2011 50th IEEE Conference on Decision and Control and
European Control Conference (CDC-ECC)
Orlando, FL, USA, December 12-15, 2011

A new class of Lyapunov functions for nonstandard switching systems: the stability analysis problem

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***Abstract*— This paper presents a new class of Lur'e type Lyapunov functions for a discrete-time switched system interconnected with a switched nonlinearity satisfying a mode-dependent cone bounded condition. This function includes the mode-dependent nonlinearity, but not its integral. Such a Lyapunov function allows to obtain sufficient conditions in terms of linear matrix inequalities (LMI), for the stability analysis in two different frameworks: global stability analysis for the considered systems and local stability analysis for these systems with an additional saturating input consisting of a switched linear state feedback. In the second case, an optimization problem based on these sufficient conditions is provided to enlarge the estimation of the basin of attraction, which may be composed of non-convex and disconnected sets, because of the presence of the nonlinearities in the Lyapunov function. Some numerical examples are presented to highlight the relevance of the new Lyapunov function and of the proposed method.**

I. INTRODUCTION

HYBRID dynamical systems consist of a finite set of subsystems or modes, with only one active mode at each time instant. This set is governed by a logical decision-making automata defining the active mode. The automata might be state-, or output- or external input-dependent. Among the class of hybrid systems we may exhibit the switching ones [12], where the switching rule is considered as *a priori* unknown signal, but possible to estimate or measure its current value. The properties to be ensured should be, so, satisfied for any arbitrary switching rule. In a large number of applications, the modes are formulated as linear subsystems.

Nonetheless this is, in general, an approximation to model physical systems with limited validity domain, because the actuators cannot provide unbounded magnitude signals, and the dynamics might be nonlinear. Thus, some nonlinearities with respect to the state and/or ones with respect to the control input, as saturation, should be taken into account to refine the modeling step and make it more realistic. These nonlinearities are naturally mode-dependent, and are here characterized by cone bounded sector conditions, which are also mode-dependent.

The stability of a system formed by a linear system connected to a cone bounded output nonlinearity has been widely studied in continuous and discrete time domains. This problem is called Lur'e problem, [11]. Among different candidates Lyapunov functions, one can point out the *Lyapunov Lur'e-type* function. This function is composed of a quadratic (in the state) term and an integral term, depending on the nonlinearity. In the continuous-time case, [9], this choice is natural because the function, by the aid of the sector condition, induces the presence of the nonlinearity in the time-derivative of Lyapunov function.

In the discrete-time domain, the same function was considered either for stability analysis (see [7], [8], [14] and references therein), or for control synthesis purposes, [3]. The Lyapunov difference of the Lur'e type functions induces a difference of integrals, which should be upper-bounded by assuming that the nonlinearity maximal variation is finite and upper-bounded.

Nevertheless, the original Lyapunov Lur'e-type function requires that the nonlinearity is time-invariant (see [11]) to ensure its positive definiteness and is thus not able to cope with the interconnection between a switched linear system and a mode-dependent nonlinearity.

Thus, in this paper, a new switching Lyapunov function is proposed in order to firstly relax the assumption related to the nonlinearity maximal variation and secondly to cope with switched systems including mode-dependent nonlinearities. Based on this new Lyapunov function, we tackle two different stability analysis problems of switching systems. In case of interconnection between a switching linear system and switching nonlinearities, sufficient conditions are given through Linear Matrix Inequalities (LMI) to ensure global stability for any switching rule. When adding a saturating switching state-feedback to such an interconnection, sufficient conditions are provided to ensure the local stability for any switching rule. In this last case, an optimization problem under LMI constraints will be presented to maximize the size of an estimation – given by a level set of the proposed Lyapunov function – of the basin of attraction.

The paper is organized as follows: Section II recalls the classical Lur'e type problem and its associated function and emphasize the impossibility to extend the framework to switched interconnections. In Section III, a new type of switched Lyapunov function dependent on the switched cone bounded nonlinearity is presented. A sufficient condition for the global stability problem is proposed in Section IV and an

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This work was partially supported by the project PICS CNRS n° 5284; CAPES-COFECUB n°701/11 and by ANR project ArHyCo, Programme "Systèmes Embarqués et Grandes Infrastructures" - ARPEGE, contract number ANR-2008 SEGI 004 01-30011459 and finally by the European Community's Seventh Framework Programme (FP7/2007-2013) under grant agreement n°257462: HYCON2 Network of Excellence "Highly-Complex and Networked Control Systems".

academic example indicates the feasibility of the proposed results. Closed-loop formulation considering a switched linear state feedback subject to input saturation is presented in Section V. An optimization problem obtaining the largest estimation of the basin of attraction, induced by a level set of the Lyapunov function is provided in Section VI and examples are given to illustrate our results. Concluding remarks are presented in Section VII.

