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Regulation and Double Price Mechanisms in Markets with Friction

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***Abstract*—In previous work we modeled the real-time power market as a dynamic system and presented an "efficiency-volatility" trade-off theorem stating that in markets with supply friction, an efficient market must have volatile prices. In this paper we introduce a novel market mechanism for power markets where there are two prices: one for the real-time power market for suppliers who have friction and another for frictionless ancillary supply with a marginal cost higher than that of regular suppliers. We show that for a given level of acceptable price volatility the double price system with the ancillary supplier is more efficient than the single price system without the frictionless ancillary supplier.**

I. INTRODUCTION

Deregulation of power industry has been long-awaited [1], [2], and in theory was supposed to increase efficiency, generator availability, and investment in the sector, and would provide protection against theft of service, and even would have positive environmental impacts [3]. The privatization attempts took place in the late 1990s in North America, starting with California, New England and Pennsylvania, New Jersey, Maryland (PJM) Independent System Operators (ISO). The results did not turn out as expected in the beginning; for instance, in the PJM market while the average price of electricity was $26.04 per MWh in Summer 1998 before deregulation, it was $37.97 per MWh the following summer, right after deregulation [4].

In the following years the deregulation of the electricity market became widespread in North America, even though the distribution and transmission sectors mostly remain regulated. During these years, volatility of the electricity prices, which was considered to be a temporal effect, became an epidemic matter [5]. Volatility is a major global problem as for instance, in China, the dominant source of energy consumed in the manufacturing sector is electricity with 40%. Even though one certainly can not neglect the exercise of market power in infamous cases such as Enron, relating volatility to mainly market power exercise [4], [6]–[13] might be a misjudgment.

Electricity markets have distinct features. The constraints due to transmission congestion, voltage and thermal constraints, Kirchoff's Laws, non-convex start-up and shut-down costs are the main reasons behind excess and lack of supply [14]. There is also the *locality* feature: production costs, fuel prices, and the overall demand vary due to location. For instance, California and New England faced very high prices in 2000-2001, whereas PJM faced relatively cheaper prices in the same period [15]. Taking into account all the characteristics of the market naturally leads to a very complex model, and may turn out to be analytically intractable, even in the relatively simpler static case.

Static optimization tools provide solutions to linear and non-linear problems with many equality and inequality constraints. Dynamic optimization goes a step further and takes dynamics into account where optimization is applied on a trajectory, increasing the complexity of the problem for the sake of superior comprehension. Efficiency in power markets is studied in [4] and [16], whereas a dynamic competitive equilibrium for a stochastic market model is studied and the role of volatility for the value of wind generation is presented in [17]. The development and the implementation of a decision tool for the coordination of electrical vehicle battery charging in a dynamic setting is studied in [18].

In a stylized dynamic model, it was shown in [19] that the occurrence of choke-up prices (the maximum price a consumer is willing to pay) is intrinsic to markets with friction, and the market is efficient. In our previous work, we modeled the power market as a dynamic optimization problem, where dynamics are subject to friction, and have shown that there is a trade-off between efficiency and non-volatility [20] such that penalizing volatility is equivalent to penalizing efficiency. Then, we have shown that the trade-off theorem also applies to a dynamic game market model where agents are coupled in their cost functions and dynamics through the price process [21].

The major factor leading to volatile prices in the analysis of [19]–[22] is *friction*. In this paper, we discuss a new market mechanism in order to mitigate the *volatility* problem. We define two price processes: the first one, p_e is the market equilibrium price, which is the market clearing price. The second one is the ancillary price, p_a, which reflects the unit price of the electricity supplied by a hypothetical frictionless ancillary power generator. We model the power market through continuous dynamics and an integrated cost function. The problem is presented as an optimal control problem, and the control action is defined as an increment process applied by the regulator on the equilibrium price process and the ancillary supply process. The HJB Equation is solved and the resulting optimal control is presented. We first show that efficiency is a monotonically decreasing function of the volatility coefficient, which penalizes volatility. We then compare two markets: with and without the frictionless supplier. We show that efficiency in the former market is higher than the latter one for any finite volatility coefficient. Moreover, when volatility coefficient tends to ∞, the efficiency in the former market equals the efficiency in the latter market.

The rest of the paper is organized as follows. In Sec. II we introduce the model. The demand $(d_t; t \geq 0)$, supply $(s_t; t \geq$

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0), ancillary supply $(s^a_t; t \geq 0)$ and the equilibrium price $(p^e_t; t \geq 0)$ processes are defined for the social cost optimizer regulator agent R with the corresponding cost function. In Sec. III we present the optimal control that leads to a volatile price process. In Sec. IV we define volatility and modify the social cost function to account for it. We solve the dynamic stochastic optimization problem for linear dynamics and a quadratic cost function and present the closed form solution. We show that there is a trade-off that can be quantified between efficiency and non-volatility, and present supporting simulations. In Sec. V we compare two markets presenting the two comparative theorems described in the previous paragraph. We present supporting simulations in Sec. VI and conclude in Sec. VII.

II. MODEL

In this section we define the optimization problem for the social cost optimizer, the regulator agent R, in the power market. We define the four dimensional state process $((d_t, s_t, s^a_t, p^e_t)^\top; t \geq 0)$. We have the demand process $(d_t; t \geq 0)$, the supply process $(s_t; t \geq 0)$, the ancillary supply $(s^a_t; t \geq 0)$, the equilibrium price process $(p^e_t; t \geq 0)$, and the ancillary price process $(p^a_t; t \geq 0)$. Also, the initial values $\{d_0, s_0, s^a_0, p^e_0, p^a_0 \in \mathbb{R}\}$, are bounded, independent of the standard Wiener processes $(w^d_t, w^s_t; t \geq 0)$. Demand, supply and the ancillary supply dynamics are defined as

$$\begin{aligned} dd_t &= f^d(d_t, p^e_t)dt + \sigma_d dw^d_t, \quad t \geq 0, \\ ds_t &= f^s(s_t, p^e_t)dt + \sigma_s dw^s_t, \quad t \geq 0, \\ ds^a_t &= u^a_t dt, \quad |u^a_t| \leq u^a_{max}, \quad t \geq 0, \end{aligned} \tag{1}$$

using deterministic continuous functions $f^d(\cdot)$ and $f^s(\cdot)$ with $(w^d_t; t \geq 0)$ and $(w^s_t; t \geq 0)$, standard Wiener processes. The function $f^d(\cdot)$ is allowed to be a function of d and p^e, values of demand and the equilibrium price, and $f^s(\cdot)$ is allowed to be a function of s and p^e, values of supply and the equilibrium price processes. We adopt the stepwise adjustment model for the ancillary supply, where the bounded input control process u^a controls the amount of the increment. Note that the ancillary supply dynamics are deterministic and directly tied to the control action to be applied by the regulator.

