$ such that $w(x^u(t)) = v(t)$ in (2). This class of admissible functions $w(\cdot)$ is denoted by $\mathcal{L}$. For each $u(\cdot) \in \mathcal{U}$ and $w(\cdot) \in \mathcal{L}$ the initial value problem (2) has a unique solution denoted by $y^w(\cdot)$. In this paper we restrict our consideration to a subclass of the introduced admissible feedback controls and consider the concrete design of the type

$$w(y(t)) = K(t)y(t), \quad K(t) \in \mathbb{R}^{m \times n}, \quad t \in \mathbb{R}_+$$

where $K(\cdot)$ is a gain matrix. This unknown gain matrix constitutes a possibility of a prescribed control design (from the class of linear functions) for the given linear system

$$\begin{aligned} \dot{y}(t) &= (f_x(x^u(t)) + B(t)K(t))y(t) + \xi(t), \\ y(0) &= 0, \end{aligned} \quad (3)$$

Our chosen design can be described as follows: the trajectory of the closed-loop linearized system (3) needs to stay in an ellipsoidal region specified by the ellipse with the center at the origin $\mathcal{E} := \{y \in \mathbb{R}^n \mid y^T P y \leq 1\}$, where P is a positive defined symmetrical $n \times n$ matrix.

Our aim is to study the auxiliary linear system (2) and to consider a geometrical characterization of the invariant/attractive ellipsoid approach. Note that various types of

linearized dynamic models have been long time recognized as a powerful tool for solving stability/stabilization problems (see e.g., [22]. We next use the above-mentioned geometric stability criterion for the linearized system (2) in the design procedure of a robust feedback controller for the original nonlinear system (1). Let us also note that the resulting uncertain linear system (1) is a non-stationary system. This non-stationarity restrict the possible direct application of the conventional LMI-based robust control design techniques from [9], [13], [17], [19].

First let us formulate a useful analytic result that makes it possible to establish the quality of the linear approximation of the original system (1) given by the auxiliary system (2). We use here the notation $\mathbb{L}_r^\infty$ for a standard Lebesgue space of measurable essentially bounded r -dimensional vector functions defined on a "big" time interval $I \subset \mathbb{R}_+$.

Theorem 1: Assume that the initial system (1) (considered on a time interval I) satisfies all the above technical assumptions. Then there exists a function $o: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that $s^{-1}o(s) \rightarrow 0$ as $s \downarrow 0$ and

$$\|x^{u+v}(\cdot) - (x^u(\cdot) + y^v(\cdot))\|_{\mathbb{L}_n^\infty} \leq o(\|v(\cdot)\|_{\mathbb{L}_m^\infty})$$

for all $u(\cdot) \in \mathcal{U}$ and $v(\cdot) \in \mathbb{L}_m^\infty$.

Proof: Let $u(\cdot) \in \mathbb{L}_m^\infty$. For an admissible $w(\cdot) \in \mathcal{L}$ we evidently have $v(\cdot) \in \mathbb{L}_m^\infty$. From the basic comparison result with the comparison functions

$$\begin{aligned} z(t) &:= x^u(t) + y^v(t), \\ \psi(t, x) &= f(x) + B(t)u + v(t), \end{aligned}$$

where $t \in \mathbb{R}_+$, we obtain

$$\begin{aligned} &\|x^{u+v}(\cdot) - (x^u(\cdot) + y^v(\cdot))\|_{\mathbb{L}_n^\infty} \leq \\ &e^C \int_I \|\dot{x}^u(t) + \dot{y}^v(t) - f(x^u(t) + y^v(t)) - \\ &B(t)(u(t) + v(t))\| dt = \\ &e^C \int_I \|\langle f_x(x^u(t)), B(t) \rangle, (y^v(t), v(t)) \rangle - \\ &[f(x^u(t) + y^v(t)) + B(t)v(t) - f(x^u(t))]\| dt \end{aligned} \quad (4)$$