Notation: The components of any vector $x \in \mathbb{R}^n$ are denoted $x_{(\ell)}, \forall \ell = 1, ..., n$. Inequalities between vectors are component-wise: $x \succeq 0$ states that $x_{(\ell)} \geq 0$ and $x \succeq y$ means that $x_{(\ell)} - y_{(\ell)} \geq 0$. $A_{(\ell)}$ (resp. $A_{i,(\ell)}$) denotes the ℓth row of matrix A (resp. A_i). For two symmetric matrices, A and B, $A > B$ means that $A - B$ is positive definite. A' denotes the transpose of matrix A. The operator $\text{diag}(x)$ means a diagonal matrix obtained from vector x. The symbol $\star$ stands for symmetric block in matrices. The ellipsoidal set $\mathcal{E}(M, \gamma)$ associated with $M > 0$ is given by $\{x \in \mathbb{R}^n; x'Mx \leq \gamma\}$ and $\mathcal{E}(M) = \mathcal{E}(M, 1)$.

II. Discrete-time Lur'e systems and candidate Lyapunov functions

A. Classical Lur'e problem

Consider the following discrete-time scalar system with a cone-bounded nonlinearity $\varphi(y_k)$

$$x_{k+1} = ax_k + f\varphi(y_k), \tag{1}$$

$$y_k = cx_k, \tag{2}$$

where $|a| < 1$, x_k, $y_k \in \mathbb{R}$ are respectively the state and output of system (1)-(2) at the instant $k \in \mathbb{N}$.

The nonlinearity $\varphi(\cdot) : \mathbb{R} \to \mathbb{R}$ is assumed to satisfy a cone bounded sector condition, *i. e.*, $\varphi(0) = 0$ and $\varphi(\cdot) \in [0, \omega]$, for some $\omega > 0$, $\forall y \in \mathbb{R}$: $\varphi(y)\,[\varphi(y) - \omega y] \leq 0$. The stability analysis of this class of systems is known as the Lur'e problem, which has been studied either in continuous- or discrete-time domain.

Two different types of Lyapunov functions are commonly considered for this problem: a purely quadratic in state function and the named *Lur'e-type* function, which is described as follows

$$v(x) = x'Px + \eta \int_0^y \varphi(s)\mathrm{d}s, \tag{3}$$

for some $\eta \geq 0$ and assuming the nonlinearity $\varphi(\cdot)$ is time-invariant, [11]. This function was, originally, proposed in the continuous-time domain, [9] and is suitable because it induces the presence of the nonlinearity in its time-derivative.

The function defined by (3), containing the nonlinearity integral, was adapted firstly to the discrete-time domain in [13] and has been discussed and refined in [4], [7], [8], [10], [14], [15].

It is well known that the stability Lyapunov condition, for discrete-time systems, is given by the Lyapunov difference $\delta v = v(x_{k+1}) - v(x_k)$. Thus, one has

$$\delta v = x'_{k+1}Px_{k+1} - x'_kPx_k + I_{k+1} - I_k, \tag{4}$$

with

$$I_k = \eta \int_0^{y_k} \varphi(s)\mathrm{d}s \tag{5}$$

and where the integral term difference may be rewritten as a single integral

$$\overline{I} = I_{k+1} - I_k = \eta \int_{y_k}^{y_{k+1}} \varphi(s)\mathrm{d}s. \tag{6}$$

In all mentioned references, one can notice that this remaining integral term is always upper-bounded, and requires an extra assumption restricting the maximal nonlinearity's variation (either its derivative, $\frac{d\varphi(y)}{dy} \leq D_{\max}$, [7], [8], [14], [15], or its discrete variation $\frac{\varphi(y_{k+1})-\varphi(y_k)}{y_{k+1}-y_k} \leq D_{\max}$, if it is not differentiable, [3], [4], [10]).

B. Lur'e-type function and switching systems

Consider, now, a discrete-time switching system composed of N nonlinear modes ($N \in \mathbb{N}$, $N \geq 1$)

$$x_{k+1} = A_{\sigma(k)}x_k + F_{\sigma(k)}\varphi_{\sigma(k)}(y_k), \tag{7}$$

$$y_k = C_{\sigma(k)}x_k, \tag{8}$$

where $x_k \in \mathbb{R}^n$, $y_k \in \mathbb{R}^p$ are respectively the state and output vector of the system (7)-(8) at the instant $k \in \mathbb{N}$. This system will satisfy the following assumptions:

Assumption 1: The switching rule $\sigma(\cdot)$ takes its values in the finite set $\mathcal{I}_N = \{1, \cdots, N\}$. $\sigma(\cdot)$ is assumed to be not known *a priori*, but its current value $\sigma(k)$ is assumed to be available in real-time.

The notation $M_{\sigma(k)}$ means that, at each time k, $M_{\sigma(k)}$ takes its value in the set $\{M_1, \cdots, M_N\}$ indexed by $\sigma(k)$. The matrices A_i, C_i and F_i have appropriate dimensions. Matrices A_i are assumed to be stable, $\forall\, i \in \mathcal{I}_N$.

Assumption 2: The N nonlinearities $\varphi_i(\cdot) : \mathbb{R}^p \to \mathbb{R}^p$ associated with each mode $i \in \mathcal{I}_N$ are assumed to satisfy their own cone bounded sector conditions and to be decentralized [11].