We employ the following assumptions on the functions $f^d(\cdot)$ and $f^s(\cdot)$ which are both subject to friction:

A1: For constant $C_1 > 0$, $f^d(0,0) \leq C_1$, $f^s(0,0) \leq C_1$ and

$$\left|\frac{\partial f^d}{\partial p}\right| + \left|\frac{\partial f^s}{\partial p}\right| \leq C_1.$$

This assumption assures that the instantaneous change in demand and supply processes is constrained. This is one of the key properties of power dynamics; the suppliers and consumers are unable to respond to abrupt changes in the system. The reason for the supplier's sluggishness is the slow ramp up in power production, whereas for the consumers it is usually not handy or very complicated and costly to startup and shutdown a running machine or a household. Note that the ancillary supply dynamics are not subject to friction.

A2: $f^d(\cdot)$ is a strictly decreasing function of p, whereas $f^s(\cdot)$ is strictly increasing.

This assumption ensures that an increase in price is reflected on the deterministic portion of decreasing demand dynamics and increasing supply dynamics.

We also adopt the assumption below for initial values of the processes and the disturbance process:

A3: $\{d_0, s_0, s^a_0, p^e_0, p^a_0 \in \mathbb{R}\}$ are bounded, and $\{w^d, w^s \in \mathbb{R}\}$ are mutually independent and independent of the initial conditions. Instantaneous variances of the disturbance processes, σ^2_d, σ^2_s, are bounded.

For the equilibrium price we adopt the stepwise price adjustment model [23] for the optimizer (so called regulator agent R), where the bounded input control process $(u^e_t; t \geq 0)$ controls the amount of the increment. The ancillary price is not controlled, that is, it is equal to the marginal cost of ancillary supply production. The price processes are defined as

$$\begin{aligned} dp^e_t &= u^e_t dt, \quad |u^e_t| \leq u^e_{max}, \quad t \geq 0, \\ p^a_t &= c^{a\prime}(s^a_t), \quad t \geq 0. \end{aligned} \tag{2}$$

The actions of R is the set $\{u = [u^e, u^a]^\top : [|u^e|, |u^a|]^\top < [u^e_{max}, u^a_{max}]^\top, u \in \mathbb{R}^2, u^e_{max}, u^a_{max} > 0\}$ which is simply constrained adjustment. R looks at the demand, supply, ancillary supply, price dynamics and taking into consideration their dynamics, cost function and the constraints, takes an action in terms of increasing or decreasing the power price and the current ancillary supply. This action is intended to linearly control market dynamics by only applying increments.

Following [19], the individual loss functions of the consumer, supplier and the ancillary supplier are defined respectively:

$$\begin{aligned} g^d(\cdot) &= p^e \cdot s + p^a \cdot s^a - v \cdot \min(d, s) + c_{bo}(r), \\ g^s(\cdot) &= c(s) - p^e \cdot s, \\ g^a(\cdot) &= c^a(s^a) - p^a \cdot s^a. \end{aligned}$$

Here, $c(s), c^a(s^a) \in \mathbf{C}^2_b : \mathbb{R} \to \mathbb{R}_+$, where $\mathbf{C}^2_b$ denotes the family of all bounded functions which are twice differentiable. The function $c(s)$ is the production cost, and is convex and strictly increasing with respect to s. One needs to find a realistic production cost function in order to have a reasonable power market model. We note that in real power markets, production cost is not a convex function. The startup and shutdown costs, transmission line constraints, weather fluctuations all affect the production cost function. However, if one neglects the startup and shutdown costs, the cost function resembles a convex function. For our model we will assume a continuous convex cost. The constant $v \in (0, \infty)$ is the value the consumer obtains for a unit of power. The blackout cost function $c_{bo}(r) \in \mathbf{C}^2_b : \mathbb{R} \to \mathbb{R}_+$ is the cost paid by the consumer in case of an unmet demand, convex, zero on $[0, \infty)$ and strictly decreasing on $(-\infty, 0)$, where r denotes the reserve, $r := s + s^a - d$. In the spot market, the consumer, D, pays $p^e \cdot s + p^a \cdot s^a$, the price of all the supply bought, to the supplier, S and the ancillary supplier S^a. Note that v is multiplied by the supplied portion of the consumer's demand.

Further note that the suppliers S and S^a pay for all the cost of production, and gain unit price multiplied with all the units of supply bought by the consumer agent D. Finally, we employ the following social cost function that is simply the addition of loss functions of D, S and S^a integrated in time:

$$J(\cdot) = \mathbb{E}\int_0^\infty e^{-\rho t}[-v \cdot \min(d_t, s_t + s_t^a) + c(s_t) + c^a(s_t^a) + c_{bo}(r_t)]dt. \quad (3)$$

In the section that follows, we consider the optimality of the cost function presented above with the dynamics (1), the control (2) and the cost function (3) under **A1**, **A2** and **A3**.

