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Stability Analysis and H_∞ Controller Design of a Class of Switched Discrete-Time Fuzzy Systems

Yong Chen, Lixian Zhang, Hamid Reza Karimi, Xudong Zhao

***Abstract*— In this paper, the problems of stability analysis and H_∞ controller design of a class of switched nonlinear systems are investigated. In a classical way, the modeling of the systems is approached by switched fuzzy systems, and both fast switching and slow switching are considered there. In particular, for slow switching scheme, a new mode-dependent average dwell time switching is proposed for the underlying switched fuzzy systems. Based on a fuzzy-basis-dependent and mode-dependent Lyapunov function, the H_∞ state-feedback controller is derived. A numerical example is given to show the validity and potential of the theoretical results.**

I. INTRODUCTION

In the past two decades, switched systems have been extensively studied since switching feature exists in quite many practical systems subject to internal parameters variations or external environmental changes [1], [2]. Switching signals, either designable or not, are crucial in analysis and design of switched systems. They are often considered as either fast (arbitrary) or slow (dwell time or average dwell time) switching. As a typical slow-switching rule, the average dwell time (ADT) switching means that the number of switches in a finite interval is bounded and the average time between consecutive switching is not less than a constant. So far, many results on ADT switching have been obtained, see [3]–[6] for instance. However, the ADT switching commonly considered in the literature is independent of the system modes and probably leads to a certain conservatism accordingly [7].

On the other hand, the most of studies are focused on switched *linear* systems with vast systematical results obtained, the investigation on switched nonlinear systems are relatively somewhat limited. It has been well recognized that, T-S fuzzy model provides an effective mathematical tool to describe nonlinear systems in a general framework [8]–[10]. Fuzzy logic control approach usually divide the nonlinear system into several local dynamics, which are represented by linear state-space models, and the overall model can be described by fuzzy fusion of these local fuzzy models. Numerous stability analysis and H_∞ control results have been obtained in literature of fuzzy control and fuzzy systems, e.g., [11]–[14], in which basic stability and H_∞ conditions are derived based on Lyapunov theory and formulated in terms of linear matrix inequalities.

Alternatively, to approximate the switched nonlinear systems, some works have been initiated by considering the switched fuzzy systems. Based on common Lyapunov function approach, the stability conditions for switched fuzzy systems have been obtained [15]. Furthermore, the relaxed stabilization conditions for discrete-time switched fuzzy systems by means of the switched quadratic Lyapunov function for all local fuzzy systems have been available in [16], [17]. In addition, some results on guaranteed cost control of switched fuzzy systems have been obtained, readers are referred to [18], [19] and the references therein for more details. It is worth mentioning that, however, for neither fast nor slow switching, the problems of l_2-gain analysis and H_∞ controller design for discrete-time switched fuzzy systems have almost not been investigated so far.

In this paper, we are concerned with using a fuzzy-basis-dependent and mode-dependent Lyapunov function to study the problems of stability analysis and H_∞ controller design for discrete-time switched fuzzy systems. The stability analysis and H_∞ controller design for arbitrary switching are firstly derived and the corresponding results are extended to the case of mode-dependent average dwell time (MDADT). This paper is organized as follows. We review the definitions and lemmas on stability and l_2-gain of switched systems in Section II. The case of switched systems under arbitrary switching is first studied in Section III, the corresponding results are extended to the case of MDADT switching in Section IV. A numerical example is given in Section V, and the paper is concluded in Section VI.

Notation: The notation used in this paper is fairly standard. The superscript "T" stands for matrix transposition, $\mathbb{R}^n$ denotes the n dimensional Euclidean space, respectively. In addition, in symmetric block matrices or long matrix expressions, we use * as an ellipsis for the terms that are induced by symmetry, and $diag\{\cdots\}$ stands for a block-diagonal matrix. Matrices, if their dimensions are not explicitly stated, are assumed to be compatible for algebraic operations. $l_2[0,\infty)$ refers to the space of square summable infinite vector sequences. $\mathbb{C}^1$ denotes the space of continuously differentiable functions. The notation $P>0$ (≥ 0) means that P is a real symmetric positive (semi-positive) definite matrix, and I and 0 represent, respectively, the identity matrix and zero matrix.

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Y. Chen and L. X. Zhang are with the Space Control and Inertial Technology Research Center, Harbin Institute of Technology, Harbin, Heilongjiang 150080, China cyong.hit@gmail.com; lixianzhang@hit.edu.cn

H. R. Karimi is with Department of Engineering, Faculty of Technology and Science, University of Agder, N-4898 Grimstad, Norway hamid.r.karimi@uia.no

X. D. Zhao is with College of Information and Control Engineering, China University of Petroleum, Qingdao, Shandong 266555, China xdzhaohit@gmail.com

II. PRELIMINARIES

The T-S fuzzy systems, suggested by Takagi and Sugeno, can represent a general class of nonlinear systems. Correspondingly, switched nonlinear systems can be approximated as switched fuzzy systems. If a switched nonlinear system is modeled by switched fuzzy system, it can be represented by the forms (include system model and controller) as follows

Subsystem i:

Local Plant Rule p:

