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Continuous Robust Control for a Class of Uncertain MIMO Nonlinear Systems

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***Abstract*— In this paper, a continuous robust feedback control is designed for a class of high-order multi-input multi-output (MIMO) nonlinear systems with two degrees of freedom containing unstructured nonlinear uncertainties in the drift vector and parametric uncertainties in the high frequency gain matrix, which is allowed to be non-symmetric in general. Given some mild assumptions on the system model, a singularity-free continuous robust tracking control law is designed that is shown to be semi-globally asymptotically stable under full-state feedback through a Lyapunov stability analysis.**

***Index Terms*— Lyapunov-based Control, Nonlinear Control, Robust Control**

I. Introduction

Over the years, numerous progress has been reported on the control design problem for multi-input and multi-output (MIMO) systems with uncertainty based on a variety of techniques. While great strides have been made in the adaptive control design problem for LTI single-input single-output (SISO) systems with uncertainty (see [1]), however, the problem gets much more complex when dealing with the corresponding MIMO system. Some early results on this topic can be found in [1], [2], and [3]. In [1], the High Frequency Gain (HFG) matrix G was assumed to be known for the control design. In [2], a control law was proposed which required the existence of a matrix S such that GS is positive definite and symmetric. Similarly, de Mathelin *et. al.* in [4] assumed that the upper bound for $\|G\|$ was known. In [5], Weller and Goodwin utilized a matrix decomposition approach based on *a priori* knowledge of the system, *i.e.,* given the decomposition $G = LU$, knowledge of the lower bounds of the diagonal entries of the upper triangular matrix U was required to be known. Under the mild assumption that the signs of the leading principal minors of the HFG matrix were known, a MIMO adaptive control law for minimum-phase systems with relative degree one has been proposed by Costa *et. al. in* [6].

When nonlinear MIMO systems with uncertainty are considered, only some special classes of MIMO control design problems can be solved. Based on the assumption that the HFG matrix was known, an adaptive backstepping technique was proposed for parametric strict feedback systems in [7]. Other adaptive control approaches were presented for a class of feedback linearizable systems in [8], [9], [10]. In [11], a robust adaptive control was designed with guaranteed performance, where an output error transformation and a neural approximator were utilized in the control design. A general procedure for designing switching adaptive controllers for multi-input nonlinear systems was proposed in [12]. In [13], Xu *et. al.* formulated a Neural Network (NN) based adaptive controller for a class of MIMO nonlinear systems, which demonstrated the local convergence of the tracking error to a residual set. Some other examples relating to NN applications in MIMO control can be found in [14] and [15]. For a class of MIMO aeroelastic system with a constant HFG matrix, an adaptive output feedback control law was designed in [16] by utilizing the backstepping technique. For a broad class of flat MIMO systems, the output tracking problem was addressed in [17] via full-state feedback adaptive control where a global asymptotic convergence result was obtained. By extending the work in [17], an adaptive output feedback control was designed in [18] but the proposed control law was susceptible to singularities owing to the existence of an algebraic loop in the controller. Later in [19], a singularity free output feedback controller was proposed based on the work in [18], which exploited the triangular structure of U obtained from the SDU decomposition. In [20], a modular output feedback controller was proposed to suppress aeroelastic vibrations on unmodeled nonlinear wing section subject to a variety of external disturbances. In [21] and [22], continuous robust control laws have been designed to stabilize the nonlinear MIMO system with unstructured nonlinearity in both the drift vector and high frequency gain matrix, which yielded semi global Uniformly Ultimately Boundedness (UUB) results.

In this paper, our goal is to design a novel continuous ($\mathcal{C}^0$) robust feedback controller for a general class of high-order MIMO nonlinear systems with two degrees-of-freedom[1] (DOFs) containing unstructured nonlinear uncertainty in the drift vector and parametric uncertainty in the non-symmetric HFG matrix. An important example of a 2-DOF problem with a non-symmetric HFG matrix is the 2D monocular visual servoing control system [23] where the HFG matrix originates from a non-symmetric transformation matrix between the task space coordinate system and the camera space coordinate system. The approach in this paper is motivated by the method for SISO systems presented in

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[1]We note here that we have not been able to extend the result to greater than 2 degrees of freedom at the present time.

[24], but the challenge here is to extend it to the MIMO system presented in this paper where the coupling of the control inputs causes the second control input to appear as a disturbance term in the closed-loop dynamics of the first degree of freedom, thereby, requiring modification to the structure of the first control input. Specifically, the coupling problem is addressed in this paper via a novel adaptive term that is designed and applied to tackle only the control coupling-related disturbance terms for which the structure is known (*i.e.,* there exists only parametric uncertainty). Assuming that the unknown state-dependent HFG matrix G is real, affine in the unknown parameters, and with nonzero leading principal minors, a matrix decomposition approach introduced and applied in [25] can be utilized to design a singularity free control design that can be shown via Lyapunov analysis to yield a semi-global asymptotic stability result for the tracking error under the proposed full state feedback robust control law.