Thus, system nonlinearities verify the following statement [11]: $\varphi_i(\cdot) \in [0_p, \Omega_i]$, *i.e.*, $\varphi_i(0) = 0$ and there exist N diagonal positive definite matrices $\Omega_i = \Omega'_i \in \mathbb{R}^{p\times p}$ such that independently, $\forall y \in \mathbb{R}^p$ and $\forall \ell = 1, \cdots, p$, $\varphi_{i,(\ell)}(y)\,[\varphi_i(y) - \Omega_i y]_{(\ell)} \leq 0$. Hence, we have the following inequality, $\forall i \in \mathcal{I}_N$:

$$\text{SC}(\varphi_i(\cdot), y, \Lambda_i) = \varphi'_i(y)\Lambda_i[\varphi_i(y) - \Omega_i y] \leq 0, \tag{9}$$

where $\Lambda_i \triangleq \text{diag}\{\lambda_{q,i}\}_{q=1;\cdots;p} \in \mathbb{R}^{p\times p}$ are any diagonal and positive matrices. Note that Ω_i is given by the designer and assumed to be known hereafter for each mode $i \in \mathcal{I}_N$.

It is simple, from *Assumption* 2, to show that the relation (9) is equivalent to $[\Omega_i y]_{(\ell)}[\varphi_i(y) - \Omega_i y]_{(\ell)} \leq 0$, $\forall \ell = 1, \cdots, p$; $\forall y \in \mathbb{R}^p$; $\forall i \in \mathcal{I}_N$, which implies, with Λ_i diagonal positive definite, that

$$0 \leq \varphi'_i(y)\Lambda_i\varphi_i(y) \leq \varphi'_i(y)\Lambda_i\Omega_i y \leq y'\Omega'_i\Lambda_i\Omega_i y,\ \forall y \in \mathbb{R}^p. \tag{10}$$

The stability analysis of systems (7)-(8) cannot be formulated by extending the function (3) because the nonlinearity is mode-dependent and also time dependent. To simplify the notations, let us assume here that $p = 1$. To obtain an integral term like (6), we should have the history dependent function, that is a function depending not only of the current output y_k, but also on all the past outputs $\{y_l\}_{l=0,\cdots,k-1}$.

$$I_k = \eta \sum_{l=0}^{k-1} \int_{y_l}^{y_{l+1}} \varphi_{\sigma(l)}(s)\mathrm{d}s. \tag{11}$$

However, the positivity of function $v(x)$ is not guaranteed because of the integral term may be negative, for instance, in case of $y_{l+1} < y_l$. In fact, the choice of a Lur'e type function containing an integral of the nonlinearity is only natural in the continuous-time domain, not in the discrete-time domain.

Another candidate function might be the one presented in [5]. This function is quadratic with respect to an augmented vector, containing the system state and the nonlinearity evaluated q (for a given integer $q \geq 1$) instants ahead. The stability conditions are derived for an auxiliary system defined recursively from the original one. However, the system nonlinearity is assumed to be time-invariant.

In the following section, a Lyapunov function candidate will be presented to relax the assumption about the maximal variation of the mode-dependent nonlinearities and to allow the time-dependency of the nonlinearities.

III. New class of Lyapunov functions

A new class of switched Lyapunov functions for discrete-time switching system (7)-(8) is proposed. This candidate function depends on the current value of the switching rule and is composed of a quadratic term with respect to the state and a cross term between the state and the switched nonlinearities.

$$V : \begin{cases} \mathcal{I}_N \times \mathbb{R}^n \times \mathbb{R}^p & \longrightarrow \quad \mathbb{R}, \\ (i, x, \phi) & \longmapsto \quad x' P_i x + 2\phi' \Delta_i \Omega_i C_i x. \end{cases} \tag{12}$$

where matrix $P_i \in \mathbb{R}^{n\times n}$ is symmetric positive definite and $\Delta_i \in \mathbb{R}^{p\times p}$ is diagonal positive semidefinite ($i \in \mathcal{I}_N$).

From inequalities (10), it is thus possible to define a lower and an upper bounds given by the quadratic functions $\underline{V}_i(x) = x'P_i x$ and $\overline{V}_i(x) = x'(P_i + 2C_i'\Omega_i'\Delta_i\Omega_i C_i)x$, respectively, such that

$$\underline{V}_i(x) \leq V(i; x; \varphi_i(C_i x)) \leq \overline{V}_i(x), \quad \forall i \in \mathcal{I}_N. \tag{13}$$

The function V can be considered as candidate, because it verifies the following properties:

- $V(i; x; \varphi_i(C_i x)) \geq 0$, $\forall x \in \mathbb{R}^n$, $\forall i \in \mathcal{I}_N$, due to the first inequality in (13),
- $V(i; x; \varphi_i(C_i x)) = 0$, if and only if $x = 0$, $\forall i \in \mathcal{I}_N$. This is induced by inequality (13) and by $P_i > 0$,
- $V(i; x; \varphi_i(C_i x))$ is unbounded, $\forall i \in \mathcal{I}_N$.

The Lyapunov difference is denoted by

$$\begin{aligned} \delta_k V \quad = \quad & V(\sigma(k+1); x_{k+1}; \varphi_{\sigma(k+1)}(C_{\sigma(k+1)} x_{k+1})) \\ & -V(\sigma(k); x_k; \varphi_{\sigma(k)}(C_{\sigma(k)} x_k)). \end{aligned} \tag{14}$$

In the following sections the Lyapunov function (12) will be used as a tool to study several stability problems.