for a constant C . From the component-wise variant of the Mean Value Theorem we next deduce

$$\begin{aligned} &f_i(x^u(t)) + y^v(t) + B(t)(u(t) + v(t)) - \\ &(f_i(x^u(t)) + B(t)u(t)) = \\ &\langle (f_i)_x(x^u(t) + v_i(t)), B(t) \rangle (y^v(t), v(t)) \end{aligned}$$

for a suitable bounded $v(\cdot)$ and with $i = 1, \dots, n$, $j = 1, \dots, m$. The Lipschitz continuity of $f(\cdot)$ on a bounded set $\mathcal{R}$ implies the existence of a function $o_1: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that $s^{-1}o_1(s) \rightarrow 0$ as $s \downarrow 0$ and

$$\begin{aligned} &\|\langle f_x(x^u(t)), B(t) \rangle, (y^v(t), v(t)) \rangle - \\ &[f(x^u(t) + y^v(t)) + B(t)v(t) - f(x^u(t))]\| \leq o_1(\|v(\cdot)\|_{\mathbb{L}_m^\infty}) \end{aligned}$$

for all $t \in I$. From (4) we finally deduce the estimation

$$\|x^{u+v}(\cdot) - (x^u(\cdot) + y^v(\cdot))\|_{\mathbb{L}_n^\infty} \leq o(\|v(\cdot)\|_{\mathbb{L}_m^\infty})$$

with $o(s) := e^C o_1(s)$. The proof is finished. $\square$

We will further apply the result from Theorem 1 in a robust control design procedure for the original nonlinear system (1) (see Section III).

Following the invariance concept from [6] we call a smooth manifold S in an Euclidean space a flow-invariant in the sense of a well-defined dynamic system

$$\begin{aligned} \dot{z}(t) &= \phi(z(t)), \quad t \in \mathbb{R}_+, \\ z(0) &= 0 \end{aligned} \quad (5)$$

if $z(t) \in S$ for all $t \geq T \in \mathbb{R}_+$. The next abstract theorem gives a general criterion of the property to be flow-invariant.

Theorem 2: A smooth manifold S is flow-invariant for system (5) iff $\phi(x)$ belongs to the tangent space T_S of S for all x from the given Euclidean space.

Note that the proof of this theorem is based on an extended Lyapunov-type technique (see [6] for details).

III. A GEOMETRICAL INTERPRETATION OF THE INVARIANT ELLIPSOID METHOD

We now introduce a new artificial dynamic variable $\theta(\cdot)$ with $\theta(0) = 0$ and define the specific manifold in the extended Euclidean state space $\mathbb{R}^{n+1}$

$$S := \{z \in \mathbb{R}^{n+1} \mid x^T P x - 1 + \theta(t) = 0\}.$$

Here $z := (x, \theta)^T$. Consider the closed-loop system (3) and introduce the additional notation $h := z^T P z - 1 + \theta$. The necessary and sufficient condition for invariance of the manifold S determined above can be now represented as follows (see Theorem 2)

$$\begin{aligned} \langle \nabla h(z), \phi(z) \rangle &= 0, \\ \theta(t) &\geq 0. \end{aligned} \quad (6)$$

By ∇h we denote the gradient of the function h introduced above. The vector field ϕ corresponds to the right hand side of the following system of equations

$$\begin{aligned} \dot{y}(t) &= (f_x(x^u(t)) + B(t)K(t))y(t) + \xi(t), \\ y(0) &= 0, \\ \dot{\theta}(t) &= -2y^T(t)P\dot{y}(t), \\ \theta(0) &= 0. \end{aligned} \quad (7)$$

Evidently, the second differential equation in (7) can be rewritten in the equivalent form using the dynamics of system (3). The resulting equation has the following form

$$\dot{\theta}(t) = -2y^T(t)P((f_x(x^u(t)) + B(t)K(t))y(t) + \xi(t)). \quad (8)$$

The above observations can be summarized as a formal result.