The paper is organized in the following manner. In Section II, we introduce the class of MIMO systems under consideration and the SDU decomposition for the input gain matrix. In Section III, error systems are developed to facilitate the subsequent control design. In Section IV, a full-state feedback continuous robust controller for the MIMO system is proposed and its stability is analyzed. Appropriate conclusions are drawn in Section V.

II. Problem Statement and Preliminaries

In this paper, the following class of MIMO nonlinear systems with two DOFs is considered

$$x^{(n)} = h\left(\mathbf{x}, x^{(n-1)}\right) + G\left(\mathbf{x}, \theta\right) u \tag{1}$$

where $x^{(i)}(t) \in \mathbb{R}^2$, $i = 0, 1, ..., n-1$ denote the system states while $\mathbf{x} \triangleq \begin{bmatrix} x^T & \dot{x}^T & ... & \left(x^{(n-2)}\right)^T \end{bmatrix}^T \in \mathbb{R}^{2n-4}$, $x(t) \in \mathbb{R}^2$ is the system output and $u(t) \in \mathbb{R}^2$ is defined to be the control input. The drift vector $h\left(\mathbf{x}, x^{(n-1)}\right) \in \mathbb{R}^2$ is assumed to be a $\mathcal{C}^2$ nonlinear function with unstructured uncertainty. The high frequency gain matrix $G(\mathbf{x},\theta) \in \mathbb{R}^{2\times 2}$ is also a $\mathcal{C}^2$ nonlinear function and affine in the unknown constant parameter vector $\theta \in \mathbb{R}^p$. For the purpose of robust control design, we assume that $G(\mathbf{x},\theta)$ is a real matrix with nonzero leading principal minors whose signs are assumed to be known. In order to facilitate the continuous robust control design, we begin by differentiating (1) which yields the following expression

$$x^{(n+1)} = f\left(\mathbf{x}, x^{(n-1)}, x^{(n)}\right) + G\left(\mathbf{x},\theta\right)\dot{u} \tag{2}$$

where $f\left(\mathbf{x}, x^{(n-1)}, x^{(n)}\right)$ is defined as

$$f(\cdot) = \dot{h}\left(\mathbf{x}, x^{(n-1)}\right) + \dot{G}\left(\mathbf{x},\theta\right) G^{-1}\left(\mathbf{x},\theta\right)\left(x^{(n)} - h\left(\mathbf{x}, x^{(n-1)}\right)\right). \tag{3}$$

Lemma 1: Any real matrix $G(\mathbf{x},\theta) \in \mathbb{R}^{2\times 2}$ with nonzero leading principal minors can be decomposed as [6]

$$G(\mathbf{x},\theta) = S(\mathbf{x},\theta)\, DU(\mathbf{x},\theta) \tag{4}$$

where $S(\mathbf{x},\theta) \in \mathbb{R}^{2\times 2}$ is a symmetric positive definite matrix, $D \in \mathbb{R}^{2\times 2}$ is a diagonal matrix with diagonal entries $+1$ or -1, $U(\mathbf{x},\theta) \in \mathbb{R}^{2\times 2}$ is a unity upper triangular matrix.

The proof for Lemma 1 can be found in [18] and [25]. Note that D needs to be known for the purposes of control design and it can be obtained from the signs of leading principal minors of $G(\mathbf{x},\theta)$. Also note that if $G(\mathbf{x},\theta)$ is a positive definite matrix, the factorization of $G(\mathbf{x},\theta)$ can be simplified in the form of $G(\mathbf{x},\theta) = S(\mathbf{x},\theta)\, U(\mathbf{x},\theta)$. After taking (4) into (2) and premultiplying $M(\mathbf{x},\theta)$ on both sides of the equation, one can get the following result

$$M(\mathbf{x},\theta)\, x^{(n+1)} = \varphi\left(\mathbf{x}, x^{(n-1)}, x^{(n)}, \theta\right) + DU(\mathbf{x},\theta)\,\dot{u} \tag{5}$$

where $S, U,$ and D have been previously defined in Lemma 1, $M(\mathbf{x},\theta) \triangleq S^{-1}(\mathbf{x},\theta) \in \mathbb{R}^{2\times 2}$ is a symmetric and positive definite matrix while $\varphi\left(\mathbf{x}, x^{(n-1)}, x^{(n)}, \theta\right) \triangleq M(\mathbf{x},\theta) \cdot f\left(\mathbf{x}, x^{(n-1)}, x^{(n)}\right) \in \mathbb{R}^2$ is an unknown auxiliary vector with unstructured uncertainty.