IV. Global Stability Analysis

Let us consider the system (7)-(8). The problem of the global stability analysis is given as follows.

Problem 1: (Global Stability Analysis) For the system (7)-(8), under Assumptions 1 and 2, determine matrices $P_i = P_i' > 0_n$ and diagonal matrices $\Delta_i \geq 0_p$ such that system (7)-(8) is globally stable, for any switching rule.

In this section, sufficient conditions to solve the Problem 1 are formulated by considering the function V.

Proposition 1: Let us consider the system (7)-(8), if there exists matrices $G_i \in \mathbb{R}^{n\times n}$, symmetric positive definite matrices $P_i \in \mathbb{R}^{n\times n}$ and positive diagonal matrices $T_i, W_i, \Delta_i \in \mathbb{R}^{p\times p}$, such that the LMI

$$\begin{bmatrix} P_j - G_j' - G_j & G_j' A_i & G_j' F_i & 0_{n\times p} \\ \star & -P_i & \Pi_1 & A_i' \Pi_2 \\ \star & \star & -2T_i & F_i' \Pi_2 \\ \star & \star & \star & -2W_j \end{bmatrix} < 0, \tag{15}$$

where

$$\Pi_1 = C_i' \bar{\Omega}_i \left[T_i - \Delta_i \right]; \quad \Pi_2 = C_j' \bar{\Omega}_j \left[W_j + \Delta_j \right], \tag{16}$$

$\forall\ (i, j) \in \mathcal{I}_N^2$, are verified, then the origin of system (7)-(8) is globally asymptotically stable, under any switching rule.

Remark 1: The inclusions $\{A_i\}_{i\in\mathcal{I}_N}$ and $\{(A_i + F_i\Omega_i C_i)\}_{i\in\mathcal{I}_N}$ should be both stable to allow the feasibility of inequality (15). These necessary conditions are obtained by considering the bounds of the sector condition (9).

Proof: If Inequality (15) holds, we have $P_j - G_j' - G_j < 0$ and $P_i > 0$. Thus, G_j is of full rank, and so $-G_j' P_j^{-1} G_j \leq P_j - G_j' - G_j$ (see [2]). This implies, combined with the change of basis $\mathrm{diag}[G_j^{-1}; I_{n+2p}]$ and a Schur complement [1], inequality (30).

In the sequel, by multiplying Inequality (30) on the right by $[x_k'\ \varphi_i'(y_k)\ \varphi_j'(y_{k+1})]'$ and on the left by its transpose and by identifying $i = \sigma(k)$ and $j = \sigma(k+1)$, it leads to inequality

$$\begin{aligned} \delta_k V - 2\mathrm{SC}(\varphi_{\sigma(k+1)}(\cdot), y_{k+1}, W_{\sigma(k+1)})) \\ -2\mathrm{SC}(\varphi_{\sigma(k)}(\cdot), y_k, T_{\sigma(k)})) \leq 0. \end{aligned} \tag{17}$$

Since nonlinearities $\varphi_{\sigma(k+1)}(y_{k+1})$ and $\varphi_{\sigma(k)}(y_k)$ verify a global sector condition, inequality (17) defines an upper-bound for the Lyapunov difference, implying $\delta_k V < 0$, for any $x_k \neq 0$. ■

Example 1: Consider system (7)-(8) with $N = n = 2$; $p = m = 1$, $\bar{\Omega}_1 = 0.5$; $\bar{\Omega}_2 = 0.7$, with

$$A_1 = \begin{bmatrix} 0.8 & 0.1 \\ 0.3 & -0.4 \end{bmatrix} F_1 = \begin{bmatrix} 0.7 \\ 0 \end{bmatrix}; C_1' = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix};$$

$$A_2 = \begin{bmatrix} 0.5 & 0.4 \\ 0.5 & 0.2 \end{bmatrix}; F_2 = \begin{bmatrix} 0.4 \\ 0 \end{bmatrix}; C_2' = \begin{bmatrix} 0.8 \\ 0 \end{bmatrix},$$

where matrices A_i and $A_i + F_i\Omega_i C_i$ are stable, $\forall i \in \mathcal{I}_N$. Thus, by applying Proposition 1, the LMI (15) is feasible, which proves the global asymptotically stability under any switching rule. The numerical results are shown below.

$$P_1 = \begin{bmatrix} 0.756 & -0.095 \\ -0.095 & 0.660 \end{bmatrix}; P_2 = \begin{bmatrix} 0.648 & 0.039 \\ 0.039 & 0.715 \end{bmatrix};$$

$$\Delta_1 = 1.402; \qquad \Delta_2 = 0.2491.$$

V. CLOSED-LOOP FORMULATION

Consider, now, a more general class of the discrete-time switching system composed of N nonlinear modes ($N \in \mathbb{N}$, $N \geq 1$), and saturating inputs

$$x_{k+1} = A_{\sigma(k)}x_k + F_{\sigma(k)}\varphi_{\sigma(k)}(y_k) + B_{\sigma(k)}\text{sat}(u_k), \quad (18)$$
$$y_k = C_{\sigma(k)}x_k. \quad (19)$$

where $u_k \in \mathbb{R}^m$ is the input vector of the system (18)-(19) at the instant $k \in \mathbb{N}$. The matrices B_i have also appropriate dimensions.