If $\xi_1(k)$ is M_{ip1}, and ..., $\xi_r(k)$ is M_{ipr}, then

$$\begin{aligned} x_{k+1} &= A_{ip}x_k + B_{ip}u_k + F_{ip}\omega_k \quad (1)\\ z_k &= C_{ip}x_k + D_{ip}u_k + G_{ip}\omega_k \quad (2)\\ u_k &= K_{ip}x_k \quad (3)\end{aligned}$$

where $\xi_j(k)$ are the premise variables, M_{ip} are the fuzzy sets, $x_k \in \mathbb{R}^{n_x}$ is the state vector, $u_k \in \mathbb{R}^{n_u}$ is the input vector, $\omega_k \in \mathbb{R}^{n_\omega}$ is the disturbance input and belongs to $l_2[0,\infty)$, $z_k \in \mathbb{R}^{n_z}$ is the output vector. The subsystem number i is determined by switching signal σ, which is a piecewise constant of time and takes its value in $\mathcal{I} = \{1,...,s\}$, $s > 1$ is the number of subsystems. At an arbitrary time k, σ may be dependent on k or x_k, or both, or other logic rules. For a switching sequence $k_0 < k_1 < k_2 < ..., \sigma$ is continuous from right everywhere. When $k \in [k_l, k_{l+1})$, we say the σ_{k_l}th subsystem is active. $(A_{ip}, B_{ip}, F_{ip}, C_{ip}, D_{ip}, G_{ip})$ is the pth local model of the ith subsystem. Through the "fuzzy fusion", the final closed-loop switched fuzzy system is given by

$$\begin{aligned} x_{k+1} &= \sum_{p=1}^{\beta(i)}\sum_{q=1}^{\beta(i)} h_{ip}h_{iq}\left(A_{ipq}x_k + F_{ip}\omega_k\right) \quad (4)\\ z_k &= \sum_{p=1}^{\beta(i)}\sum_{q=1}^{\beta(i)} h_{ip}h_{iq}\left(C_{ipq}x_k + G_{ip}\omega_k\right) \quad (5)\end{aligned}$$

where $A_{ipq} = A_{ip} + B_{ip}K_{iq}$, $C_{ipq} = C_{ip} + D_{ip}K_{iq}$, and

$$\eta_{ip} = \prod_{l=1}^{r} M_{ipl}(\xi_l(k)),\ h_{ip} = \frac{\eta_{ip}}{\sum_{p=1}^{\beta(i)}\eta_{ip}} \quad (6)$$

in which $M_{ipl}(\xi_l(k))$ is the grade of membership function of $\xi_l(k)$ in M_{ipl}. It is assumed that $\eta_{ip} \geq 0$ for all k and $p = 1,2,...,\beta(i)$. Therefore, the normalized membership function h_{ip} satisfies

$$h_{ip} \geq 0,\ \sum_{p=1}^{\beta(i)} h_{ip} = 1,\ \text{for all } k \quad (7)$$

Our objective in this paper is to design a state-feedback H_∞ controller for the switched fuzzy system (4)-(5) under arbitrary or mode-dependent average dwell time switching. The following definitions are first recalled.

Definition 1: [7] For a switching signal $\sigma(t)$ and any $T \geq t \geq 0$, let $N_{\sigma i}(t,T)$ be the switching numbers that the ith subsystem is activated over the interval $[t,T]$, and T_i denote the running time of ith subsystem over the interval $[t,T]$, $i \in \mathcal{I}$. We say that $\sigma(t)$ has a mode-dependent average dwell time τ_{ai}, if there exist positive number N_{0i} (we call N_{0i} mode-dependent chatter bounds here), and τ_{ai} such that

$$N_{\sigma i}(t,T) \leq N_{0i} + \frac{T_i}{\tau_{ai}},\ \forall T \geq t \geq 0 \quad (8)$$

Remark 1: Definition 1 has been proposed in [7], it means that if there exist positive numbers τ_{ai}, $i \in \mathcal{I}$ such that a switching signal has the MDADT property, we only require the average time among the intervals associated with the ith subsystem is larger than τ_{ai}.

Definition 2: [1] The switched system (4)-(5) with $\omega_k \equiv 0$ is globally uniformly asymptotically stable (GUAS), if there exists a class $\mathcal{KL}$ function α such that for all initial condition x_0, the solutions of (4)-(5) satisfy the inequality $\|x_k\| \leq \alpha(\|x_0\|, k)$, $\forall k \geq 0$.

Definition 3: For $\gamma > 0$, system (4)-(5) is said to be GUAS with l_2-gain no greater than γ, if under zero initial condition, system (4)-(5) is GUAS and the inequality $\sum_{k=0}^{\infty} z_k^T z_k \leq \sum_{k=0}^{\infty}\gamma^2\omega_k^T\omega_k$ holds for all nonzero $\omega_k \in l_2[0,\infty)$.

Definition 4: [6] For $\gamma_s > 0$, system (4)-(5) is said to be GUAS with weighted l_2-gain no greater than γ_s, if under zero initial condition, system (4)-(5) is GUAS and the inequality $\sum_{k=0}^{\infty}(1-\lambda)^k z_k^T z_k \leq \sum_{k=0}^{\infty}\gamma_s^2\omega_k^T\omega_k$, $0 < \lambda < 1$ holds for all nonzero $\omega_k \in l_2[0,\infty)$.

In addition, the following lemmas are required for later development.