III. CENTRALIZED CONTROL FORMULATION

In this section we analyze the optimal control problem in terms of the state vector $x := [d, s, s^a, p^e]^\top$. As stated before, this is a centralized control problem for the regulator agent R. In principle, R's objective is to regulate demand, supply and the ancillary supply processes so that the best social outcome is achieved. In this section we show that the optimal control of the regulator is a "bang-bang" control. We rewrite (1) and (2) in vector form with stochastic dynamics as

$$dx = \psi dt + G dw, \quad t \geq 0, \quad (4)$$

where w is a 4×1 standard Wiener process. We set $x := [d, s, s^a, p^e]^\top$, and write

$$\psi = \begin{pmatrix} f^d(d, p^e) \\ f^s(s, p^e) \\ u^a \\ u^e \end{pmatrix}, \ G = \begin{pmatrix} \sigma_d & 0 & 0 & 0 \\ 0 & \sigma_s & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The loss function of (3) is rewritten here as $g(x) = g(d, s, s^a, p^e) = -v \cdot \min(d, s + s^a) + c(s) + c^a(s^a) + c_{bo}(s + s^a - d)$. The admissible control for the regulator is specified as $\mathcal{U} = \{u : u$ adapted to $\sigma(x_s, s \leq t), t \geq 0, [|u^e|, |u^a|]^\top < [u^e_{max}, u^a_{max}]^\top\}$. Therefore, the regulator can at most increase or decrease the price and ancillary supply with unit u_{max} and $-u_{max}$ at each iteration. Finally, the cost associated with (4) and a control u is specified to be $J(x, u) = \mathbb{E}[\int_0^\infty e^{-\rho t} g(d_t, s_t, s_t^a, p_t^e)dt]$. Further, we set the value function

$$V(x) \triangleq \inf_{u \in \mathcal{U}} J(x, u). \quad (5)$$

The theorem that follows claims the existence and uniqueness of the optimal control to the problem (5).

Theorem 3.1: There exists a unique $\hat{u} \in \mathcal{U}$ such that $J(x, \hat{u}) = \inf_{u \in \mathcal{U}} J(x, u)$, and if $\tilde{u} \in \mathcal{U}$ is another control such that $J(x, \tilde{u}) = J(x, \hat{u})$, then $\mathbb{P}_\Omega(\tilde{u}_s \neq \hat{u}_s) > 0$ only on a set of times $s \in [0, T]$ of Lebesgue measure zero.

Proof: For $(v_t, \eta_t; t \geq 0)$ two standard mutually independent Wiener processes, we can define alternative control actions in the form of two stochastic differential equations $d(p_t^e)^p = u_t^e dt + \epsilon dv_t$, $d(s_t^a)^p = u_t^a dt + \epsilon d\eta_t$. The resulting value function can be shown to be a viscosity solution to an HJB Equation, and this solution is unique (see Chapter 4, [24]). For fixed u^a and u^e, we have $P\{\lim_{\epsilon \to 0} \sup_{t \geq 0} |(s^a)^p - s^a| = 0\} = 1$, and $P\{\lim_{\epsilon \to 0} \sup_{t \geq 0} |(p^e)^p - p^e| = 0\} = 1$, and Lebesgue's Dominated Convergence Theorem is employed to obtain $|J^p(x, u) - J(x, u)| \to 0$, as $\epsilon \to 0$. Therefore, $V^p(x) \to V(x)$ as $\epsilon \to 0$ follows. An application of Arzela-Ascoli Theorem leads to $V^p(x) \to V(x)$, as $\epsilon \to 0$. ■

Now that we have shown the existence and uniqueness of the optimal control, we consider approaches for computing the optimal solution. For a function class $\mathcal{G}$: (i) $V \in \mathbf{C}^{1,2}([0, \infty) \times \mathbb{R}^4)$, (ii) $|V| \leq C(1 + d^{k_1} + s^{k_2} + (s^a)^{k_3})$ where C, k_1, k_2, k_3 depend on V, we write the HJB Equation

$$\rho V + \sup_{u \in \mathcal{U}} \left\{ -\frac{\partial^\top V}{\partial x}\psi \right\} - \frac{1}{2}\text{Tr}\left(\frac{\partial^2 V}{\partial x^2} G G^\top \right) - g(\cdot) = 0. \quad (6)$$

A classical solution to the HJB Equation (6) may not exist as $GG^\top$ is not of full rank in (4) [25]. Therefore, viscosity solutions are adopted.

A. Perturbation Method

Following [23] in order to make the $GG^\top$ matrix full rank, we add $(1/2)\epsilon^2(\partial^2 V/\partial s^{a2}) + (1/2)\epsilon^2(\partial^2 V/\partial p^{e2})$ to (6). For the function class $\mathcal{G}$:

$$\begin{aligned} &\rho V - \frac{\partial V^p}{\partial d} f^d(d, p) - \frac{\partial V^p}{\partial s} f^s(s, p) \\ &+ \sup_{u \in \mathcal{U}} \left\{ -\frac{\partial V^p}{\partial s^a} u^a - \frac{\partial V^p}{\partial p^e} u^e \right\} - \frac{1}{2}\sigma_d^2 \frac{\partial^2 V^p}{\partial d^2} - \frac{1}{2}\sigma_s^2 \frac{\partial^2 V^p}{\partial s^2} \\ &- \frac{1}{2}\epsilon^2 \frac{\partial^2 V^p}{\partial s^{a2}} - \frac{1}{2}\epsilon^2 \frac{\partial^2 V^p}{\partial p^{e2}} - g(d, s, p) = 0. \quad (7) \end{aligned}$$

It can be easily shown that Equation (7) has a solution with an argument similar to Theorem 3.1. Also, it can be proved that $|J^p(x, u) - J(x, u)| \to 0$ uniformly as $\epsilon \to 0$ and using Lebesgue's dominated convergence theorem. Therefore, one can prove that $V^p(x) \to V(x)$ as $\epsilon \to 0$. This gives us the result below.

Corollary 1: For the function class $\mathcal{G}$ the solution $u^* \in \mathcal{U}$ to the perturbed HJB Equation (7) is found as:

$$u^* = \arg\min_{u \in \mathcal{U}} \frac{\partial^\top V^p}{\partial x}\psi = \begin{bmatrix} -\text{sgn}\left(\frac{\partial V^p}{\partial p^e}\right) \cdot u^e_{max} \\ -\text{sgn}\left(\frac{\partial V^p}{\partial s^a}\right) \cdot u^a_{max} \end{bmatrix}, \quad (8)$$

where $\{u^e, u^a\}$ were previously defined as $ds_t^a = u_t^a dt$, $dp_t^e = u_t^e dt$, $t \geq 0$, $|u_t| \leq u_{max}$.

Hence, the optimal control is found as a double switch. When we look at the the perturbed HJB Equation (7), the value function is differentiable everywhere in the function class $\mathcal{G}$, and due to the constraint defined on the control action, the optimal control is represented as a bang-bang control. The bound $|V| \leq C(1 + d^{k_1} + s^{k_2} + s^{a^{k_3}})$ is a direct estimate, and $V \in \mathbf{C}^{1,2}([0, \infty) \times \mathbb{R}^4)$ is satisfied due to the stochasticity of the dynamics (4) with the full rank disturbance process [24]. One can numerically solve the value function for implementation.