Because of the control inputs are bounded in magnitude, the standard saturation function is considered: $\text{sat}(u(k))_{(\ell)} = \text{sign}(u_{(\ell)}(k))\min(\rho_{(\ell)}, |u_{(\ell)}(k)|)$, $\forall \ell = 1, \dots, m$, where $\rho_{(\ell)} > 0$ denotes the symmetric saturation level relative to the ℓth control. The vector ρ is supposed to be given.

Throughout this paper, the class of control law considered is the switching linear state feedback:

$$u_k = K_{\sigma(k)}x_k \quad (20)$$

where the $m \times n$ matrix $K_{\sigma(k)}$ is the switching control gain.

The saturation is modelled through the dead-zone nonlinearity $\Psi(u_k) = u_k - \text{sat}(u_k)$. By replacing u_k given in (20) and using $\Psi(u_k)$ into (18), the closed-loop model is, thus

$$x_{k+1} = A_{\text{cl},\sigma(k)}x_k + F_{\sigma(k)}\varphi_{\sigma(k)}(y_k) - B_{\sigma(k)}\Psi(u_k), \quad (21)$$

where $A_{\text{cl},i} = A_i + B_iK_i$, $\forall i \in \mathcal{I}_N$.

Let us define the following set necessary to associate the dead-zone with a generalized sector condition. For a given set of matrices $H_i \in \mathbb{R}^{m\times(n+p)}$, $i \in \mathcal{I}_N$, one defines

$$\mathcal{S}(\{H_i\}_{i\in\mathcal{I}_N}, \rho) = \left\{\theta \in \mathbb{R}^{n+p}; -\rho \leq H_i\theta \leq \rho, \forall i \in \mathcal{I}_N\right\}. \quad (22)$$

Lemma 1: Consider $m \times (n+p)$-matrices, $\hat{K}_i = [K_i \;\; 0_{m\times p}]$ and $\hat{J}_i = [J_{1,i} \;\; J_{2,i}]$. If the vector $\hat{x}_k = [x_k' \;\; \varphi_{\sigma(k)}'(y_k)]'$ is an element of $\mathcal{S}(\{\hat{K}_i - \hat{J}_i\}_{i\in\mathcal{I}_N}, \rho)$, then, with the control law $u_k = K_{\sigma(k)}x_k$, the nonlinearity $\Psi(u_k)$ satisfies the following sector condition

$$\text{SC}_{u_k} = \Psi'(u_k)U_i\left[\Psi(u_k) - J_{1,i}x_k - J_{2,i}\varphi(y_k)\right] \leq 0, \quad (23)$$

for any diagonal definite matrix $0_m < U_i \in \mathbb{R}^{m\times m}$, $\forall i \in \mathcal{I}_N$.

Proof: The proof is straightforward from Lemma 1 in [16]. ∎

Remark 2: The switching auxiliaries gains $J_{2,i}$ related to $\varphi_i(C_ix)$ are considered in the generalized sector condition because the proposed Lyapunov function $V(i; x; \varphi_i(C_ix))$ depends on the modal nonlinearities.

Let us define, also, the level sets associated with the proposed Lyapunov function and a given $\gamma > 0$, which will be used, in the sequel, to estimate the basin of attraction of system (21). Consider the set

$$L_V(\gamma) = \{x \in \mathbb{R}^n; V(i; x; \varphi_i(C_ix)) \leq \gamma, \forall i \in \mathcal{I}_N\}, \quad (24)$$

which is naturally related to the two ellipsoids intersections associated with the upper and lower-bounds $\underline{V}_i(x)$ and $\overline{V}_i(x)$

$$\bigcap_{i\in\mathcal{I}_N} \mathcal{E}(P_i + 2C_i'\Omega_i'\Delta_i\Omega_iC_i, \gamma) \subset L_V(\gamma) \subset \bigcap_{i\in\mathcal{I}_N} \mathcal{E}(P_i, \gamma). \quad (25)$$

Remark 1: Due to the presence of nonlinearities $\varphi_i(\cdot)$, the set $L_V(1)$ may be non-convex and disconnected. These are important properties, justified by the fact that in discrete-time case, the transition between x_k and x_{k+1} is not continuous. In addition, not only ellipsoidal sets can be considered in the estimation of the basin of attraction.

VI. LOCAL STABILITY ANALYSIS

In this section, we present a solution for the Problem of local stability analysis related to the system (18)-(19) stated as

Problem 2: (Local Stability Analysis) Given a switched gain K_i, ($i \in \mathcal{I}_N$), of the control law (20), determine a region in the state space, as large as possible included in the basin of attraction $\mathcal{B}_0$ of the system (18)-(19), for any switching rule.

The following proposition solves the problem 2 by using the function V.