Lemma 1: [5] The following arbitrarily switched system

$$\begin{aligned} x_{k+1} &= A_i x_k + F_i\omega_k \quad (9)\\ z_k &= C_i x_k + G_i\omega_k \quad (10)\end{aligned}$$

is GUAS with l_2-gain no greater than γ, if there exist $\mathbb{C}^1$ functions $V_{\sigma(k)}: \mathbb{R}^n \to \mathbb{R}$, $\sigma(k) \in \mathcal{I}$, class $\mathcal{K}_\infty$ functions κ_{1i} and κ_{2i}, $i \in \mathcal{I}$, and a constant $\gamma > 0$, satisfying

$$\kappa_{1i}(\|x_k\|) \leq V_i(x_k) \leq \kappa_{2i}(\|x_k\|) \quad (11)$$

$$\Delta V_{\sigma(k)} + z_k^T z_k - \gamma^2\omega_k^T\omega_k < 0 \quad (12)$$

Lemma 2: [7] Consider the switched system (9)-(10) with $\omega_k = 0$, and let $0 < \lambda_i < 1$, $\mu_i > 1$, $i \in \mathcal{I}$. Suppose that there exist $\mathbb{C}^1$ functions $V_{\sigma(k)}: \mathbb{R}^n \longrightarrow \mathbb{R}$, $\sigma(k) \in \mathcal{I}$, and class $\mathcal{K}_\infty$ functions κ_{1i} and κ_{2i}, $i \in \mathcal{I}$, such that

$$\begin{aligned} \kappa_{1i}(\|x_k\|) &\leq V_i(x_k) \leq \kappa_{2i}(\|x_k\|) \quad (13)\\ \Delta V_i(x_k) &\leq -\lambda_i V_i(x_k) \quad (14)\end{aligned}$$

and $\forall(\sigma(k_v) = i,\ \sigma(k_{v-1}) = j) \in \mathcal{I}\times\mathcal{I}$, $i \neq j$,

$$V_i(x_{k_v}) \leq \mu_i V_j(x_{k_v}) \quad (15)$$

then the system (9)-(10) is GUAS with $\omega_k = 0$ for any switching signal with MDADT

$$\tau_{ai} > \tau_{ai}^* = -\ln\mu_i/\ln(1-\lambda_i) \quad (16)$$

Lemma 3: Consider the switched system (9)-(10) with $x_0 = 0$, and let $0 < \lambda_i < 1$, $\mu_i > 1$, $i \in \mathcal{I}$. If the system satisfying (13), (14), (15), and $\forall i \in \mathcal{I}$, denoting $\Gamma(k) = z_k^T z_k - \gamma^2\omega_k^T\omega_k$

$$\Delta V_i(x_k) + \lambda_i V_i(x_k) + \Gamma(k) \leq 0 \quad (17)$$

then the switched system (9)-(10) is GUAS and has weighted l_2-gain $\sum_{k=0}^{\infty}(1-\overline{\lambda})^k z_k^T z_k < \gamma_s^2 \sum_{k=0}^{\infty} \omega_k^T \omega_k$ no greater than $\gamma_s = \sqrt{\frac{\overline{\lambda}}{\underline{\lambda}}}\gamma$ for switching signal with MDADT (16), where $\overline{\lambda} = \max\{\lambda_i\}$, $\underline{\lambda} = \min\{\lambda_i\}$.

Remark 2: The proof of Lemma 3 is cumbersome. We omit them here due to space limitation, and we will include them in the journal version of the paper.

Lemma 4: Given two matrices $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{m\times n}$, and symmetric positive definite matrix $P \in \mathbb{R}^{m\times m}$, then

$$A^T PB + B^T PA \leq A^T PA + B^T PB \tag{18}$$

III. STABILITY ANALYSIS AND H_∞ CONTROLLER DESIGN FOR FAST-SWITCHED SYSTEMS

A. Stability and l_2-gain Analysis

The aim here is to find less-conservative conditions ensuring the globally uniformly asymptotic stability and l_2-gain of system (4)-(5). As a natural idea, choosing a common quadratic Lyapunov function of the form $V(x_k) = x_k^T \mathcal{P} x_k$, we can obtain these conditions. However, such an approach often yields overly conservative results. An improved method is to construct fuzzy-basis-dependent and mode-dependent quadratic Lyapunov function (FMDQLF) for switched fuzzy systems. The form of FMDQLF is given by $V(x_k) = x_k^T \mathcal{P}_{\sigma(k)}(\xi(k))x_k$. Now, based on the FMDQLF, we shall present our first result on l_2-gain conditions of switched fuzzy systems (4)-(5) in the following theorem.