III. Open-Loop Error System Development

In this paper, the objective of the control design is to guarantee the asymptotic convergence of the tracking error as well as to ensure boundedness for all signals during closed-loop operation. To facilitate the following control design, one can first design the bounded desired trajectory $x_d(t) \in \mathbb{R}^2$ to be smooth enough such that

$$x_d^{(i)}(t) \in \mathcal{L}_\infty, \quad \forall\ i = 1, ..., n+2 \tag{6}$$

and $\mathbf{x}_d \triangleq \begin{bmatrix} x_d^T & \dot{x}_d^T & ... & \left(x_d^{(n-1)}\right)^T \end{bmatrix}^T \in \mathbb{R}^{2n-2}$. Then, the tracking error $e_1 \in \mathbb{R}^2$ can be defined as follows

$$e_1 = x_d - x. \tag{7}$$

Furthermore, the following auxiliary error signals $e_i \in \mathbb{R}^2\ \forall\ i = 2, ...n$ are utilized

$$\begin{aligned} e_2 &= \dot{e}_1 + e_1, \\ e_3 &= \dot{e}_2 + e_2 + e_1, \\ &\ \vdots \\ e_n &= \dot{e}_{n-1} + e_{n-1} + e_{n-2}. \end{aligned} \tag{8}$$

The result in [24] shows that e_i can be expressed as

$$e_i(t) = \sum_{j=0}^{i-1} c_{ij} e_1^{(j)}(t) \qquad \forall\ i = 2, 3, ..., n \tag{9}$$

where the known constant coefficients c_{ij} are generated via a Fibonacci number series [24]. Based on above definitions, the filtered error signal $r(t) \in \mathbb{R}^2$ and $z(t) \in \mathbb{R}^{2n+2}$ can be defined as follows

$$\begin{aligned} r &= \dot{e}_n + \alpha e_n, \\ z &\triangleq \begin{bmatrix} e_1^T & e_2^T & ... & e_n^T & r^T \end{bmatrix}^T \end{aligned} \tag{10}$$

where α is a positive gain constant. After taking the time derivative of r in (10) and utilizing (5), (7), (8), and (9), one can obtain the open-loop dynamics as follows

$$\begin{aligned} M\dot{r} = \quad & M\left(x_d^{(n+1)} + \sum_{j=0}^{n-2} c_{ij} e_1^{(j+2)} + \alpha \dot{e}_n\right) \\ & -\varphi\left(\mathbf{x}, x^{(n-1)}, x^{(n)}, \theta\right) + e_n + \Pi \\ & -D\dot{u} - e_n \end{aligned} \tag{11}$$

where $\bar{U}(\mathbf{x},\theta) \in \mathbb{R}^{2\times 2}$ is a strictly upper triangular matrix while $\Pi \in \mathbb{R}^2$ is an auxiliary vector with the following definitions

$$\begin{aligned} &\bar{U}(\mathbf{x},\theta) \triangleq D - DU(\mathbf{x},\theta), \\ &\Pi \triangleq \bar{U}(\mathbf{x},\theta)\,\dot{u} = \begin{bmatrix}\bar{U}_{12}(\mathbf{x},\theta)\,\dot{u}_2 & 0\end{bmatrix}^T. \end{aligned} \tag{12}$$

In order to facilitate the full state control design for above open-loop dynamics, (11) can be rewritten in a compact form as

$$M\dot{r} = -\frac{1}{2}\dot{M}r + N + \Pi - D\dot{u} - e_n \tag{13}$$

where $N(\cdot) \in \mathbb{R}^2$ in (13) is defined as

$$\begin{aligned} N = \quad & M\left(x_d^{(n+1)} + \sum_{j=0}^{n-2} c_{ij} e_1^{(j+2)} + \alpha \dot{e}_n\right) \\ & -\varphi\left(\mathbf{x}, x^{(n-1)}, x^{(n)}, \theta\right) + e_n + \tfrac{1}{2}\dot{M}r \\ = \quad & N_d + \tilde{N}_0 \end{aligned} \tag{14}$$

where $N_d = N\left(\mathbf{x}_d, x_d^{(n)}, x_d^{(n+1)}\right) \in \mathbb{R}^2$ and $\tilde{N}_0 = N - N_d \in \mathbb{R}^2$. Then, it can be easily verified that $\|N_d\|$, $\left\|\dot{N}_d\right\| \in \mathcal{L}_\infty$ given the smoothness of the desired trajectory as given by (6) and the fact that $\varphi\left(\mathbf{x}, x^{(n-1)}, x^{(n)}, \theta\right)$ is a $\mathcal{C}^1$ function. Furthermore, by using the fact that N is continuously differentiable, $\left\|\tilde{N}_0\right\|$ can be upperbounded as

$$\left\|\tilde{N}_0\right\| \leq \rho_0(\|z\|)\,\|z\| \tag{15}$$

where $\rho_0(\cdot)$ is a global invertible nondecreasing function and will be used in the ensuing stability analysis.