Theorem 3: The ellipsoidal set $\mathcal{E}$ is an invariant set for the closed-loop system (3) if and only if the variable $\theta(t)$ is non-negative for all $t \in \mathbb{R}_+$.

Proof: Consider the invariance condition (6) and compute the scalar product

$$\langle \nabla h(z), \phi(z) \rangle \Big|_{z=z(t)}.$$

Evidently, we have

$$\begin{aligned} \langle (2y^T(t)P, 1), (\dot{y}(t), -2y^T(t)P\dot{y}(t)) \rangle &= \\ 2y^T(t)P\dot{y}(t) - 2y^T(t)P\dot{y}(t) &= 0. \end{aligned}$$

As we can see, the first condition from (6) is always true. Therefore, (6) is reduced to the second condition, namely, to the non-negativity of the variable $\theta(t)$. The proof is completed. $\square$

The solution of the initial-value problem on a time interval I for the artificial variable $\theta(\cdot)$ has the usual integral representation

$$\theta(t) = \int_I [-2y^T(t)P(f_x(x^u(t)) + B(t)K(t))y(t) + y^T(t)P\xi(t)]dt.$$

Therefore, the non-negativity of $\theta(t)$ is equivalent to the condition

$$\int_I [y^T(t)P(f_x(x^u(t)) + B(t)K(t))y(t) + y^T(t)P\xi(t)]dt \leq 0. \quad (9)$$

Let us now find a suitable gain matrix $K(\cdot)$ from the condition (9). The non-positivity condition of the integral in (9) is evidently true for a nonnegative integrand. That means

$$y^T(t)P(f_x(x^u(t)) + B(t)K(t))y(t) + y^T(t)P\xi(t) \leq 0.$$

Using the boundness condition of the uncertain parameter $\xi(\cdot)$ we obtain

$$y^T(t)P(f_x(x^u(t)) + B(t)K(t))y(t) + M\|y(t)\|\|P\| \leq 0.$$

Evidently, on a bounded time interval I the last inequality is a consequence of the following (more general) matrix inequality

$$Pf_x(x) + PB(t)K(t) + MY\|P\|E \leq 0, \quad (10)$$

with respect to the unknown matrices P and $K(\cdot)$. Note that (10) is considered over all $x \in \mathbb{R}^n$ and not only for the reference state vectors $x^u(t)$. We use here the additional notation $Y := \sup_{t \in I} \|y(t)\|$. By E we denote the unit matrix. Note that the constant Y is well-defined. Recall that the solution $y(\cdot)$ of the linearized system (3) is an absolutely continuous function that attains its maximum on a compact set I . The upper bound Y introduced above can be constructively found in some specific cases. For example, we can deduce the following classic estimation (see e.g., [21])

$$\begin{aligned} Y &\leq \max_{t \in I} \{De^{\omega t}\}, \\ \omega &> \max \sup_{t \in I} \{\text{Re} \lambda(t) \mid \lambda(t) \in \\ &\sigma((f_x(x^u(t)) + B(t)K(t)))\}, \end{aligned}$$

where $D > 0$ is a constant that depends on the given uncertainties bound M . Here $\sigma(G)$ denotes the set of all eigenvalues of a matrix G and $\lambda(t)$ are the eigenvalues of the matrix $((f_x(x^u(t)) + B(t)K(t)))$ for all $t \in I$. Let us introduce an additional restriction for the norm of the matrix P . Let us additionally consider the natural requirement $\|P\| \leq \rho$, where $\rho > 0$ is a prescribed constant. The last condition and a suitable selection of the constant ρ evidently determine the

size of the invariant ellipsoid $\mathcal{E}$. From (10) we deduce the resulting bilinear matrix inequality (BMI)

$$Pf_x(x) + PB(t)K(t) + MY\rho E \leq 0, \quad (11)$$

The BMI-type matrix inequality (11) provides a theoretic fundament for possible implementable numerical procedures. A solution of (11) depends on the prescribed (reference) state variable in the following sense $K(t) = \tilde{K}(x''(t))$ with respect to the reference trajectory $x''(\cdot)$ related the initial system (1). The next section contains a conceptual algorithm for a constructive numerical treatment of the obtained BMI (11).