Theorem 1: The closed-loop discrete-time switched fuzzy system (4)-(5) is GUAS with l_2-gain no greater than γ, if there exist a set of matrices $P_{ip} > 0$, symmetric matrices X_{ipq}, W_{ipq}, and matrices U_{ipq}, such that $\forall(i,j) \in \mathcal{I}\times\mathcal{I}$, $1 \leq p \leq q \leq \beta(i)$, $1 \leq l \leq \beta(j)$.

$$\Delta_{ipq}^T \begin{bmatrix} P_{jl} & 0 \\ 0 & I \end{bmatrix} \Delta_{ipq} - \begin{bmatrix} P_{ip} & 0 \\ 0 & \gamma^2 I \end{bmatrix} + \Theta_{ipq} < 0 \tag{19}$$

$$\hat{\Xi}_i = \begin{bmatrix} \Theta_{i11} & \Theta_{i12} & \cdots & \Theta_{i1\beta(i)} \\ * & \Theta_{i22} & \cdots & \Theta_{i2\beta(i)} \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & \Theta_{i\beta(i)\beta(i)} \end{bmatrix} > 0 \tag{20}$$

where $\Delta_{ipq} = (\Gamma_{ipq} + \Gamma_{iqp})/2$,

$$\Gamma_{ipq} = \begin{bmatrix} A_{ipq} & F_{ip} \\ C_{ipq} & G_{ip} \end{bmatrix}, \Theta_{ipq} = \begin{bmatrix} X_{ipq} & * \\ U_{ipq} & W_{ipq} \end{bmatrix}$$

Proof. Assume $\mathcal{P}_i(\xi(k))$ to be of the form $\mathcal{P}_i(\xi(k)) = \sum_{l=1}^{\beta(i)} h_{il} P_{il}$, where $P_{il} > 0$, thus, one obtains $\mathcal{P}_i(\xi(k)) > 0$, and the FMDQLF satisfies

$$\kappa_{1i}(\|x_k\|) \leq V_i(x_k) \leq \kappa_{2i}(\|x_k\|)$$

for some class $\mathcal{K}_\infty$ functions κ_{1i} and κ_{2i}. Consider the closed-loop system (4)-(5) with $\omega_k = 0$, $\forall(i,j) \in \mathcal{I}\times\mathcal{I}$, taking the difference between $V(x_{k+1})$ and $V(x_k)$, we have

$$\begin{aligned}
\Delta V &= x_k^T \sum_{l=1}^{\beta(j)} h_{jl} \sum_{p=1}^{\beta(i)}\sum_{q=1}^{\beta(i)}\sum_{m=1}^{\beta(i)}\sum_{n=1}^{\beta(i)} h_{ip}h_{iq}h_{im}h_{in} \\
&\quad \times \left(A_{ipq}^T P_{jl} A_{imn} - P_{ip}\right) x_k \\
&= x_k^T \sum_{l=1}^{\beta(j)} h_{jl} \sum_{p=1}^{\beta(i)}\sum_{q=1}^{\beta(i)}\sum_{m=1}^{\beta(i)}\sum_{n=1}^{\beta(i)} \frac{1}{4} h_{ip}h_{iq}h_{im}h_{in} \\
&\quad \times \left[(A_{ipq}+A_{iqp})^T P_{jl}(A_{imn}+A_{inm}) - 4P_{ip}\right] x_k \\
&= x_k^T \sum_{l=1}^{\beta(j)} h_{jl} \sum_{p=1}^{\beta(i)}\sum_{q=1}^{\beta(i)}\sum_{m=1}^{\beta(i)}\sum_{n=1}^{\beta(i)} \frac{1}{8} h_{ip}h_{iq}h_{im}h_{in} \\
&\quad \times \left[(A_{ipq}+A_{iqp})^T P_{jl}(A_{imn}+A_{inm}) - 8P_{ip}\right. \\
&\quad \left.+ (A_{imn}+A_{inm})^T P_{jl}(A_{ipq}+A_{iqp})\right] x_k
\end{aligned}$$

then, according to Lemma 4, we obtain

$$\begin{aligned}
&\Delta V \\
\leq\ & x_k^T \sum_{l=1}^{\beta(j)} h_{jl} \sum_{p=1}^{\beta(i)}\sum_{q=1}^{\beta(i)}\sum_{m=1}^{\beta(i)}\sum_{n=1}^{\beta(i)} \frac{1}{8} h_{ip}h_{iq}h_{im}h_{in} \\
& \times \left[(A_{ipq}+A_{iqp})^T P_{jl}(A_{ipq}+A_{iqp}) - 8P_{ip}\right. \\
& \left.+ (A_{imn}+A_{inm})^T P_{jl}(A_{imn}+A_{inm})\right] x_k \\
=\ & x_k^T \sum_{l=1}^{\beta(j)} h_{jl} \sum_{p=1}^{\beta(i)}\sum_{q=1}^{\beta(i)} \frac{1}{4} h_{ip}h_{iq} \\
& \times \left[(A_{ipq}+A_{iqp})^T P_{jl}(A_{ipq}+A_{iqp}) - 4P_{ip}\right] x_k \\
=\ & x_k^T \sum_{l=1}^{\beta(j)} h_{jl} \sum_{p=1}^{\beta(i)} \frac{1}{4} h_{ip}^2 \\
& \times \left[(A_{ipp}+A_{ipp})^T P_{jl}(A_{ipp}+A_{ipp}) - 4P_{ip}\right] x_k \\
& +2x_k^T \sum_{l=1}^{\beta(j)} h_{jl} \sum_{p=1}^{\beta(i)}\sum_{q>p}^{\beta(i)} \frac{1}{4} h_{ip}h_{iq} \\
& \times \left[(A_{ipq}+A_{iqp})^T P_{jl}(A_{ipq}+A_{iqp}) - 4P_{ip}\right] x_k
\end{aligned}$$