IV. EFFICIENCY–VOLATILITY TRADE-OFF

Efficiency and non-volatility are two desirable properties of power markets. In this section we show that these two notions contradict each other in a market model with friction. Therefore, one has to trade-off efficiency and non-volatility in designing the market mechanism.

The optimal control policy for the system (1) and the price processes due to the nature of the optimal control (8) were shown in the previous section. Since the demand and supply processes are defined by stochastic differential equations, they fluctuate on their trajectories and the regulator modifies the ancillary supply and the equilibrium price process for the optimal outcome.

We consider an objective function that penalizes the control action

$$u := [u^e, u^a]^\top. \quad (9)$$

Recall the loss function defined in (3) and the input control process $(u_t;\, t \geq 0)$ defined in (2). The cost associated with the system is defined as

$$J(x,u) = \mathbb{E}\int_0^\infty e^{-\rho t}[g(d_t, s_t, s_t^a, p_t^e) + u_t^\top R u_t]dt, \quad (10)$$

where we add $u_t^\top R u_t$ to the term (3). We have $R > 0$ in the form of

$$R \triangleq \begin{bmatrix} r_e & 0 \\ 0 & r_a \end{bmatrix}, \quad (11)$$

where r_e is the *volatility coefficient*. We will show that if the *volatility coefficient* decreases, the expected cost decreases. In other words, if high volatility is not penalized, the social cost defined in (3) increases.

We define *efficiency* as the quantity obtained when the expected cost is multiplied by -1 and the control action penalizing part is taken out: $-\mathbb{E}\int_0^\infty e^{-\rho t}[g(d_t, s_t, s_t^a, p_t^e)]dt$. *Volatility* on the other hand is defined by the price fluctuation measured by $\mathbb{E}\int_0^\infty e^{-\rho t}\|u_t^e\|^2 dt$.

We require two more assumptions here:

A4: The supply process $(s_t; t \geq 0)$ and the demand process $(d_t;\, t \geq 0)$ are linear mean-reverting processes that have bounded variances.

A5: The production costs $c(\cdot)$, $c^a(\cdot)$ and the blackout cost $c_{bo}(\cdot)$ functions are convex functions in the form of $c(\cdot)$, $c^a(\cdot)$, $c_{bo}(\cdot) \in \mathbf{C}_b^2 : \mathbb{R} \to \mathbb{R}_+$.

As a special case, we study quadratic cost functions and introduce a penalty term for u in the cost function, removing the bound on the control input:

$$J(x,u) = \mathbb{E}\int_0^\infty e^{-\rho t}[x_t^\top Q x_t + 2x_t^\top D + u_t^\top R u_t]dt, \quad (12)$$

where $x := (d, s, p)^\top$, $Q \geq 0$, $R > 0$ and D is a continuous vector valued function. Employing **A4**, we have the dynamics

$$dx(t) = (Ax(t) + Bu(t) + c)\, dt + G dw, \quad t \geq 0, \quad (13)$$

where $\Psi(x,u) \triangleq (Ax(t) + Bu(t) + c)$, w is a 4×1 standard Wiener process, $x(0) = x_0$, and A, B, G are in the form of

$$A = \begin{pmatrix} * & 0 & 0 & * \\ 0 & * & 0 & * \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix},$$
$$c = \begin{pmatrix} * \\ * \\ 0 \\ 0 \end{pmatrix}, \quad G = \begin{pmatrix} \sigma_d & 0 & 0 & 0 \\ 0 & \sigma_s & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (14)$$

The coefficients $\theta = [A, B, c] \in \mathbf{\Theta} \in \mathbb{R}^{n(n+m+1)}$, will be called the dynamics parameters.

A. Existence and Uniqueness of the Optimal Control

From now on, we will work on (12) and (13). We take the admissible control set $\mathcal{U}_2 = \{u : u \text{ adapted to } \sigma(x_s, s \leq t),\, t \geq 0,\, \int_0^T e^{-\rho t}\|u_t\|^2 dt < \infty,\, \rho > 0\}$. The minimum cost-to-go from any initial state (x) is described by the *value function* which is defined by $V(x) = \inf_{u \in \mathcal{U}_2} J(x,u)$. The optimal control problem is well defined with the Hamilton-Jacobi-Bellman (HJB) Equation,

$$\rho V + \sup_{u \in \mathcal{U}_2}\left\{-\frac{\partial V}{\partial x}^\top \Psi - r\|u\|^2\right\} - \frac{1}{2}\mathrm{Tr}\left(\frac{\partial^2 V}{\partial x^2} G G^\top\right) - x^\top Q x - 2x^\top D = 0. \quad (15)$$

Theorem 4.1: Equation (15) has a unique solution for the admissible control set $\mathcal{U}_2$.

Proof: The proof follows a generic argument. A typical treatment can be found in [26].

B. Closed Form Solution

Standard arguments due to Anderson and Moore (1989) [27] show that $J(x,u)$ is quadratic in x. Furthermore, at any point $x \in \mathbb{R}^4$, the minimum cost-to-go is quadratic in x. Consequently, one can model V of the form $V(x) = x^\top K x + 2x^\top S + q$, for all $x \in \mathbb{R}^4$. Substituting $V(x)$ in (15) and applying first order optimization gives

$$u^*(t) = -R^{-1}B^\top[Kx(t) + S]. \quad (16)$$

Solving the closed loop expression we get the Algebraic Riccati Equation and the two equations:

$$-\rho K + KA + A^\top K - KBR^{-1}B^\top K + Q = 0, \quad (17)$$
$$-\rho S + (A - BR^{-1}B^\top K)^\top S + Kc + D = 0, \quad (18)$$
$$-\rho q + 2S^\top c - S^\top BR^{-1}B^\top S + \mathrm{Tr}(KGG^\top) = 0. \quad (19)$$

The linear quadratic optimal control problem admits a unique optimum feedback controller given by (16) which obtains the minimum value of the cost function: $J(x,u^*) = x^\top K x + 2x^\top S + q$.