We now back to the original nonlinearly affine control system (1) considered on a time interval I and ask for an admissible control design $v(t, y)$ such that the combined function $v(t, y(t))$ is an essentially bounded measurable function (an admissible control from $\mathcal{U}$). Moreover, the chosen control function need to possesses the robustness property in the following sense: the closed-loop variant of system (1)

$$\begin{aligned} \dot{x}(t) &= f(x(t)) + B(t)v(t, y(t)) + \xi(t) \text{ a.e. on } I, \\ x(0) &= x_0, \end{aligned} \quad (12)$$

is assumed to be stable in a practical sense. In this contribution we restrict our consideration to the practically stable control design strategy that guarantee an invariancy property of an ellipsoidal set $\mathcal{E}_0$ associated with (1). We now show that $\mathcal{E}_0$ is related to the invariant ellipsoid $\mathcal{E}$ for the linearized system (3).

Theorem 4: Assume that the initial system (1) satisfies all the technical assumptions from Section II and an admissible control $u(\cdot)$ generates a bounded reference trajectory $x''(\cdot)$ $\|x''(\cdot)\|_{\mathbb{L}_m^\infty} \leq \chi$. Let P and $K(\cdot)$ are chosen from the BMI (11). Then the ellipsoidal set

$$\mathcal{E}_0 := (\|K(\cdot)\| + 1) \mathcal{E} + \chi,$$

where $\mathcal{E}$ is the invariant ellipsoid determined by P , is an invariant set for the closed-loop system (12), where the practically stabilizing feedback $v(\cdot, \cdot)$ is defined as follows

$$v(t, y(t)) := u(t) + w(t, y(t)), \quad w(t, y(t)) := K(t)y(t), \quad (13)$$

and $y(\cdot)$ is a solution to the linearized system (3).

Proof: Using the result of Theorem 1, we obtain the corresponding estimation

$$\|x^{u+v}(\cdot) - (x''(\cdot) + y^v(\cdot))\|_{\mathbb{L}_m^\infty} \leq o(\|K(\cdot)y^v(\cdot)\|_{\mathbb{L}_m^\infty}).$$

The last one implies the following

$$\begin{aligned} \|x^{u+v}(\cdot) - x''(\cdot)\|_{\mathbb{L}_m^\infty} &\leq o(\|K(\cdot)y^v(\cdot)\|_{\mathbb{L}_m^\infty}) + \\ \|y^v(\cdot)\|_{\mathbb{L}_n^\infty} &\leq (\|K(\cdot)\| + 1)\|y^v(\cdot)\|_{\mathbb{L}_n^\infty} \end{aligned}$$

and

$$\|x^{u+v}(\cdot)\| \leq (\|K(\cdot)\| + 1)\|y^v(\cdot)\|_{\mathbb{L}_n^\infty} + \|x''(\cdot)\|_{\mathbb{L}_m^\infty}.$$

Since $x''(\cdot)$ is bounded and $\mathcal{E}$ is an invariant set for (3), we deduce the invariance property of the ellipsoid $\mathcal{E}_0$ associated with (12)-(13). The proof is completed. $\square$

Evidently, the obtained Theorem 4 makes it possible to describe constructively an invariant ellipsoidal set for the original system (1) using an invariant ellipsoid associated with the linearized system (3). Moreover, it also specifies the corresponding stabilizing (in the practical sense) feedback-type control strategy (13).