from (19), it is not difficult to get

$$(A_{ipq}+A_{iqp})^T P_{jl}(A_{ipq}+A_{iqp}) - 4P_{ip} + 4X_{ipq} < 0$$

with the above inequality, we have

$$\begin{aligned}
&\Delta V \\
<\ & -x_k^T \sum_{l=1}^{\beta(j)} h_{jl} \left(\sum_{p=1}^{\beta(i)} h_{ip}^2 X_{ipp}\right. \\
& \left. + 2\sum_{p=1}^{\beta(i)}\sum_{q>p}^{\beta(i)} h_{ip}h_{iq}X_{ipq}\right) x_k \\
=\ & -x_k^T \begin{bmatrix} h_{i1}I \\ h_{i2}I \\ \vdots \\ h_{i\beta(i)}I \end{bmatrix}^T \hat{X}_i \begin{bmatrix} h_{i1}I \\ h_{i2}I \\ \vdots \\ h_{i\beta(i)}I \end{bmatrix} x_k
\end{aligned}$$

where

$$\hat{X}_i = \begin{bmatrix} X_{i11} & X_{i12} & \cdots & X_{i1\beta(i)} \\ * & X_{i22} & \cdots & X_{i2\beta(i)} \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & X_{i\beta(i)\beta(i)} \end{bmatrix}$$

by inequality (20), one has

$$\hat{X}_i > 0$$

that is, $\Delta V < 0$, the stability of system (4)-(5) with $\omega_k = 0$ is guaranteed. On the other hand, by some mathematical operations, l_2-gain of system (4)-(5) can be proved. Consider the closed-loop system (4)-(5) with zero initial condition, we have

$$\begin{aligned}\Delta V &= \zeta_k^T \sum_{l=1}^{\beta(j)} h_{jl} \sum_{p=1}^{\beta(i)}\sum_{q=1}^{\beta(i)}\sum_{m=1}^{\beta(i)}\sum_{n=1}^{\beta(i)} h_{ip}h_{iq}h_{im}h_{in} \\ &\quad \times \left(\begin{bmatrix} A_{ipq}^T \\ F_{ip}^T \end{bmatrix} P_{jl} \begin{bmatrix} A_{imn} & F_{im} \end{bmatrix} - \begin{bmatrix} P_{ip} & 0 \\ 0 & 0 \end{bmatrix} \right) \zeta_k\end{aligned}$$

$$\begin{aligned}z_k^T z_k - \gamma^2 \omega_k^T \omega_k &= \zeta_k^T \sum_{p=1}^{\beta(i)}\sum_{q=1}^{\beta(i)}\sum_{m=1}^{\beta(i)}\sum_{n=1}^{\beta(i)} h_{ip}h_{iq}h_{im}h_{in} \\ &\quad \times \left(\begin{bmatrix} C_{ipq}^T \\ G_{ip}^T \end{bmatrix} \begin{bmatrix} C_{imn} & G_{im} \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & \gamma^2 I \end{bmatrix} \right) \zeta_k\end{aligned}$$

where

$$\zeta_k = \begin{bmatrix} x_k^T & \omega_k^T \end{bmatrix}^T$$

it follows from Lemma 1 that

$$\begin{aligned}&\Delta V + z_k^T z_k - \gamma^2 \omega_k^T \omega_k \\ &= \zeta_k^T \sum_{l=1}^{\beta(j)} h_{jl} \sum_{p=1}^{\beta(i)}\sum_{q=1}^{\beta(i)}\sum_{m=1}^{\beta(i)}\sum_{n=1}^{\beta(i)} h_{ip}h_{iq}h_{im}h_{in} \\ &\times \left(\begin{bmatrix} A_{ipq}^T & C_{ipq}^T \\ F_{ip}^T & G_{ip}^T \end{bmatrix} \begin{bmatrix} P_{jl} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_{imn} & F_{im} \\ C_{imn} & G_{im} \end{bmatrix} \right. \\ &\qquad \left. - \begin{bmatrix} P_{ip} & 0 \\ 0 & \gamma^2 I \end{bmatrix} \right) \zeta_k\end{aligned}$$

using the same technique as stability analysis, we can obtain (12). Then, according to Lemma 1, system (4)-(5) has the l_2-gain no great than γ. This completes the proof. □

B. H_∞ Controller Design

Now, based on stability and l_2-gain results in Theorem 1, the conditions of a switched fuzzy-dependent controller ensuring the GUAS and H_∞ performance of system (4)-(5) can be obtained, which is given in the following theorem.

Theorem 2: If there exist a set of matrices $Q_{ip} > 0$, symmetric matrices X_{ipq}, W_{ipq}, and matrices Ω_i, U_{ipq}, Y_{ip}, such that $\forall (i,j) \in \mathcal{I} \times \mathcal{I}$, $1 \le p \le q \le \beta(i)$, $1 \le l \le \beta(j)$.