IV. Control Development

A. Controller Design and Closed-Loop Error System

By assuming that all the state variables $\mathbf{x}$ are measurable, we can design a continuous robust feedback control law as follows

$$\begin{aligned} u(t) = \quad & D^{-1}\big\{(K+I_2)\,e_n(t) - (K+I_2)\,e_n(0) \\ & + \textstyle\int_0^t \left[\hat{\Phi} + (K+I_2)\,\alpha e_n(\tau) + \Gamma \text{sign}(e_n(\tau))\right] d\tau\big\} \end{aligned} \tag{16}$$

where $K = K_p + diag\{K_{d,1}, 0\} \in \mathbb{R}^{2\times 2}$ and $\Gamma \in \mathbb{R}^{2\times 2}$ are both diagonal gain matrices, $I_2 \in \mathbb{R}^{2\times 2}$ is an identity matrix, $\hat{\Phi}(t) \triangleq \begin{bmatrix} Y\hat{\theta} & 0\end{bmatrix}^T \in \mathbb{R}^2$, while $Y(\cdot)$ and $\hat{\theta}(t)$ will be defined later. In view of (16), the time derivative of $u(t)$ yields

$$\begin{aligned} \dot{u}_1 = \quad & D_{1,1}^{-1}\left[Y\hat{\theta} + (K_{1,1}+1)\,r_1 + \Gamma_{1,1}\text{sign}(e_{n,1})\right], \\ \dot{u}_2 = \quad & D_{2,2}^{-1}\left[(K_{2,2}+1)\,r_2 + \Gamma_{2,2}\text{sign}(e_{n,2})\right] \end{aligned} \tag{17}$$

where $\dot{u}_i(t)$ denotes the i^{th} element in $\dot{u}(t)$, $D_{i,i}$, $K_{i,i}$, and $\Gamma_{i,i}$ denote the i^{th} diagonal element in the matrices D, K, and Γ, respectively, while $e_{n,i}(t)$ and $r_i(t)$ represent the i^{th} element in auxiliary error signal $e_n(t)$ and filtered error signal $r(t)$, respectively. Note that $u_2(t)$ is readily implementable since $e_{n,2}(t)$ is measurable. $Y\hat{\theta}$ in $u_1(t)$ is designed to tackle the coupling-related disturbance terms $\bar{U}_{12}(\mathbf{x},\theta)\,\dot{u}_2$, which we write explicitly as follows

$$\begin{aligned} \Pi \quad &= \begin{bmatrix} \bar{U}_{12}(\mathbf{x},\theta)\, D_{2,2}^{-1}\left[(K_{2,2}+1)\,r_2 + \Gamma_{2,2}\text{sign}(e_{n,2})\right] \\ 0 \end{bmatrix} \\ &= \Lambda + \Phi \end{aligned} \tag{18}$$

where we have obtained the expression in (18) by substituting for $\dot{u}_2(t)$ from (17) into (12). Furthermore, $\Phi \in \mathbb{R}^2$ is a discontinuous auxiliary vector defined as follows

$$\Phi = \begin{bmatrix} Y\theta & 0\end{bmatrix}^T \tag{19}$$

while $\Lambda \in \mathbb{R}^2$ is an auxiliary vector defined as follows

$$\Lambda = \begin{bmatrix} \Lambda_1 & 0\end{bmatrix}^T \tag{20}$$

where $Y \triangleq D_{2,2}^{-1}\Gamma_{2,2}\text{sign}(e_{n,2})\,Y_{12} \in \mathbb{R}^{1\times p}$ is a regression vector, while θ is an unknown parameter vector and we have utilized the fact that $\bar{U}_{12}(\mathbf{x},\theta)$ can be parameterized as $\bar{U}_{12}(\mathbf{x},\theta) = Y_{12}(\mathbf{x})\,\theta$. We note here that the portion of the disturbance represented by (19) cannot be handled via a robustifying term because of its discontinuous nature; however, since Φ is affine in the uncertainty, it can be handled via adaptation as will be shown subsequently. Also note that $\Lambda_1 \triangleq \Delta(\mathbf{x})\,r_2 \in \mathbb{R}$ where $\Delta(\mathbf{x}) \triangleq D_{2,2}^{-1}\bar{U}_{1,2}(\mathbf{x},\theta)\,(K_{2,2}+1)$. After adding and subtracting the term $\Delta_d \triangleq \Delta(\mathbf{x}_d) \in \mathbb{R}$ to Δ, one can obtain