IV. THE NUMERICAL ASPECTS

Our last section is devoted to a numerical illustration of the proposed analytic robust control design technique. We firstly discuss shortly a possible computational approach to the practical treatment of the BMI (11). Let N be a sufficiently large positive integer number and

$$G_N := \{t_0 = 0, t_1, \dots, t_N\},$$

$$\max_{0 \leq k \leq N-1} |t_{k+1} - t_k| \leq r_N$$

be a (possible non-equidistant) partition of I . Let us define $\Delta_{k+1} := t_{k+1} - t_k$, $k = 0, \dots, N-1$ and consider (11) for $t \in G_N$. That means we examine a system of N matrix inequalities of the type

$$\begin{aligned} Pf_x(x) + PB(t_k)K(t_k) + MY\rho E &\leq 0, \\ k &= 0, \dots, N-1. \end{aligned} \quad (14)$$

Note that the computed value of the matrix $K(t_k)$ corresponds to the value $B(t_k)$ of the matrix function $B(\cdot)$. The initial BMI (11) and the discrete variant (14) of this inequality can be reduced to the following linear matrix inequalities (LMIs)

$$\begin{aligned} f_x(x) + B(t)K(t) + MY\rho Z &\leq 0, \\ f_x(x) + B(t_k)K(t_k) + MY\rho Z &\leq 0, \end{aligned} \quad (15)$$

where $Z := P^{-1}$. The symmetrized form of (15) can be written as

$$\begin{aligned} (f_x(x) + f_x(x)^T) + (B(t)K(t) + K^T(t)B^T(t)) + \\ 2MY\rho Z &\leq 0, \\ (f_x(x) + f_x(x)^T) + (B(t_k)K(t_k) + \\ K^T(t_k)B^T(t_k)) + 2MY\rho Z &\leq 0, \end{aligned} \quad (16)$$

The inequalities in (15) and (16) are considered with respect to the pair $(Z, K(\cdot))$ of unknown matrix variables. For every index $k = 0, \dots, N-1$ and the associated $t_k \in G_N$ the discrete LMI from (15) can be solved, for example, using the standard numerical MATLAB routines.

In this paper, we additionally propose a new conceptual approach to a practical solution of a LMI. Our method use an equivalent variational description of (16) and can also be applied to some LMIs of the general form (see e.g., [5]). Consider the continuous LMI from (16) and select a suitable diagonal $2n \times 2n$ matrix $\mathcal{A} < 0$ with the $n \times n$ -dimensional blocks $\mathcal{A}_{11} = \mathcal{A}_{22}$. The system

$$\begin{aligned} F(Z, \mathcal{B}(\cdot)) &:= \text{diag}[(f_x(x) + f_x(x)^T) + (\mathcal{B}(t) + \\ \mathcal{B}^T(t)) + 2MY\rho Z] &= \mathcal{A}, \quad Z = Z^T > 0, \end{aligned}$$

where

$$\mathcal{B}(t) := B(t)K(t), \quad t \in I,$$

determines a possible solution $(\hat{Z}, \hat{\mathcal{B}}(\cdot))$ of the original LMI. This solution depends of the concretely selected matrix $\mathcal{A}$.

The corresponding gain matrix $\hat{K}(\cdot)$ can now be found from the relation

$$B(t)\hat{K}(t) = \mathcal{B}(t).$$

The above linear matrix equation admits the equivalent variational representation:

$$\begin{aligned} & \frac{1}{2} \langle F(Z, \mathcal{B}(\cdot)), (Z, \mathcal{B}(\cdot)) \rangle_{\mathcal{S}^{2n}} - \\ & \langle F(Z, \mathcal{B}(\cdot)), \mathcal{A} \rangle_{\mathcal{S}^{2n}} \rightarrow \max \\ & \text{subject to } (Z, \mathcal{B}(\cdot)) \in \mathcal{S}^{2n}, \\ & Z = Z^T > 0, \end{aligned} \quad (17)$$

where $\mathcal{B}(t) := BK(t)$, $t \in I$ and $\mathcal{S}^{2n}$ is a Hilbert space of the $2n \times 2n$ -dimensional matrices. The scalar product in that space can be defined using the trace: $\langle S_1, S_2 \rangle_{\mathcal{S}^{2n}} := \text{tr}\{S_1 S_2\}$ for some S_1, S_2 from $\mathcal{S}^{2n}$. The discrete variant of (17) related to the corresponding LMI in (16) can now be written as