$$\begin{bmatrix} O_{ipq} & * & * & * \\ U_{ipq} & W_{ipq} - \gamma^2 I & * & * \\ M_{ipq} & \frac{F_{ip}+F_{iq}}{2} & -Q_{jl} & * \\ N_{ipq} & \frac{G_{ip}+G_{iq}}{2} & 0 & -I \end{bmatrix} < 0 \tag{21}$$

$$\hat{\Xi}_i > 0 \tag{22}$$

where

$$\begin{aligned} O_{ipq} &= Q_{ip} - \left(\Omega_i + \Omega_i^T\right) + X_{ipq} \\ M_{ipq} &= \left(A_{ip}\Omega_i + A_{iq}\Omega_i + B_{ip}Y_{iq} + B_{iq}Y_{ip}\right)/2 \\ N_{ipq} &= \left(C_{ip}\Omega_i + C_{iq}\Omega_i + D_{ip}Y_{iq} + D_{iq}Y_{ip}\right)/2 \end{aligned}$$

holds. Then the closed-loop system (4)-(5) is GUAS with H_∞ performance index γ, and the H_∞ controller is given by

$$K_{ip} = Y_{ip}\Omega_i^{-1} \tag{23}$$

Proof. Consider the closed-loop system (4)-(5), applying the change of variable $Y_{ip} = K_{ip}\Omega_i$ to (21), one has

$$\begin{bmatrix} O_{ipq} & * & * & * \\ U_{ipq} & W_{ipq} - \gamma^2 I & * & * \\ \frac{A_{ipq}+A_{iqp}}{2}\Omega_i & \frac{F_{ip}+F_{iq}}{2} & -Q_{jl} & * \\ \frac{C_{ipq}+C_{iqp}}{2}\Omega_i & \frac{G_{ip}+G_{iq}}{2} & 0 & -I \end{bmatrix} < 0$$

with the inequality $(Q_{ip} - \Omega_i)^T Q_{ip}^{-1} (Q_{ip} - \Omega_i) \ge 0$, which is equivalent to $\Omega_i^T Q_{ip}^{-1} \Omega_i \ge \Omega_i + \Omega_i^T - Q_{ip}$, yields

$$\begin{bmatrix} X_{ipq} - \Omega_i^T Q_{ip}^{-1}\Omega_i & * & * & * \\ U_{ipq} & W_{ipq} - \gamma^2 I & * & * \\ \frac{A_{ipq}+A_{iqp}}{2}\Omega_i & \frac{F_{ip}+F_{iq}}{2} & -Q_{jl} & * \\ \frac{C_{ipq}+C_{iqp}}{2}\Omega_i & \frac{G_{ip}+G_{iq}}{2} & 0 & -I \end{bmatrix} < 0$$

it follows from (21) , (22) that $0 < Q_{ip} < \Omega_i + \Omega_i^T - X_{ipp}$, $X_{ipp} > 0$, one has $0 < Q_{ip} < \Omega_i + \Omega_i^T$, which implies that Ω_i is invertible. Pre-multiplying $diag(\Omega_i^{-T}, I, I, I)$ and post-multiplying $diag(\Omega_i^{-1}, I, I, I)$ to above inequality, lead to

$$\begin{bmatrix} \Omega_i^{-T} X_{ipq}\Omega_i^{-1} - Q_{ip}^{-1} & * & * & * \\ U_{ipq}\Omega_i^{-1} & W_{ipq} - \gamma^2 I & * & * \\ \frac{A_{ipq}+A_{iqp}}{2} & \frac{F_{ip}+F_{iq}}{2} & -Q_{jl} & * \\ \frac{C_{ipq}+C_{iqp}}{2} & \frac{G_{ip}+G_{iq}}{2} & 0 & -I \end{bmatrix} < 0$$

setting $Q_{ip}^{-1} = P_{ip}$, $\Omega_i^{-T} X_{ipq}\Omega_i^{-1} = \hat{X}_{ipq}$, $U_{ipq}\Omega_i^{-1} = \hat{U}_{ipq}$. We have

$$\begin{bmatrix} \hat{X}_{ipq} - P_{ip} & * & * & * \\ \hat{U}_{ipq} & W_{ipq} - \gamma^2 I & * & * \\ \frac{A_{ipq}+A_{iqp}}{2} & \frac{F_{ip}+F_{iq}}{2} & -P_{jl}^{-1} & * \\ \frac{C_{ipq}+C_{iqp}}{2} & \frac{G_{ip}+G_{iq}}{2} & 0 & -I \end{bmatrix} < 0$$

by Schur complement, the above inequality is equivalent to (19). In addition, pre-multiplying $diag\{\Omega_i^{-T},I,\Omega_i^{-T},I,...,\Omega_i^{-T},I\}$ and post-multiplying $diag\{\Omega_i^{-1},I,\Omega_i^{-1},I,...,\Omega_i^{-1},I\}$ to (22), one has (20). Therefore, by Theorem 1, switched fuzzy system (4)-(5) is GUAS and has a H_∞ performance index γ. □

Remark 3: Note that, if we omit subscript i and j in Theorem 2, we will get the corresponding H_∞ controller design conditions for non-switched fuzzy systems, which have been proposed in [13].