Proposition 2: For given matrices $K_i \in \mathbb{R}^{m\times n}$ and fixed $U_i \in \mathbb{R}^{m\times m}$ ($i \in \mathcal{I}_N$), consider optimization variables as matrices $G_i \in \mathbb{R}^{n\times n}$, $J_{1,i} \in \mathbb{R}^{m\times n}$, $J_{2,i} \in \mathbb{R}^{m\times p}$, symmetric positive definite matrices $P_i \in \mathbb{R}^{n\times n}$ and positive diagonal matrices $R_i, Q_i, T_i, W_i, \Delta_i \in \mathbb{R}^{p\times p}$, and a scalar μ. The optimization problem

$$\min_{G_i,\, P_i,\, J_{1,i},\, J_{2,i},\, Q_i,\, R_i,\, T_i,\, W_i,\, \Delta_i,\, \mu} \mu$$

subject to LMIs:

$$\begin{bmatrix} \mu I_n - P_i & -\Pi_4 \\ \star & 2R_i \end{bmatrix} > 0, \quad \forall i \in \mathcal{I}_N, \quad (26)$$

$$\begin{bmatrix} P_i & \Pi_3 & (K_i - J_{1,i})'_{(\ell)} \\ \star & 2Q_i & -J'_{2,i,(\ell)} \\ \star & \star & \rho^2_{(\ell)} \end{bmatrix} > 0, \quad (27)$$

$\forall\, i \in \mathcal{I}_N$; $\forall\, \ell = 1, \cdots, m$, and

$$\begin{bmatrix} P_j - G_j' - G_j & G_j'A_{\text{cl},i} & G_j'F_i & -G_j'B_i & 0_{n\times p} \\ \star & -P_i & \Pi_1 & J'_{1,i}U_i & A'_{\text{cl},i}\Pi_2 \\ \star & \star & -2T_i & J'_{2,i}U_i & F_i'\Pi_2 \\ \star & \star & \star & -2U_i & -B_i'\Pi_2 \\ \star & \star & \star & \star & -2W_j \end{bmatrix} < 0, \quad (28)$$

$\forall\, (i,j) \in \mathcal{I}_N^2$, where Π_1, Π_2 defined by (16) and

$$\Pi_3 = C_i'\bar{\Omega}_i\left(\Delta_i - Q_i\right); \quad \Pi_4 = C_i'\bar{\Omega}_i\left[R_i + \Delta_i\right] \quad (29)$$

allows to obtain an estimation $L_V(1)$ of $\mathcal{B}_0$, induced by the Lyapunov function (12).

Proof: If Inequality (28) holds, we have $P_j - G_j' - G_j < 0$ and $P_i > 0$. Thus, G_j is of full rank, and so $-G_j'P_j^{-1}G_j \leq P_j - G_j' - G_j$ (see [2]). This implies, combined with the change of basis $\text{diag}[G_j^{-1}; I_{n+2p+m}]$ and a Schur complement [1] inequality (31).

$$\begin{bmatrix} A_i' \\ F_i' \\ 0 \end{bmatrix} P_j \begin{bmatrix} A_i' \\ F_i' \\ 0 \end{bmatrix}' + \begin{bmatrix} -P_i & C_i'\Omega_i\left[T_i - \Delta_i\right] & A_i'C_j'\Omega_j\left[W_j + \Delta_j\right] \\ \star & -2T_i & F_i'C_j'\Omega_j\left[W_j + \Delta_j\right] \\ \star & \star & -2W_j \end{bmatrix} < 0. \tag{30}$$

$$\begin{bmatrix} A_{\text{cl},i}' \\ F_i' \\ -B_i' \\ 0 \end{bmatrix} P_j \begin{bmatrix} A_{\text{cl},i}' \\ F_i' \\ -B_i' \\ 0 \end{bmatrix}' + \begin{bmatrix} -P_i & C_i'\Omega_i\left[T_i - \Delta_i\right] & J_{1,i}'U_i & A_{\text{cl},i}'C_j'\Omega_j\left[W_j + \Delta_j\right] \\ \star & -2T_i & J_{2,i}'U_i & F_i'C_j'\Omega_j\left[W_j + \Delta_j\right] \\ \star & \star & -2U_i & -B_i'C_j'\Omega_j\left[W_j + \Delta_j\right] \\ \star & \star & \star & -2W_j \end{bmatrix} < 0. \tag{31}$$

$$\begin{bmatrix} P_i & \Pi_3 \\ \star & 2Q_i \end{bmatrix} - \frac{1}{\rho_{(\ell)}^2} \begin{bmatrix} (K_i - J_{1,i})_{(\ell)}' \\ -J_{2,i,(\ell)}' \end{bmatrix} \begin{bmatrix} (K_i - J_{1,i})_{(\ell)}' \\ -J_{2,i,(\ell)}' \end{bmatrix}' > 0. \tag{32}$$

$$V(\sigma(k); x_k; \varphi_{\sigma(k)}(y_k)) + 2\text{SC}(\varphi_{\sigma(k)}(\cdot), y_k, Q_{\sigma(k)})) \geq \frac{1}{\rho_{(\ell)}^2} \left\| (\hat{K}_i - \hat{J}_i)_{(\ell)} \hat{x}_k \right\|^2. \tag{33}$$

In sequence, by multiplying Inequality (31) on the right by $[x_k' \; \varphi_i'(y_k) \; \Psi'(u_k) \; \varphi_j'(y_{k+1})]'$ and on the left by its transpose and by identifying $i = \sigma(k)$ and $j = \sigma(k+1)$, it leads to inequality

$$\begin{aligned} \delta_k V - 2\text{SC}_{u_k} - 2\text{SC}(\varphi_{\sigma(k+1)}(\cdot), y_{k+1}, W_{\sigma(k+1)})) \\ -2\text{SC}(\varphi_{\sigma(k)}(\cdot), y_k, T_{\sigma(k)})) \leq 0, \end{aligned} \tag{34}$$

which defines an upper-bound for the Lyapunov difference.