$$\Delta = \tilde{\Delta} + \Delta_d \tag{21}$$

where $\tilde{\Delta} = \Delta(\mathbf{x}) - \Delta_d(\mathbf{x}_d) \in \mathbb{R}$ and $\|\Delta_d\| \in \mathcal{L}_\infty$ based on the boundedness of $\mathbf{x}_d$. By using the fact that $U(\mathbf{x},\theta)$ is continuously differentiable, $\left\|\tilde{\Delta}\right\|$ can be further bounded as

$$\left\|\tilde{\Delta}\right\| \leq \rho_\Delta(\|z\|)\,\|z\| \tag{22}$$

where $\rho_\Delta(\cdot)$ is a global invertible nondecreasing function. Thus, $\Lambda_1 = \left[\tilde{\Delta} + \Delta_d(\mathbf{x}_d)\right] r_2$ can be upperbounded as

$$\begin{aligned} \|\Lambda_1\| \quad & \leq \left\|\tilde{\Delta} + \Delta_d(\mathbf{x}_d)\right\| \|r_2\| \\ & \leq \left[\rho_\Delta(\|z\|)\,\|z\| + \|\Delta_d\|\right] \|z\| \\ & \leq \rho_1(\|z\|)\,\|z\| \end{aligned} \tag{23}$$

where $\rho_1(\cdot)$ is a global invertible nondecreasing function which depends on the gain $K_{2,2}$ – this fact would be utilized in the ensuing stability analysis. We note that the coupling-related disturbance term $\bar{U}_{12}(\mathbf{x},\theta)\,\dot{u}_2$ has been separated into two parts Φ and Λ. While the latter term (which is continuously differentiable) will be compensated by nonlinear damping and the sign function based robustifying term, the former term (which is discontinuous) needs to be dealt with adaptively. Thus, one can define the parameter dynamic estimate as $\hat{\theta} \in \mathbb{R}^p$ and the corresponding mismatch as $\tilde{\theta} = \theta - \hat{\theta} \in \mathbb{R}^p$. Motivated by structure of Y and the following stability analysis, the adaptation law for $\hat{\theta}$ can be designed as follows

$$\begin{aligned}\hat{\theta}(t) &= \int_{t_0}^{t} \Gamma_Y Y r_1 d\tau \\ &= \int_{t_0}^{t} \Gamma_Y Y \dot{e}_{n,1} d\tau + \int_{t_0}^{t} \Gamma_Y Y \alpha e_{n,1} d\tau\end{aligned} \tag{24}$$

where $\Gamma_Y \triangleq \gamma_Y I$ and $I \in \mathbb{R}^{p\times p}$ is a identity matrix while γ_Y is a positive constant. It is important to note that r_1 is unmeasurable since it depends on $\dot{e}_{n,1}$ which in turn depends on $x^{(n)}$ which is not a state variable for the original system model given by (1) and is therefore considered unmeasurable. Therefore, the adaptation law cannot be implemented directly in the form shown in (24). Based on the known value of sign$(e_{n,2})$ and using additivity of integration on intervals, the integral term associated with unknown value $\dot{e}_{n,1}$ in (24) can be rewritten as

$$\begin{aligned}\int_{t_0}^{t} \Gamma_Y Y \dot{e}_{n,1} d\tau = \quad & k \sum_{j=1}^{n} \int_{t_{j,0}^{+}}^{t_{j,f}^{+}} Y_{12} \dot{e}_{n,1} d\tau \\ & -k \sum_{k=1}^{m} \int_{t_{k,0}^{-}}^{t_{k,f}^{-}} Y_{12} \dot{e}_{n,1} d\tau\end{aligned} \tag{25}$$

where $k = \Gamma_Y D_{2,2}^{-1} \Gamma_{2,2}$ and

$$\text{sign}(e_{n,2}) = \begin{cases} 1, & \forall\, t \in \left(t_{j,0}^{+},\, t_{j,f}^{+}\right), \quad j = 1, ..., n \\ -1, & \forall\, t \in \left(t_{k,0}^{-},\, t_{k,f}^{-}\right), \quad k = 1, ..., m \\ 0, & \text{otherwise.} \end{cases} \tag{26}$$