IV. EXTENSION TO SLOW-SWITCHED SYSTEMS

Based on the similar techniques used in the previous developments, we consider a class of MDADT switching defined as (8), and have the following propositions:

Proposition 1: Consider the closed-loop system (4)-(5), and let $0 < \lambda_i < 1$, $\mu_i \ge 1$ be the given constant. If there exist matrices $P_{ip} > 0$, symmetric matrices X_{ipq},

W_{ipq}, and matrices U_{ipq}, such that $\forall(i,j) \in \mathcal{I} \times \mathcal{I}$, $i \neq j$, $1 \leq p \leq q \leq \beta(i)$, $1 \leq l \leq \beta(i)$.

$$\Delta_{ipq}^T \begin{bmatrix} P_{il} & 0 \\ 0 & I \end{bmatrix} \Delta_{ipq} - \begin{bmatrix} (1-\lambda_i) P_{ip} & 0 \\ 0 & \gamma^2 I \end{bmatrix} + \begin{bmatrix} X_{ipq} & * \\ U_{ipq} & W_{ipq} \end{bmatrix} \leq 0 \quad (24)$$

$$\hat{\Xi}_i \geq 0 \quad (25)$$

$$P_{jl} - \mu_j P_{ip} \leq 0 \quad (26)$$

where

$$\gamma_s = \sqrt{\frac{\overline{\lambda}}{\underline{\lambda}}}\gamma, \ \Delta_{ipq} = \frac{\Gamma_{ipq} + \Gamma_{iqp}}{2}, \ \Gamma_{ipq} = \begin{bmatrix} A_{ipq} & F_{ip} \\ C_{ipq} & G_{ip} \end{bmatrix}$$

then system (4)-(5) under MDADT switching (16) is GUAS with weighted l_2-gain no greater than γ_s.

Proposition 2: Consider the closed-loop system (4)-(5), and let $0 < \lambda_i < 1$, $\mu_i \geq 1$ be the given constant. If there exist a set of matrices $Q_{ip} > 0$, symmetric matrices X_{ipq}, W_{ipq}, matrices Ω_i, U_{ipq}, Y_{ip}, such that $\forall(i,j) \in \mathcal{I} \times \mathcal{I}$, $i \neq j$, $1 \leq p \leq q \leq \beta(i)$, $1 \leq l \leq \beta(i)$.

$$\begin{bmatrix} O_{ipq} & * & * & * \\ U_{ipq} & W_{ipq} - \gamma^2 I & * & * \\ M_{ipq} & \frac{F_{ip}+F_{iq}}{2} & -Q_{il} & * \\ N_{ipq} & \frac{G_{ip}+G_{iq}}{2} & 0 & -I \end{bmatrix} \leq 0 \quad (27)$$

$$\hat{\Xi}_i \geq 0 \quad (28)$$

$$Q_{ip} - \mu_j Q_{jl} \leq 0 \quad (29)$$

where

$$\begin{aligned} O_{ipq} &= Q_{ip}/(1-\lambda_i) - \left(\Omega_i + \Omega_i^T\right) + X_{ipq} \\ M_{ipq} &= \left(A_{ip}\Omega_i + A_{iq}\Omega_i + B_{ip}Y_{iq} + B_{iq}Y_{ip}\right)/2 \\ N_{ipq} &= \left(C_{ip}\Omega_i + C_{iq}\Omega_i + D_{ip}Y_{iq} + D_{iq}Y_{ip}\right)/2 \end{aligned}$$

holds. Then system (4)-(5) is GUAS with weighted l_2-gain no greater than $\gamma_s = \sqrt{\frac{\overline{\lambda}}{\underline{\lambda}}}\gamma$ under MDADT switching (16) and the controller is given by (23).

Remark 4: Incorporating the slow switching considered in Lemma 2 and Lemma 3, the proof for the aforementioned propositions can be obtained using the same techniques in previous section, and we will include them in the journal version of the paper.

V. NUMERICAL EXAMPLE

In this section, to illustrate the effectiveness of the results developed in previous sections, we consider the following discrete-time switched fuzzy system (4)-(5) consists of two subsystems, and each subsystem has two fuzzy rules.

$$A_{11} = \begin{bmatrix} 0.9 & 0.28 \\ -0.02 & 0.72 \end{bmatrix}, A_{12} = \begin{bmatrix} 0.8 & 0.1 \\ 0.09 & 0.84 \end{bmatrix},$$

$$A_{21} = \begin{bmatrix} 0.85 & 0.1 \\ 0.11 & 0.85 \end{bmatrix}, A_{22} = \begin{bmatrix} 0.86 & 0.11 \\ 0.41 & 0.95 \end{bmatrix},$$

$$B_{11} = \begin{bmatrix} 0.6 \\ -0.6 \end{bmatrix}, B_{12} = \begin{bmatrix} 0.5 \\ 0.3 \end{bmatrix}, B_{21} = \begin{bmatrix} 0.4 \\ 0.5 \end{bmatrix}, B_{22} = \begin{bmatrix} 0.3 \\ 0.8 \end{bmatrix},$$

$$F_{11} = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix}, F_{12} = \begin{bmatrix} 0.4 \\ 0 \end{bmatrix}, F_{21} = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}, F_{22} = \begin{bmatrix} 0.6 \\ 2 \end{bmatrix},$$

$$C_{11} = \begin{bmatrix} 0.5 & 0.2 \end{bmatrix}, C_{12} = \begin{bmatrix} 0.8 & 0.4 \end{bmatrix},$$

$$C_{21} = \begin{bmatrix} 0.9 & 0.8 \end{bmatrix}, C_{22} = \begin{bmatrix} 0.85 & 0.7 \end{bmatrix},$$

$$D_{11} = 0.1, D_{12} = 0.72, D_{21} = 0.3, D_{22} = 0.92,$$

$$G_{11} = 0.2, G_{12} = 0.7, G_{21} = 1.3, G_{22} = 0.8.$$

The fuzzy membership functions are taken as

$$h_{11} = \sin^2(x_1 + 0.5), \ h_{12} = 1 - h_{11},$$

$$h_{21} = 1 - \frac{1}{1 + \exp(-3x_2/(0.5 - \pi/2))} \times \frac{1}{1 + \exp(-3x_2/(0.5 + \pi/2))}, \ h_{22} = 1 - h_{21}$$

Our purpose here is to design a set of fuzzy-basis-dependent and mode-dependent state-feedback controllers such that the resulting closed-loop system is stable with an optimized H_∞ disturbance attenuation performance.