$$\begin{aligned} & \frac{1}{2} \langle F(Z, \mathcal{B}(t_k)), (Z, \mathcal{B}(t_k)) \rangle_{\mathcal{S}^{2n}} - \\ & \langle F(Z, \mathcal{B}(t_k)), \mathcal{A} \rangle_{\mathcal{S}^{2n}} \rightarrow \max \\ & \text{subject to } (Z, \mathcal{B}(t_k)) \in \mathcal{S}^{2n}, \\ & Z = Z^T > 0, \end{aligned} \quad (18)$$

for

$$\begin{aligned} F(Z, \mathcal{B}(\cdot)) &:= \text{diag}[(f_x(x) + f_x(x)^T) + \\ & (\mathcal{B}(t) + \mathcal{B}^T(t)) + 2MY\rho Z \end{aligned}$$

Evidently, (17) and (18) constitute the quadratic optimization problems in the space of matrices. These problems can be reduced to the conventional maximization problems associated with the quadratic, bilinear and linear combinations of components of the matrices from (17) and (18). The discrete-type variational problem (18) can provide a fundament for the constructive computational treatment of the discretized LMI from (16).

We now apply the proposed theoretic and numeric techniques to an illustrative example and consider the mathematical model of a separately excited DC motor with the dynamics of the type (1) (see [12])

$$\begin{aligned} J \frac{d\Omega}{dt} &= c\Phi_s I_r - B\Omega - \eta, \\ L_r \frac{dI_r}{dt} &= U_r - R_r I_r - c\Phi_s \Omega, \\ \frac{d\Phi_s}{dt} &= U_s - R_s \Phi_s. \end{aligned} \quad (19)$$

The variable Ω denotes the angular velocity of the shaft, I_r and I_s are the currents of the rotor circuit, and R_r and R_s denote the corresponding resistances. The rotor and stator voltages are expressed by U_r and U_s . The rotor inductance is denoted here by L_r and Φ_s is the stator flux. The given parameters J and B in the above model express the moment of inertia of the rotor and the viscous friction coefficient, respectively. Finally, η denotes a parametrical uncertainty (see the general model (1)) and c represents a constant parameter that depends on the spatial architecture of the drive. The control parameters in the above model (19) are denoted by U_r and U_s . The initial conditions in (19) are

selected as follows $(\Omega^0, I_r^0, \Phi_s^0)^T = (1, 1, 1)^T$. The linearized model (2) can be written as follows

$$\begin{aligned} \dot{y}_1 &= -\frac{B}{J}y_1 + \frac{c}{J}\Phi_s^{ref}y_2 + \frac{cI_r^{ref}}{J}y_3 - \eta \\ \dot{y}_2 &= -\frac{c}{L_r}\Phi_s^{ref}y_1 - \frac{R_r}{L_r}y_2 - \frac{c}{L_r}\Omega^{ref} + \frac{1}{L_r}U_r \\ \dot{y}_3 &= -R_s + U_s \end{aligned} \quad (20)$$

where $y := (y_1, y_2, y_3)^T$ is a state vector of the linearized model. An invariant ellipsoid for the linear system (20) was computed using the numerical approach from [9]. The resulting linear-type state space representation is characterized by the following systems matrices:

$$A = \begin{bmatrix} -\frac{B}{J} & \frac{c\Phi_s^{ref}}{J} & \frac{cI_r^{ref}}{J} \\ -\frac{c\Phi_s^{ref}}{L_r} & -\frac{R_r}{L_r} & -\frac{c\Omega^{ref}}{L_r} \\ 0 & 0 & -R_s \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ \frac{1}{L_r} & 0 \\ 0 & 1 \end{bmatrix},$$

and $Q_\eta = I_{3 \times 3}$.