Also note that $(0,\, t] = T^{+} \cup T^{-}$ where $T^{+} = \bigcup_{j=1}^{n} \left(t_{j,0}^{+},\, t_{j,f}^{+}\right]$ and $T^{-} = \bigcup_{k=1}^{m} \left(t_{k,0}^{-},\, t_{k,f}^{-}\right]$. Then, integration by parts can be utilized in each interval in T^{+} and T^{-} as

$$\begin{aligned}\hat{\theta}(t) = \quad & k \sum_{j=1}^{n} \left[Y_{12} e_{n,1} \Big|_{t_{j,0}^{+}}^{t_{j,f}^{+}} - \int_{t_{j,0}^{+}}^{t_{j,f}^{+}} \dot{Y}_{12} e_{n,1}(\tau)\, d\tau \right] \\ & -k \sum_{k=1}^{m} \left[Y_{12} e_{n,1} \Big|_{t_{k,0}^{-}}^{t_{k,f}^{-}} - \int_{t_{k,0}^{-}}^{t_{k,f}^{-}} \dot{Y}_{12} e_{n,1}(\tau)\, d\tau \right] \\ & + \int_{0}^{t} \Gamma_Y Y \alpha e_{n,1} d\tau.\end{aligned} \tag{27}$$

Since $e_{n,1}, Y_{12}(\mathbf{x}), \dot{Y}_{12}\left(\mathbf{x}, x^{(n-1)}\right)$ are measurable, thus $\hat{\theta}(t)$ is implementable in the form shown above. Finally, after substituting (17) into (13), one can obtain the following closed loop error dynamics

$$\begin{aligned}M\dot{r} = \quad & -\tfrac{1}{2}\dot{M}r + N_d + \tilde{N}_0 + \Lambda + \tilde{\Phi} \\ & -(K + I)\, r - \Gamma \text{sign}(e_n) - e_n\end{aligned} \tag{28}$$

where N_d and $\tilde{N}_0$ have been defined previously and $\tilde{\Phi} \triangleq \begin{bmatrix} Y\tilde{\theta} & 0 \end{bmatrix}^T$.

B. Stability Analysis

Before we proceed to analyze the stability of the closed-loop system under the control design proposed in the previous section, we state the following two lemmas

Lemma 2: For the following auxiliary function $L(t) \in \mathbb{R}$

$$L = r^T \left(N_d - \Gamma \text{sign}(e_n)\right), \tag{29}$$

if the control gain matrix Γ is chosen as

$$\Gamma_{i,i} > \|N_{d,i}\|_{\mathcal{L}\infty} + \frac{1}{\alpha} \left\|\dot{N}_{d,i}\right\|_{\mathcal{L}\infty} \qquad \forall\ i = 1, 2 \tag{30}$$

where $N_{d,i}$ is the i^{th} element in the vector N_d, then we can obtain

$$\int_0^t L(\tau)\, d\tau \le \varsigma_L \tag{31}$$

where $\varsigma_L = \sum_{i=1}^{2} \Gamma_{i,i} \left|e_{n,i}(0)\right| - e_{n,i}(0)\, N_{d,i}(0)$.

Proof: The proof for this lemma can be adapted readily from [24]. ■

Lemma 3: Consider a system $\dot{\eta} = h(\eta, t)$ where $h : \mathbb{R}^m \times \mathbb{R}_{\ge 0} \to \mathbb{R}^m$ and the solution exists. Defining the region $D \subset \mathbb{R}^m$ and $D := \{\eta \in \mathbb{R}^m |\, \|\eta\| < \varepsilon\}$ where ε is some positive constant, if there exists a continuously differentiable function $V : D \times \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ such that

$$W_1(\eta) \le V(\eta, t) \le W_2(\eta) \quad \text{and} \quad \dot{V}(\eta, t) \le -W(\eta) \tag{32}$$

where $W_1(\cdot)$ and $W_2(\cdot)$ are continuous positive-definite functions while $W(\cdot)$ is a uniformly continuous positive semidefinite function, and if $\eta(0) \in S$ where the region of attraction is defined as

$$S := \left\{ \eta \in D | W_2(\eta) < \min_{\|\eta\| = \varepsilon} W_1(\eta) \right\},$$

then, it can be shown that

$$W(\eta) \to 0 \quad \text{as} \quad t \to \infty. \tag{33}$$

Proof: The proof for this lemma can be found in Theorem 8.4 of [26]. ■

Theorem 1: Provided the control gain matrix K defined in (16) is chosen to be large enough, $\alpha > 1/2$, and Γ is selected according to (30), the proposed robust control design ensures that all the error signals $e_1^{(i)} \to 0$ as $t \to \infty \quad \forall\ i = 1, ..., n$.