By using the LMI toolbox in Matlab to solve the conditions in Theorem 2, we can get $\gamma = 3.1371$ and the H_∞ controllers gains as

K_{11}=[-0.1794 -0.0300], K_{12}=[-0.7554 -0.5348],
K_{21}=[-1.3135 -1.3078], K_{22}=[-1.0707 -0.9476]. (30)

Applying the controllers in (30) and generating an arbitrary switching sequence, one can get the steady-response of the resulting closed-loop system as shown in Fig.1 for $\omega(k) = 1 \times \exp(-0.5k)$.

Then, we consider the MDADT switching. Given $\lambda_1 = 0.1, \lambda_2 = 0.2, \mu_1 = 1.4, \mu_2 = 2$ and solving the conditions in Proposition 2. We can obtain $\gamma_s = 7.3746$ and the controllers gains as

K_{11}=[-0.0867 0.0176], K_{12}=[-0.1182 -0.0147],
K_{21}=[-1.0359 -1.1985], K_{22}=[-1.0192 -0.9932]. (31)

Using the controllers and generating switching sequences with MDADT property $\tau_{1a} = 3.1935$, $\tau_{2a} = 3.1063$. The steady-response of the system is given in Fig.2 with $\omega(k) = 1 \times \exp(-0.5k)$. Fig.3 gives the comparison on H_∞ performance indices that the resulting closed-loop systems can achieve when applying (30) and (31), respectively. It can be seen from Fig.3 that the designed controllers in (30) and (31) under the admissible switching signals are effective.

VI. CONCLUSIONS AND FUTURE WORKS

In this paper, the problems of stability analysis and H_∞ controller design of a class of switched nonlinear systems, which are switched discrete-time fuzzy systems, have been studied. A fuzzy-basis-dependent and mode-dependent Lyapunov function is proposed for the system, and a new H_∞

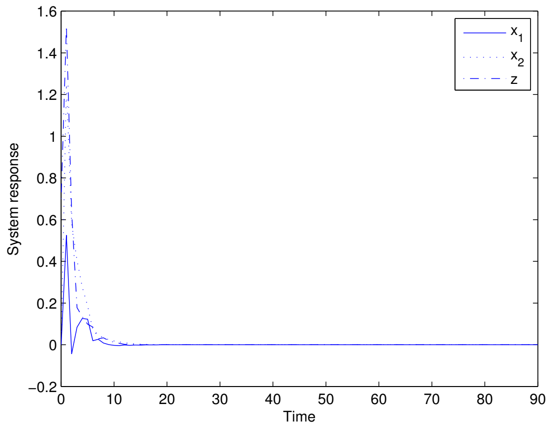

Fig. 1. System response of the closed-loop system in arbitrary switching

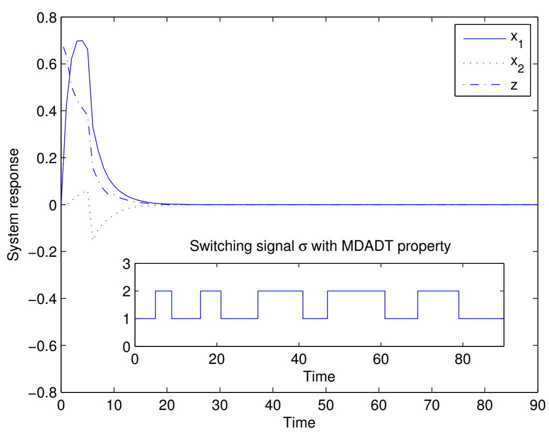

Fig. 2. System response of the closed-loop system in MDADT switching

controller is derived via linear matrix inequalities formulations. We also remarked that the obtained conditions cover the case of non-switched fuzzy systems, which has been widely studied. Future works will be concerned with the investigations on the applications of switched fuzzy systems to real systems, such as vehicle suspension systems, robotic fish systems, etc.

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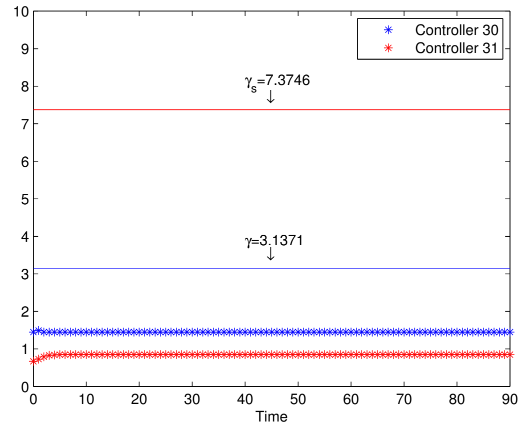

Fig. 3. H_∞ performance indices of the closed-loop systems

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