C. Efficiency–Volatility Trade-off

We would like to look at the relation between r_e (11), the *volatility coefficient* and the state penalizing part of the cost function obtained when the volatility term is removed from the cost function. We define the *state penalizing cost* as

$$J^*_{sp}(x,u^*) \triangleq \mathbb{E}\int_0^\infty e^{-\rho t}[x^\top(t)Qx(t)+2x^\top(t)D]dt, \quad (20)$$

which is termed *efficiency* when multiplied by -1.

Theorem 4.2: Suppose **A1**-**A5** hold. For all $x \in \mathbb{R}^4$, the state penalizing cost portion (20) of the cost function (12) applying the optimal control u^* is an increasing function of r_e. Moreover $\lim_{r_e\to\infty} J^*_{sp}(x,u^*) < \infty$.

Proof: First we show that $J_{sp}(x,u^*;r_e) \in \mathbf{C}_b$. The quadratic cost function was shown (12) to be $J(x,u^*) = \int_0^T\{x^{*\top}(t)Qx^*(t)+2x^{*\top}(t)D+\|u^{*\top}(t)Ru^*\|^2\}dt$. We seek to compute $dJ_{sp}(x,u^*)/dr_e$. However, the calculations are easier for $dJ_{sp}(x,u^*)/d\gamma$, where $\gamma = r_e^{-1}$. We have

$$\frac{dJ^*_{sp}}{dt}\Big|_{t=T} = f^J(x) = \Big(x^{*\top}(t,\gamma)Qx^*(t,\gamma)+2x^{*\top}(t,\gamma)D\Big)\Big|_{t=T}, \quad (21)$$

with the initial condition $J^*_{sp}(0)=0$. We solve the stochastic differential equation, arrange the terms, and obtain

$$\begin{aligned} R_1(t) &= \int_s^t e^{\left(A-\mathbb{B}\gamma\mathbb{B}^\top K(\tau,\gamma)\right)^\top(\tau-s)}\Big(-\mathbb{B}\gamma\mathbb{B}^\top\frac{dK(\tau,\gamma)}{d\gamma} \\ &\quad -\mathbb{B}\mathbb{B}^\top K(\tau,\gamma)\Big)^\top\cdot e^{\left(A-\mathbb{B}\gamma\mathbb{B}^\top K(\tau,\gamma)\right)^\top(s-\tau)}d\tau, \\ R_2(t) &= \int_0^t e^{\left(A-\mathbb{B}\gamma\mathbb{B}^\top K(\tau,\gamma)\right)(t-s)}GG^\top R_1(t) \\ &\quad e^{\left(A-\mathbb{B}\gamma\mathbb{B}^\top K(\tau,\gamma)\right)^\top(t-s)}ds, \\ \frac{dJ_{sp}(x,u^*)}{d\gamma} &= 2\int_0^T \mathrm{Tr}(R_2(t)*Q)dt. \end{aligned} \quad (22)$$

All the terms in (22) are positive except for $(-\mathbb{B}\gamma\mathbb{B}^\top dK(\tau,\gamma)/d\gamma - \mathbb{B}\mathbb{B}^\top K(\tau,\gamma))^\top$. Using a similar approach shown in [22] we get $R_1(t) < 0$. Therefore, one obtains $dJ_{sp}(x,u^*)/d\gamma < 0$ for all $\gamma > 0$. As $dJ_{sp}(x,u^*)/dr_e = -(dJ_{sp}(x,u^*)/d\gamma)r_e^{-2}$, one obtains $dJ_{sp}(x,u^*)/dr_e > 0$ for all $r_e > 0$. Also, due to **A4**, the system is a mean reverting process. Therefore, in the open loop, the system is non-divergent. The argument $\lim_{r_e\to\infty} J^*_{sp}(x,u^*) < \infty$ follows. ■

Increasing the *volatility coefficient* increases social cost, therefore decreases *efficiency*, while decreasing the coefficient decreases the cost, hence increases *efficiency*. On the other hand increasing the *volatility coefficient* decreases volatility, whereas decreasing *volatility coefficient* increases volatility. Therefore, there is a trade-off between social *efficiency* and *non-volatility*.

V. COMPARISON WITH SINGLE PRICING SCHEME

In our previous work [20] we introduced the three dimensional market model where we proved the efficiency non-volatility trade-off. In this paper, we extend the model by adding an extra state, the supply of a hypothetical frictionless ancillary supplier that produces electricity by a higher marginal cost than the regular supplier. From now on we will call the model without the ancillary supplier the *single price* model and the one with the ancillary supplier the *double price* model, due to the two price processes involved in the system: p^e and p^a. In this section we present the main theorem of the paper: we show that even though the marginal cost of production of the ancillary supplier is uniformly higher than that of the regular one, the efficiency obtained in the double price model is higher than that of the single price model for a given level of acceptable volatility.

First we introduce two new assumptions for the system:

A6: $c'(s) < c^{a\prime}(s)$ for all $s \geq 0$.

A7: The set of dynamics parameters $\mathbf{\Theta}$ is a compact set in the form of $\mathbf{\Theta} \subset \mathbb{R}^{n(n+m+1)}$.

Recall (11), and let us denote the *efficiency function* $E_1(r_e)$ for the single price model described in [20], and $E_2(r_e)$ for the double price model described in this paper. We first show that these two functions are continuous and bounded:

Lemma 5.1: Under **A1**-**A7**, the functions $E_1(r_e) \in \mathbf{C}_b[0,\infty)$ and $E_2(r_e) \in \mathbf{C}_b[0,\infty)$.

Plan of the Proof: First we show that

$$|E(r_e) - E({r_e}')| \leq C_B(|r_e - {r_e}'|),$$

where $r_e, {r_e}' \in Q_B \triangleq \{r_e : r_e > 0, |r_e| \leq B\}$, $B > 0$. By extending the time horizon to infinite, it can be verified that $E(r_e) = \lim_{T\to\infty} E^T(r_e)$ for all r_e. The upper bounds for $K(x), S(x), q(x)$ are obtained by direct estimates of the growth rate of E. ■

Next we show that the efficiency for the double price model described in this paper, E_2 is uniformly higher than the single price model, E_1:

Theorem 5.2: Let **A1**-**A7** hold, and $s^a(0) = 0$. We have $E_2(r_e) > E_1(r_e)$ for all $r_e \in (0,\infty)$.