Proof: First, a non-negative Lyapunov function candidate V_0 is defined as

$$V_0(y, t) = \frac{1}{2} \sum_{i=1}^{n} e_i^T e_i + \frac{1}{2} r^T M r + \frac{1}{2} \tilde{\theta}^T \Gamma_Y^{-1} \tilde{\theta} + P \tag{34}$$

where the non-negative auxiliary function P can be defined as follows

$$P = \varsigma_L - \int_0^t L(\tau)\, d\tau \tag{35}$$

and $y = \begin{bmatrix} z & \tilde{\theta} & \sqrt{P} \end{bmatrix}^T \in \mathbb{R}^{2n+4}$. Based on the fact that $M(\mathbf{x}, \theta)$ is positive definite, one can prove that $\underline{M} \le M(\mathbf{x}, \theta) \le \bar{M}(\|y\|)$ where $\underline{M}$ is a positive constant and $\bar{M}(\cdot)$ is a nondecreasing function. Thus, V_0 in (34) can be bounded as follows

$$\begin{gathered}\lambda_1 \|y\|^2 \le V_0(y, t) \le \lambda_2(\|y\|) \|y\|^2 \\ W_1(y) = \lambda_1 \|y\|^2 \quad \text{and} \quad W_2(y) = \lambda_2(\|y\|) \|y\|^2\end{gathered}$$

where $\lambda_1 = \frac{1}{2} \min\left\{1,\ \underline{M},\ \Gamma_Y^{-1}\right\}$, and $\lambda_2 = \frac{1}{2} \max\left\{2,\ \bar{M}(\|y\|),\ \Gamma_Y^{-1}\right\}$. Upon taking the time derivative of (34) and utilizing (35), we obtain

$$\dot{V}_0 = \sum_{i=1}^{n} e_i^T \dot{e}_i + r^T M \dot{r} + \frac{1}{2} r^T \dot{M} r + \tilde{\theta}^T \Gamma_Y^{-1} \dot{\tilde{\theta}}^T - L. \quad (36)$$

By substituting from (7), (8), (10), (24), (28), (29), and utilizing the fact that $ab \leq \frac{1}{2} a^T a + \frac{1}{2} b^T b$, an upper bound for (36) can be obtained as

$$\begin{aligned} \dot{V}_0 \leq & -\sum_{i=1}^{n-2} e_i^T e_i - \tfrac{1}{2} e_{n-1}^T e_{n-1} - \left(\alpha - \tfrac{1}{2}\right) e_n^T e_n - \|r\|^2 \\ & + \|r\| \left\| \tilde{N}_0 \right\| + \|r\| \|\Lambda\| - \lambda_K \|r\|^2 - K_{d,1} r_1^2 \end{aligned} \quad (37)$$

where $\alpha > 1/2$ and λ_K is the maximum eigenvalue for the gain matrix K_p. Thus, $\dot{V}_0$ can be further upperbounded as

$$\begin{aligned} \dot{V}_0 \leq & -\lambda_3 \|z\|^2 + \rho_0 (\|z\|) \|r\| \|z\| \\ & -\lambda_K \|r\|^2 - K_{d,1} r_1^2 + r_1 \rho_1 (\|z\|) \|z\| \end{aligned} \quad (38)$$

where $\lambda_1 = \min\{1/2, \ (\alpha - 1/2)\}$. Then, by adding and subtracting term $\frac{\rho_0^2(\|z\|)}{4\lambda_K} \|z\|^2$ and $\frac{\rho_1^2(\|z\|)}{4K_{d,1}} \|z\|^2$ to the right hand side of the above inequality and utilizing a nonlinear damping argument, one can further upperbound $\dot{V}_0$ as follows

$$\begin{aligned} \dot{V}_0 \leq & -\lambda_4 \|z\|^2 - \left(\frac{\lambda_3 - \lambda_4}{2} - \frac{\rho_0^2(\|z\|)}{4\lambda_K} \right) \|z\|^2 \\ & - \left(\frac{\lambda_3 - \lambda_4}{2} - \frac{\rho_1^2(\|z\|)}{4K_{d,1}} \right) \|z\|^2 . \end{aligned} \quad (39)$$

Given a positive constant $\lambda_4 < \lambda_3$, one can first choose K_p such that $\lambda_K > \frac{\rho_0^2(\|z\|)}{2(\lambda_3 - \lambda_4)}$ or equivalently $z(t) \in \mathcal{D}_1$ where

$$\mathcal{D}_1 \triangleq \left\{ z \mid \|z\| < \rho_0^{-1} \left(\sqrt{2\lambda_K (\lambda_3 - \lambda_4)} \right) \right\}.$$