Further, by applying a Schur complement on Inequality (27), with respect to the last block, we obtain inequality (32). By multiplying the inequality (32) on the right by $\hat{x}_k = [\; x_k' \quad \varphi_i'(y_k) \;]'$ and on the left by its transpose and by identifying $i = \sigma(k)$, it leads to inequality (33)

The nonlinearity $\varphi_{\sigma(k)}(\cdot)$ verifying the sector bounded condition, and by noting $\hat{K}_i$ and $\hat{J}_i$ as defined in Lemma 1, we have

$$V(\sigma(k); x_k; \varphi_{\sigma(k)}(y_k)) \geq \frac{1}{\rho_{(\ell)}^2} \left\| (\hat{K}_i - \hat{J}_i)_{(\ell)} \hat{x}_k \right\|^2, \tag{35}$$

which induces the inclusion

$$L_V(1) \subset \mathcal{S}\big(\{(\hat{K}_i - \hat{J}_i)\}_{i \in \mathcal{I}_N}, \rho\big). \tag{36}$$

Thus, inside the Lyapunov level set $L_V(1)$, the sector condition (23), related to the dead-zone function is verified.

By multiplying the inequality (26) on the right by $\hat{x}_k = [\; x_0' \quad \varphi_i'(C_i x_0) \;]'$ and on the left by its transpose, one gets the following inequality

$$\mu x_0' x_0 + 2\text{SC}(\varphi_i(\cdot), C_i x_0, R_i) \geq V(i; x_0; \varphi_i(Cix_0)). \tag{37}$$

Due to the fact that the nonlinearity $\varphi_i(\cdot)$ verifies the sector bounded condition, $\forall\, i \in \mathcal{I}_N$, we have

$$\mathcal{E}(\mu I_n) \subset L_V(1). \tag{38}$$

Because of inclusion (36), the local sector condition $\text{SC}_{u_k} \leq 0$ for the dead-zone is verified inside $L_V(1)$, which implies, in addition of the inequality (34), that $\delta_k V < 0$, $\forall\, x \neq 0$. That is asymptotic stability is proved inside $L_V(1)$. Finally, by minimizing μ it implies the maximization of the radius of the ball included in $L_V(1)$, thanks to Inequality (38). ■

Remark 3: A possible choice for setting matrices U_i may be $U_i = \alpha I_m$ (for a fixed $\alpha > 0$) or, if available, the values obtained by the algorithm which has designed the control law, like in [6].

Two different examples are shown to highlight that Proposition 2 is able to improve the size of the estimation of the basin of attraction. In both examples, the switching state-feedback gains are designed by following the method proposed in [6].

Example 2: Consider $N = n = 2$; $p = m = 1$; $\rho = 1$, $\Omega_1 = 1$; $\Omega_2 = 1.1$, with $U_1 = U_2 = I_m$

$$A_1 = \begin{bmatrix} 0.4 & 0.5 \\ 0.3 & 0.9 \end{bmatrix}; B_1 = \begin{bmatrix} 0.9 \\ 0 \end{bmatrix}; F_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}; C_1' = \begin{bmatrix} -0.9 \\ 1 \end{bmatrix};$$

$$A_2 = \begin{bmatrix} 0.9 & 0.6 \\ 0.4 & 0.7 \end{bmatrix}; B_2 = \begin{bmatrix} 1.1 \\ 0 \end{bmatrix}; F_2 = \begin{bmatrix} 1.3 \\ 0 \end{bmatrix}; C_2' = \begin{bmatrix} 0.9 \\ -0.5 \end{bmatrix}.$$

The nonlinearities are $\varphi_1(y) = \Omega_1 \frac{y(1+\cos(40y))}{2}$ and $\varphi_2(y) = \Omega_2 \frac{y(1+\exp(-y^2/2))}{2}$. By applying the algorithm in [6], based on a switched quadratic Lyapunov function, we obtain $\mu = 0.5498$; $U_1 = 1.2846$; $U_2 = 1.0987$; $K_1 = \begin{bmatrix} -0.012 & -0.934 \end{bmatrix}$; $K_2 = \begin{bmatrix} -0.574 & -0.836 \end{bmatrix}$. Proposition 2 provides $\mu = 0.5485$; $\Delta_1 = 0.0381$; $\Delta_2 = 5.3189.10^{-8}$;

$$P_1 = \begin{bmatrix} 0.2640 & 0.0925 \\ 0.0925 & 0.4674 \end{bmatrix}; \; P_2 = \begin{bmatrix} 0.3436 & 0.1327 \\ 0.1327 & 0.4626 \end{bmatrix}.$$

As shown in Fig. 1, our estimation of $\mathcal{B}_0$, which is the $L_V(1)$ sets, contains the one (consisting in the intersection of ellipsoids) of [6]. Moreover, the estimation is composed of disconnected sets and the global area $\mathcal{A}_{L_V(1)} = 7.99$ provided for the proposed method, is 12% larger than the one of [6] ($\mathcal{A}_{\mathcal{E}} = 7.12$).