Proof: The proof has three steps. Recall that u^a is defined as in (9), and we denote u^* (16) as the optimal control action that minimizes (12). First, we show that for u^*, $P(\mathbb{E}\int_0^\infty e^{-\rho t}1_{u^a_t\neq 0}dt > 0) = 1$ on the probability space $(\Omega,\mathcal{F},P)$. Secondly, we show that $u^a(t) = 0, t \in [0,\infty)$ implies $E_2(r_e) = E_1(r_e)$ for any $r_e > 0$. Lastly, it is shown that the unique minimizing control solution u^* can not have $u^a(t) = 0, t \in [0,\infty)$.

Step 1: In this step we show that there exists a neighborhood $\mathcal{N}_{r_e}$ such that $d, s, s^a \in \mathcal{N}_{r_e}$ implies $u^a > 0$. Then we show that $\mathbb{E}\int_0^\infty e^{-\rho t}1_{s_t,d_t\in\mathcal{N}_{r_e}}dt > 0$ w.p.1.

It can be shown that u^a is a function of K and S defined in (17) and (18), therefore a function of d, s, s^a. Employing **A4** for demand and supply processes we have the mean reverting processes with time varying means

$$dd(t) = [\theta^d(t) - \rho^d d(t)]dt + \sigma^d dw^d(t), \quad (23)$$

$$ds(t) = [\theta^s(t) - \rho^s s(t)]dt + \sigma^s dw^s(t), \quad (24)$$

where $w^d(t), w^s(t)$ are mutually independent Wiener processes due to **A3**. Note that $\theta^d(t)$ and $\theta^s(t)$ are correlated and adapted to $\sigma(d(s), s(s), s \leq t)$. The mean reversion friction parameters ρ^d, ρ^s and the volatility parameters σ^d, σ^s are assumed to be constant, while $\theta^d(t), \theta^s(t)$ are bounded functions for $r_e > 0$. Solving the Itō formula, the unique solutions are given by the Gaussian processes

$$\begin{aligned} d(t) &= e^{-\rho^d t} d_0 + e^{-\rho^d t} \int_0^t e^{\rho^d \tau} \theta^d(\tau) d\tau \\ &\quad + e^{-\rho^d t} \sigma^d \int_0^t e^{\rho^d \tau} dw^d(\tau), \\ s(t) &= e^{-\rho^s t} s_0 + e^{-\rho^s t} \int_0^t e^{\rho^s \tau} \theta^s(\tau) d\tau \\ &\quad + e^{-\rho^s t} \sigma^s \int_0^t e^{\rho^s \tau} dw^s(\tau), \end{aligned} \tag{25}$$

with mean and covariance functions

$$\begin{aligned} \mathbb{E}[d(t)] &= e^{-\rho^d t} d_0 + e^{-\rho^d t} \int_0^t e^{\rho^d \tau} \theta^d(\tau) d\tau, \\ Cov[d(\tau), d(t)] &= \frac{{\sigma^d}^2}{2\rho^d} (e^{-\rho^d |\tau - t|} - e^{-\rho^d (\tau + t)}), \\ \mathbb{E}[s(t)] &= e^{-\rho^s t} s_0 + e^{-\rho^s t} \int_0^t e^{\rho^s \tau} \theta^s(\tau) d\tau, \\ Cov[s(\tau), s(t)] &= \frac{{\sigma^s}^2}{2\rho^s} (e^{-\rho^s |\tau - t|} - e^{-\rho^s (\tau + t)}). \end{aligned} \tag{26}$$

We make the observation that for an arbitrary $r_e > 0$, there exists a neighborhood $\mathcal{N}_{r_e}$ such that $d, s, s^a \in \mathcal{N}_{r_e}$ implies $u^a > 0$. We want to show that $\mathbb{E} \int_0^\infty e^{-\rho t} 1_{s_t, d_t \in \mathcal{N}_{r_e}} dt > 0$ w.p.1 given $\sigma^d, \sigma^s > 0$. On the probability space $(\Omega, \mathcal{F}, P)$ we define the random variable

$$\omega := \left(\mathbb{E} \int_s^\infty e^{-\rho t} 1_{s_t, d_t \in \mathcal{N}_{r_e}} dt | d_s = d, s_s = s \right), \tag{27}$$

such that $P(\omega > 0) \geq P(\omega_1 > 0)$ where $P(w_1 > 0) = \min[P(\omega_2 > 0), P(\omega_3 > 0)]$, and ω_2 and ω_3 are defined as

$$\begin{aligned} \omega_2 &:= \left(\mathbb{E} \int_s^\infty e^{-\rho t} 1_{d_{t,1}, d_{t,2} \in \mathcal{N}_{r_e}} dt | d_{s,1} = d, d_{s,2} = d \right), \\ \omega_3 &:= \left(\mathbb{E} \int_s^\infty e^{-\rho t} 1_{s_{t,1}, s_{t,2} \in \mathcal{N}_{r_e}} dt | s_{s,1} = s, s_{s,2} = s \right), \end{aligned} \tag{28}$$

where d_1, d_2 are two distinct realizations of (23) and s_1, s_2 are distinct realizations of (24). As the probability is shown to be positive for a single realization, due to the mutual independence of the two realizations for ω_2 and ω_3, employing (26) we obtain $P(\omega_2 > 0) = 1$ and $P(\omega_3 > 0) = 1$, which implies

$$P(\omega_1 > 0) = 1. \tag{29}$$

Therefore $P(\omega > 0) = 1$, and consequently we have $\mathbb{E} \int_0^\infty e^{-\rho t} 1_{u^a \neq 0} dt > 0$ w.p.1.

Step 2: The next thing is to show that $u^a(t) = 0,\ t \in [0, \infty)$ implies $E_2(r_e) = E_1(r_e)$ for any $r_e > 0$. Remember that B is in the form given in (14), therefore $s^a(0) = 0$ implies $s^a(t) = 0, t \in [0, \infty)$ given $u^a(t) = 0,\ t \in [0, \infty)$. Therefore, $E_2(r_e) = E_1(r_e)$ for any $r_e > 0$ given that $u^a(t) = 0,\ t \in [0, \infty)$.