This ensures that the first parenthesized term in (39) is non-negative. Since $K \triangleq K_p + diag\{K_{d,1}, 0\}$, it is clear to see that $K_{2,2}$ is determined only by K_p and is independent of $K_{d,1}$. Then, based on the fact that ρ_1 depends on $K_{2,2}$, one can select $K_{d,1}$ large enough such that $K_{d,1} > \frac{\rho_1^2(\|z\|)}{2(\lambda_3 - \lambda_4)}$ or $z(t) \in \mathcal{D}_2$ where

$$\mathcal{D}_2 \triangleq \left\{ z \mid \|z\| < \rho_1^{-1} \left(\sqrt{2K_{d,1} (\lambda_3 - \lambda_4)} \right) \right\},$$

and $\mathcal{D}_1 \cap \mathcal{D}_2$ is non-empty. Motivated by Lemma 3 and the definition of y, $\mathcal{D}_1$, and $\mathcal{D}_2$, a region $\mathcal{D}$ can be defined as

$$\begin{aligned} \mathcal{D} \triangleq & \left\{ y \mid \|y\| < \rho_0^{-1} \left(\sqrt{2\lambda_K (\lambda_3 - \lambda_4)} \right) \right\} \\ \cap & \left\{ y \mid \|y\| < \rho_1^{-1} \left(\sqrt{2K_{d,1} (\lambda_3 - \lambda_4)} \right) \right\}. \end{aligned}$$

Thus, it is straightforward to prove that

$$\dot{V}_0 \leq -\lambda_4 \|z\|^2 = -W(y), \quad \forall\, y \in \mathcal{D}. \quad (40)$$

From (34) and (40), it is known that $V_0 \in \mathcal{L}_\infty$, and it is also straightforward to see that e_i, r, $\tilde{\theta}$, $\hat{\theta} \in \mathcal{L}_\infty$ $\forall\ i = 1, ..., n$. Then, by using (9), one can easily see that $e_1^{(i)} \in \mathcal{L}_\infty$ $\forall$ $i = 1, ..., n-1$. Then, by using (8) and (10), one can easily see that $\dot{e}_i \in \mathcal{L}_\infty$ $\forall\ i = 1, ..., n$ which further implies that $e_1^{(n)} \in \mathcal{L}_\infty$ Next, given the fact that x_d is $\mathcal{C}^{n+2}$ smooth and $e_1^{(i)} \in \mathcal{L}_\infty$ $\forall\ i = 1, ..., n$, it is possible to show that $x^{(i)} \in \mathcal{L}_\infty$ $\forall\ i = 1, ..., n$ and $f\left(\mathbf{x}, x^{(n-1)}, x^{(n)}\right)$, $G(\mathbf{x}, \theta) \in \mathcal{L}_\infty$ by using the definition in (7). Now, by utilizing (1), one can show that $u \in \mathcal{L}_\infty$. Based on the fact that $r \in \mathcal{L}_\infty$, we can see that $\dot{u}_2 \in \mathcal{L}_\infty$ according to (17). $Y \in \mathcal{L}_\infty$ based on the boundedness on x_d and e_i. Then, according to previous boundedness result on $\hat{\theta}$, one can also prove that $\dot{u}_1 \in \mathcal{L}_\infty$ given the definition in (17), which further implies $\dot{r} \in \mathcal{L}_\infty$ by using the definition in (11). Thus, given the facts that e_i, $\dot{e}_i$, r, $\dot{r} \in \mathcal{L}_\infty$ $\forall\ i = 1, ..., n$, one can draw the conclusion that $\dot{W} = -\lambda_4 z^T \dot{z} \in \mathcal{L}_\infty$ which implies that $W(y)$ is uniformly continuous.

Based on the definition of $\mathcal{D}$, one can also define a region S as

$$\begin{aligned} S \triangleq & \left\{ y \in \mathcal{D} \mid W_2(y) < \lambda_1 \left(\rho_0^{-1} \left(\sqrt{2\lambda_K (\lambda_3 - \lambda_4)} \right) \right)^2 \right\} \\ \cap & \left\{ y \in \mathcal{D} \mid W_2(y) < \lambda_1 \left(\rho_1^{-1} \left(\sqrt{2K_{d,1} (\lambda_3 - \lambda_4)} \right) \right)^2 \right\}. \end{aligned}$$

Now, one can use Lemma 3 to prove $\|z\| \to 0$ as $t \to \infty$ $\forall\ y(0) \in S$. From (10), one can see that $e_i(t)$, $r(t) \to 0$ as $t \to \infty$ $\forall\ i = 1, ..., n$. By using (9), one can recursively prove that that $e_1^{(i)} \to 0$ $\forall\ i = 1, ..., n$, as $t \to \infty$. Also note that region of attraction S in this problem can be made arbitrarily large to include any initial condition by choosing a large enough control gain. The above facts imply that our stability result is semi-global. ■

V. Conclusion

In this paper, the tracking control design problem for a class of uncertain MIMO nonlinear systems with two degrees of freedom has been considered. Based on mild assumptions about the smoothness of the unknown drift vector and the high frequency gain matrix (which is allowed to be non-symmetric in general), a continuous robust state feedback control strategy was proposed. A Lyapunov based stability analysis was pursued to ensure a semi-global asymptotic stability result for the tracking error under this control. Simulation results in [32] have demonstrated the tracking performance of the proposed control algorithm. Our future work will focus on extending this work to higher degrees of freedom.

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