The invariant ellipsoid for the original system 19 is an over extension $\mathcal{E}_0$ (see Theorem 4) of the ellipsoid $\mathcal{E}$ for the linearized system (20). The obtained resulting ellipsoid for system 19 has the following characteristic parameter (the maximal eigenvalue of the matrix P associated with the linearized system) $\max \lambda_P \simeq 0.5 \cdot 10^3$. Recall that the maximal eigenvalue of the ellipsoid matrix P determines the maximal semi-axis of the ellipsoid $\mathcal{E}$. The computational results associated with the invariant ellipsoid for the original nonlinear model (19) and the resulting dynamical behavior of the designed system are presented on Fig. 1 and Fig. 2.

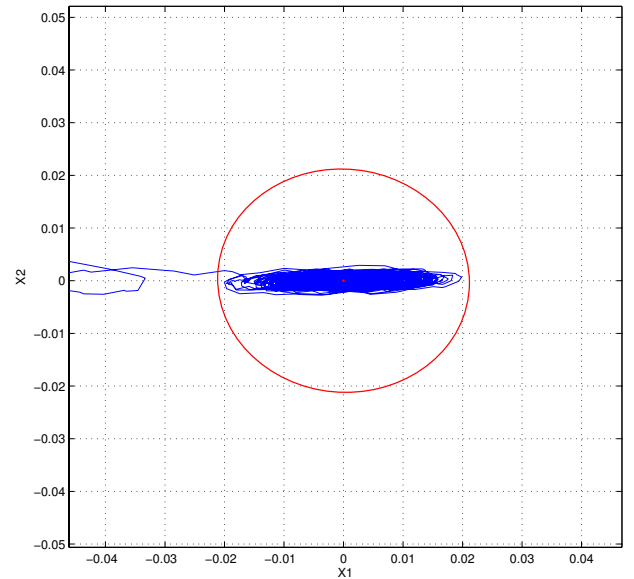


Fig. 1. The dynamics projection on the subspace (x_1, x_2)

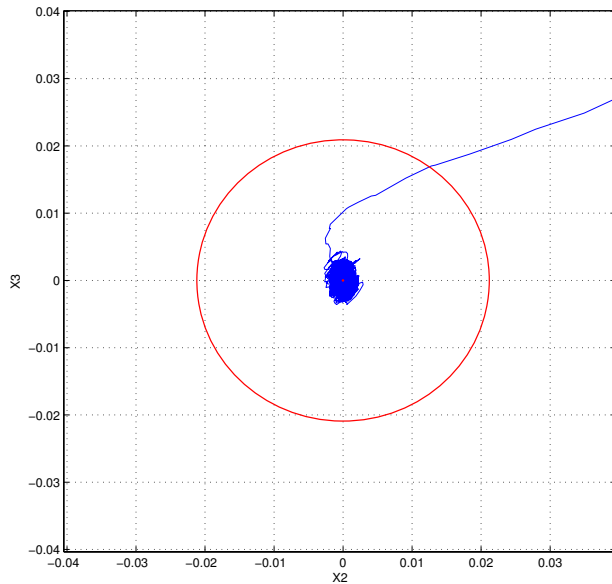


Fig. 2. The dynamics projection on the subspace (x_2, x_3)

Finally note that the implementation of the conceptual computational techniques proposed in this paper was carried out, using the standard MATLAB packages.

V. CONCLUDING REMARKS

The presented paper studies a theoretical approach to a robust control design for a class of systems with uncertainties. We have considered a general class of nonstationary nonlinearly affine systems and constructed the ellipsoidal invariant sets associated with these dynamic models. The main result contains a newly elaborated extension and a geometrical interpretation of the invariant ellipsoid method.

The presented version of the conventional invariant ellipsoid method in combination with the proposed linearization scheme can also be applied to more general nonlinear control systems, for example, to some classes of models with hybrid dynamics (see e. g., [2]). One also can incorporate into the developed analytical framework the alternative types of constrained uncertainties and some practically motivated nonlinear feedback-type control strategies.

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