Step 3: In Step 1 we have shown that for $\sigma^d, \sigma^s > 0$, we have $\mathbb{E} \int_0^\infty e^{-\rho t} 1_{u^a \neq 0} dt > 0$. The term (16) is the unique minimizing control, u^*, therefore $u^a(t) = 0, t \in [0, \infty)$ can not be the unique minimizing control. Hence, $E_2(r_e) > E_1(r_e)$ for all $r_e \in (0, \infty)$. ■

Finally, we show asymptotically, as the *volatility coefficient* tends to 0, that means as volatility tends to infinity, the efficiency obtained in both systems are equal to each other:

Theorem 5.3: Let **A1**-**A7** hold. We have

$$\lim_{r_e \to 0^+} \sup_{\theta \in \mathbf{\Theta}} E_1(r_e; \theta) = E_1^* < \infty, \text{ and,} \tag{30}$$

$$\lim_{r_e \to 0^+} \sup_{\theta \in \mathbf{\Theta}} E_2(r_e; \theta) = E_2^* < \infty, \tag{31}$$

for all $\theta \in \mathbf{\Theta}$; and moreover, $\lim_{r_e \to 0^+} E_1(r_e; \theta) = \lim_{r_e \to 0^+} E_2(r_e; \theta)$ for all $\theta \in \mathbf{\Theta}$.

Proof: The system is $[A_\theta, B_\theta]$ controllable and $[A_\theta, Q^{1/2}]$ observable due to the specific structure of the matrices given in (14). Therefore for any $r_e > 0$, $E_1(r_e), E_2(r_e) \in \mathbf{C}_b[0, \infty)$. It was shown in Theorem 4.2 that (20) is an increasing function of r_e and $\lim_{r_e \to \infty} J_{sp}^*(x, u^*) < \infty$. Therefore, $\lim_{r_e \to 0^+} J_{sp}^*(x, u^*) < \infty$ follows which implies $\lim_{r_e \to 0^+} E(r_e) < \infty$, and E_1 and E_2 are monotonically decreasing functions of r_e. This proves the claim $\lim_{r_e \to 0^+} \sup_{\theta \in \mathbf{\Theta}} E_1(r_e) = E_1^* < \infty$ and $\lim_{r_e \to 0^+} \sup_{\theta \in \mathbf{\Theta}} E_2(r_e) = E_2^* < \infty$. We employ **A7**, therefore the suprema exist. Also, due to the particular structure of the matrices (14) u^a can be shown to be a function of r^e through K and S such that all the third row entries of K and S decay to 0 as $r_e \to 0^+$. One can partition the Q matrix as follows:

$$Q = \begin{bmatrix} q11 & q12 & q13 & 0 \\ q21 & q22 & q23 & 0 \\ q31 & q32 & q33 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where $q33 > q11$ due to **A6**. The solution to the algebraic Riccati equation gives the matrix K, a 4×4 matrix, where $k44 > k34$ and $k44 > k33$ such that $k44/k33 \to \infty$ as $r_e \to 0^+$. In the closed loop the control action affects the system dynamics with $BR^{-1}B^\top(Kx + S)$, therefore as $r_e \to 0^+$, $u_4/u_3 \to \infty$. As the system is controllable and observable in the defined set $\mathbf{\Theta}$, we have $u_4 < \infty$, therefore $u_3 \to 0$. Also, as shown in the proof of Theorem 5.2, $E_2(r_e; \theta) = E_1(r_e; \theta)$ for any $r_e > 0$ given that $u^a(t) = 0,\ t \in [0, \infty)$. Hence, $E_1^*(\theta) = E_2^*(\theta)$ for all $\theta \in \mathbf{\Theta}$. ■

These results can be summarized as follows: for the models described, friction leads to volatility. A new model was introduced in this paper by adding an ancillary power producer, which produces power with a very high cost, but is not subject to friction, and does not affect the future dynamics of the supply and demand processes. It has been shown that this new system performs better in terms of efficiency even though the marginal cost of production is higher for the introduced ancillary supplier. Moreover, it was shown that if volatility is allowed to swing freely, then the systems have the same performance. The intuition for this is as follows: there is a

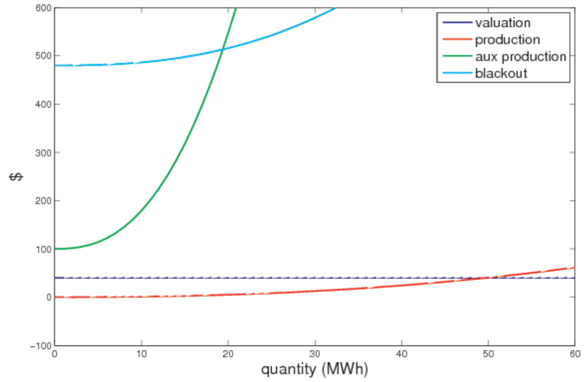

Fig. 1. Marginal Costs

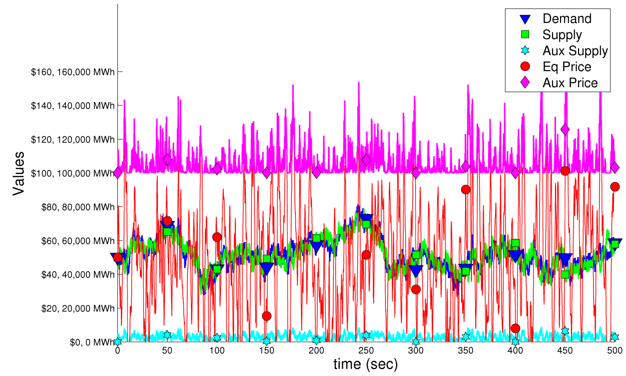

Fig. 2. $r_e = 0.01$

very high correlation between friction and volatility, therefore if volatility is allowed to go very high, not punished, then the problem due to friction disappears, as the system will be practically frictionless. Hence, there is no incentive to use expensive ancillary power generators.