Two trajectories with initial conditions located in the disconnected sets of $L_V(1)$ are also depicted in Fig. 1. One can notice both trajectories that converge to the origin. Also,

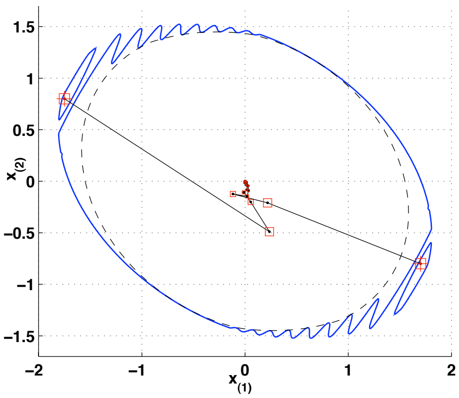

Fig. 1. Disconnected $L_V(1)$ (blue solid line) obtained by Proposition 2 and Level Sets given by [6] (dashed line). Trajectories for two different initial conditions inside disconnected $L_V(1)$, with initial condition given by a cross and a square. The following points are given by a square.

it should be pointed out that every point of the trajectories are located inside of the set $L_V(1)$.

Example 3: Consider, now, $N = n = 2$; $p = m = 1$; $\rho = 1.5$, $\Omega_1 = 0.9$; $\Omega_2 = 1.9$, $U_1 = U_2 = 5I_m$, where $\varphi_1(y) = \Omega_1 \frac{y(1+\cos(40y))}{2}$ and $\varphi_2(y) = \Omega_2 \frac{y(1+\sin(35y))}{2}$. The matrices are

$$A_1 = \begin{bmatrix} -1.25 & 0.2 \\ 0.2 & 0.2 \end{bmatrix}; B_1 = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix}; F_1 = \begin{bmatrix} 1.2 \\ 0 \end{bmatrix}; C_1' = \begin{bmatrix} 1 \\ -0.9 \end{bmatrix};$$

$$A_2 = \begin{bmatrix} 0.4 & 0.3 \\ 0.6 & 1.1 \end{bmatrix}; B_2 = \begin{bmatrix} 0 \\ 0.7 \end{bmatrix}; F_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}; C_2' = \begin{bmatrix} 0.7 \\ 0.5 \end{bmatrix}.$$

The algorithm in [6], provides $\mu = 4.0253$ and $K_1 = \begin{bmatrix} 1.2087 & 0.8253 \end{bmatrix}$; $K_2 = \begin{bmatrix} -1.3854 - 1.9464 \end{bmatrix}$.

By applying Proposition 2, one gets $\mu = 3.6451$, $\Delta_1 = 2.0459.10^{-10}$; $\Delta_2 = 0.3027$;

$$P_1 = \begin{bmatrix} 0.9263 & 0.5606 \\ 0.5606 & 3.5261 \end{bmatrix}; \; P_2 = \begin{bmatrix} 1.2219 & 1.2628 \\ 1.2628 & 2.1955 \end{bmatrix}.$$

In addition, in respect with basin of attraction $\mathcal{B}_0$ estimation, it is possible to see, as shown in Fig. 2, that the proposed method estimation contains the one given by [6]. The resulting areas of the estimation, are respectively

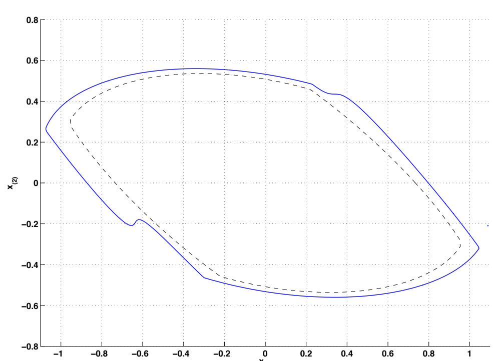

Fig. 2. $L_V(1)$ (blue solid line) obtained by Proposition 2 containing the Level Sets given by [6] (dashed line).

$\mathcal{A}_{L_V(1)} = 1.5860$ for the proposed method, and $\mathcal{A}_{\mathcal{E}} = 1.3497$, which means a region enlargment of 17%.

VII. Conclusion

The extension of Lur'e problem to the switched framework is investigated via a new Lyapunov function which is mode-dependent and which includes the switching nonlinearity by avoiding the integral term. For an interconnection between a switching linear system and a mode-dependent nonlinearity satisfying a sector condition, sufficient conditions are given to ensure the global stability via this Lyapunov function. If a saturating switched state feedback is added to this interconnection, an optimization problem is presented to maximize an estimation of the basin of attraction based on a level set of the Lyapunov function. This estimation may be non-convex and disconnected due to the presence of the nonlinearity. Numerical examples are given to highlight the obtained improvements.

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