VI. SIMULATIONS

The suppliers' marginal production cost functions, consumer's unit valuation and the marginal blackout cost in case of an unmet demand are depicted in Fig. 1. The dynamics of the consumer, supplier, ancillary supplier, equilibrium price and the ancillary price are:

$$\begin{aligned}
d_{k+1} &= d_k - \rho^d \left(d_k - (\beta - p_k^e)\right) \Delta t + \sigma w_k^d \sqrt{\Delta t}, \\
s_{k+1} &= s_k - \rho^s \left(s_k - (p_k^e - \gamma)\right) \Delta t + \sigma w_k^s \sqrt{\Delta t}, \\
s_{k+1}^a &= s_k^a + u_k^a \Delta t, \\
p_{k+1}^e &= p_k^e + u_k^e \Delta t, \\
p_{k+1}^a &= c^{a\prime},
\end{aligned}$$

where $\rho^d = \rho^s = 0.05$, $\Delta t = 0.001$, $\beta = 100$, $\gamma = 0$, $\sigma = 4$, $t_{final} = 500$, with the initial conditions $x_0 = (d_0, s_0, s^a, p_0)^\top = (50, 50, 0, 50)^\top$.

In Fig. 2 and Fig. 3 we present the trajectories of a single realization of demand, supply, ancillary supply and the price processes for a double price system. Fig. 2 shows the dynamics when $r_e = 0.01$ and Fig. 3 displays for $r_e = 100$. We see that for a low value of r_e, the equilibrium price is very volatile, but the ancillary price moves within a short range, whereas for a high value of r_e, the system is less volatile at the expense of very high ancillary supply prices.

In Fig. 4 and Fig. 5 we compare two systems: the single price system with only supply and demand processes and the double price system with the addition of the ancillary supply and the corresponding ancillary price process. Same variables presented are used for both systems with $r_e = 1$. We see that the double price system heavily uses the ancillary supplier despite its very costly production, and moreover as proved in Theorem 5.2, the efficiency in Fig. 5 is higher than in Fig. 4.

Lastly in Fig. 6 we compare the efficiencies for the single price system and the double price system for any given

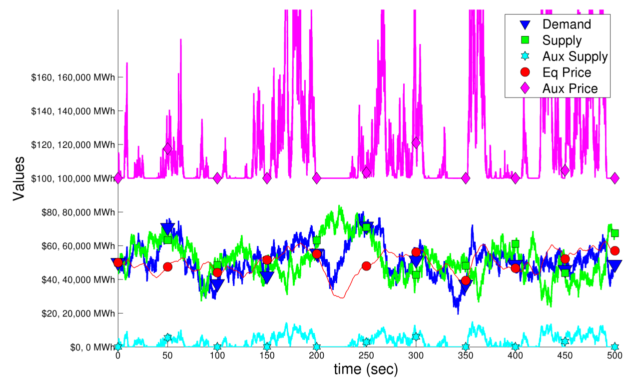

Fig. 3. $r_e = 100$

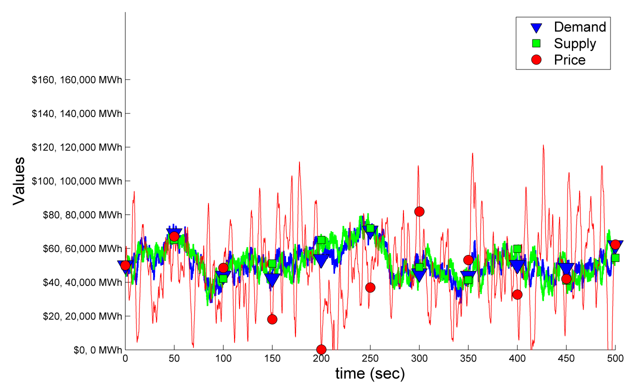

Fig. 4. Single Pricing $r_e = 1$

volatility obtained analytically via Theorem 5.3. One can clearly see that for any given volatility the double price system performs better than the single price system. The asymptotical equality of the systems as shown in Theorem 5.3 also can be seen.

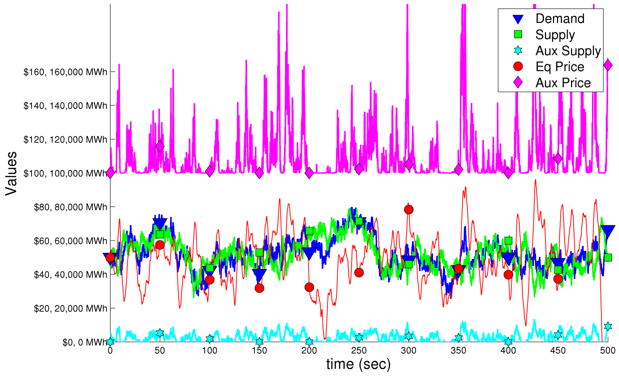

Fig. 5. Double pricing $r_e = 1$

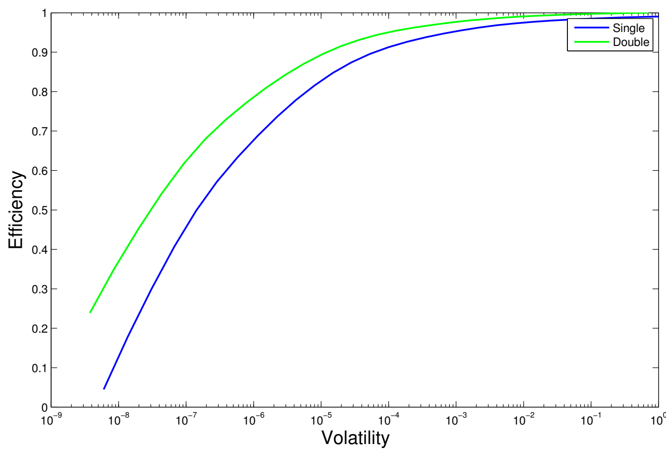

Fig. 6. Trade-off Comparisons

VII. Conclusion

This paper extends the model given in [20] by adding an ancillary supplier to the system. It is shown that even though the ancillary supplier's marginal cost of production is considerably higher than *regular* producers, its ability to respond to abrupt changes quickly makes it very useful. Note that in a static system, it would by no means be advantageous to let the ancillary supply price spike up, while keeping the real-time equilibrium within reasonable limits. However, a dynamic analysis of the system shows that this action is indeed the optimal solution, as it is commonly applied by the regulator as shown in the simulation results, once more showing that a static analysis of the dynamic grid system might be misleading.

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